Polynomial Interpolation and Marcinkiewicz-Zygmund Inequalities on the Unit Circle*

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The objective of this paper is to derive an intimate relationship among three important mathematical tools, namely: polynomial interpolation, Marcinkiewicz-Zygmund inequalities, and $A_p$-weights. In particular, it is shown that minimum $p$-separation of sample points on the unit circle together with certain uniform $A_p$-weights generated by these sample points constitute a necessary and sufficient condition for the validity of the Marcinkiewicz-Zygmund inequality evaluated at these points, which in turn, is equivalent to the Jackson-type estimate, using the Popov-Andreev module of continuity, of polynomial interpolation, again at these sample points.

Key Words: Rate of convergence; polynomial interpolation; $H^p$-norm estimation; Marcinkiewicz-Zygmund inequalities; $A_p$-weights.

1. INTRODUCTION

Let $U: |z| < 1$ denote the open unit disk in the complex plane with boundary $T$, and consider a family $(t_{n,k}: k = 0, \ldots, n) , n = 1, 2, \ldots,$ of real numbers that satisfy

$$0 \leq t_{n,0} < \cdots < t_{n,n} < 2\pi ,$$

(1.1)

together with its corresponding family $(z_{n,k}: k = 0, \ldots, n) , n = 1, 2, \ldots,$ of points on $T$, defined by

$$z_{n,k} = e^{it_{n,k}} .$$

(1.2)

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In this paper, we will be concerned with the approximation properties of
the Lagrange polynomial interpolation operators $L_n$ defined by

$$
\begin{align*}
L_n &= L_n\{z_{n,k}\}, \quad A \to \pi_n, \\
(L_n f - f)(z_{n,k}) &= 0, \quad k = 0, \ldots, n, \quad f \in A,
\end{align*}
$$

with sample points $(z_{n,k})$, where $A$ denotes, as usual, the disk algebra of
all functions analytic in $U$ and continuous on the closure of $U$, and $\pi_n$
denotes the space of all (complex-valued) polynomials of degree not exceeding $n$. To study how well the operators approximate we will use the
Hardy space norms

$$
\|f\|_p = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta \right)^{1/p}, \quad \text{for } 1 \leq p < \infty;
$$

$$
\max_{|z|=1} |f(z)|, \quad \text{for } p = \infty,
$$

for $f \in A$, and the notion of “mean module of continuity”

$$
\tau(f; \delta)_p := \left( \frac{1}{2\pi} \int_0^{2\pi} \left( \Omega(f; e^{i\theta}, \delta) \right)^p \, d\theta \right)^{1/p},
$$

with

$$
\Omega(f; \theta, \delta) := \sup_{t_1, t_2 \in [\theta - \delta, \theta + \delta]} |f(e^{it_1}) - f(e^{it_2})|,
$$

where $\delta > 0$.

The reason for considering the metric of measurement $\|\cdot\|_p$, $p < \infty$, which is weaker than the uniform norm $\|\cdot\|_\infty$ for studying the convergence
of $L_n f$ to $f$, $f \in A$, is due to a result of Vertesi [10] which says that for any family $(z_{n,k}: k = 0, \ldots, n)$ of sample points on $T$, there exists some $f_0 \in A$, for which

$$
\limsup_{n \to \infty} |(L_n f)(z)| = \infty, \text{ a.e. } T.
$$

The first result on the convergence in $U$ of $L_n f$ to $f$ for every $f \in A$ is
Lozinski’s theorem [5], which says that when the $(n+1)$st roots of unity

$$
\omega_{n,k} := e^{2\pi ik/(n+1)}, \quad k = 0, \ldots, n,
$$

are used as sample points, then the Lagrange polynomial interpolation
operators $L_n = L_n\{\omega_{n,k}\}$ in (1.3), with $z_{n,k} = \omega_{n,k}$, have the property

$$
\|L_n f - f\|_p \to 0, \quad f \in A,
$$

for $f \in A$.\)
as long as $p < \infty$. An elegant proof of (1.6) for $p = 2$ was provided by Walsh and Sharma [11], and generalizations of (1.6) to rational approximation were investigated by Saff and Walsh [8], and also by Sharma and Vétesi [9]. More recently, in a joint work with Shen [2], we have established the following result on the rate of convergence of (1.6).

**Theorem A.** Let $1 < p < \infty$, and $L_n = L_n((\omega_n,k),\cdot)$ be the Lagrange polynomial interpolation operators defined in (1.3) with $z_{n,k} = \omega_{n,k}$. Then there exists some positive constant $c_p$, such that

$$
\|L_nf - f\|_p \leq c_p \tau \left( f; \frac{1}{n+1} \right)_p, \quad f \in A, \quad n = 1,2,\ldots, \quad (1.7)
$$

where $\tau(f; \delta)_p$ is the mean module of continuity defined in (1.4).

Hence, if the derivative $f'$ of $f \in A$ is a function in the Hardy space $H^p$, then it follows from the estimate

$$
\tau \left( f; \frac{1}{n+1} \right)_p \leq \frac{c_p}{n} \|f'\|_p \quad (1.8)
$$

of Popov and Andreev [7], that

$$
\|L_nf - f\|_p \leq \frac{c_p}{n} \|f'\|_p. \quad (1.9)
$$

(Of course the constants $c_p$ in (1.7)–(1.9) may be different.) Therefore, if the weaker estimate (1.9) in place of (1.7) in Theorem A is used, then we have the following result, which reflects the “sharpness” of Theorem A.

**Corollary A.** Let $1 < p < \infty$ and $1/p < \alpha < 1$, and consider the function class

$$
W^\alpha_p := \{ f \in A : \|f'(rz)\|_p = O((1 - r)^{\alpha-1}), \quad r < 1 \}. \quad (1.10)
$$

Then $f \in W^\alpha_p$ if and only if

$$
\|L_nf - f\|_p \leq \frac{c_p f}{n^{\alpha}}, \quad (1.11)
$$

where $(L_n)$ is the sequence of Lagrange polynomial interpolatory operators with sample points at the $n$th roots of unity.

The key technique in the proof of Theorem A is the Marcinkiewicz-Zygmund inequalities

$$
c_p \|P_n\|_p^p \leq \frac{1}{n+1} \sum_{k=0}^{n} |P_n(\omega_n,k)|_p^p \leq d_p \|P_n\|_p^p, \quad P_n \in \pi_n, \quad (1.12)
$$
where 0 < c_p ≤ d_p < ∞. (See [13, p. 28] for a proof for the trigonometric polynomials.)

In this paper, we let the roots of unity perturb. In other words, the family of sample points \( \{z_{n,k}\} \) in (1.2) is allowed to be different from \( \{\omega_{n,k}\} \). In this regard, let us first recall the well-known fact (cf. Walsh [11, Chap. 7]) which says that the Lagrange polynomial operators in (1.3) satisfy

\[ \|L_n f - f\|_\infty \to 0 \]

for all functions \( f \) analytic on the closure of \( U \) if and only if the family of sample points \( z_{n,k} \) in (1.2) is uniformly distributed on \( T \), meaning that the corresponding family \( \{t_{n,k}\} \) in (1.1) satisfies

\[ \max_{0 \leq k \leq n} \left| t_{n,k} - \frac{2\pi k}{n+1} \right| \to 0. \quad (1.13) \]

Since our concern is the convergence of \( L_n f, f \in A \), we have to use a norm measurement such as \( \| \cdot \|_p, \ p < \infty \), which is weaker than \( \| \cdot \|_\infty \); and in our other joint work [1] with Shen, we showed that if the rate of convergence to 0 in (1.13) is \( O\left(\frac{1}{n}\right) \), then the interpolation operators \( L_n = L_n(\{z_{n,k}\}, \cdot) \) in (1.13) have the approximation property

\[ \|L_n f - f\|_p \leq c_p \inf_{P_n \in \pi_n} \| f - P_n \|_\infty, \quad f \in A, \quad (1.14) \]

where \( 1 < p < \infty \) and \( c_p \) depends only on \( p \). On the other hand, we have also shown in [1] that under the same rate of convergence \( O\left(\frac{1}{n}\right) \) in (1.13), the Marcinkiewicz-Zygmund inequalities

\[ c_p \|P_n\|_p^p \leq \frac{1}{n + 1} \sum_{k=0}^{n} |P_n(z_{n,k})|^p \leq d_p \|P_n\|_p^p, \quad P_n \in \pi_n, \quad (1.15) \]

where 0 < c_p ≤ d_p < ∞ and 1 < p < ∞, are also valid.

It turns out, as we will show in this paper, that by using the mean module of continuity to measure the rate of \( L^p \)-convergence of the Lagrange interpolation operators, as in Theorem A for the special case \( z_{n,k} = \omega_{n,k} \), the validity of the Marcinkiewicz-Zygmund inequalities (1.15), for the general setting of \( \{z_{n,k}\} \), is equivalent to the rate of convergence, \( \tau(f; \frac{1}{n+1}) \), of \( L_n f = L_n(\{z_{n,k}\}, f) \) to every \( f \in A \), without any assumption on the distribution of the family of sample points \( \{z_{n,k}\} \) on \( T \). Hence, the Marcinkiewicz-Zygmund inequalities give a precise tool for the study of the convergence of the Lagrange interpolation operators, when the mean module of continuity is used to measure the error of approximation. It remains to give a sharp characterization of the distribution of the family of
sample points \( \{z_{n,k}\} \) on \( T \) for which the Marcinkiewicz-Zygmund inequalities hold.

Motivated by our proof of (1.14) in [1] (see, in particular, [1; Lemma 1]), we consider the following notion of uniform \( A_p \)-weights for the purpose of characterizing the distribution of these sample points on \( T \).

**Definition.** A family of functions \( f_n \in A \) is said to provide uniform \( A_p \)-weights, where \( 1 < p < \infty \), if there exists some positive constant \( c_p \), such that

\[
\left\{ \frac{1}{|I|} \int_I |f_n(e^{it})|^p dt \right\}^{1/p} \left\{ \frac{1}{|I|} \int_I |f_n(e^{it})|^{-q} dt \right\}^{1/q} < c_p,
\]

for all intervals \( I \subset [0, 2\pi) \), where \( |I| \) denotes the length of \( I \) and \( q = p/(p - 1) \).

We will characterize the distribution of \( \{z_{n,k}\} \subset T \) by applying the condition of uniform \( A_p \)-weights of the polynomials

\[
W_n(z) := \prod_{k=0}^{n} \left( z - \frac{n}{n+1} z_{n,k} \right)
\]

(1.17)

generated by \( \{z_{n,k}\} \). Since the uniform distribution of \( z_{n,k} = e^{i\alpha_n} \) on \( T \), as described by (1.13), is also equivalent to the uniform convergence of

\[
|w_n(z)|^{1/(n+1)} \to |z|,
\]

(1.18)
on all compact subsets of \( |z| > 1 \), where

\[
w_n(z) := \prod_{k=0}^{n} (z - z_{n,k}),
\]

(1.19)
the uniform \( A_p \)-weight condition on \( W_n \) in (1.17) may be considered as an analog of uniform distribution of \( \{z_{n,k}\} \) on \( T \) for the study of \( L^p \) convergence of \( L_n f \) to \( f \), \( f \in A \). The main result in this paper is the following.

**Theorem 1.** Let \( 1 < p < \infty \), \( \{z_{n,k}\} \subset T \), and \( L_n = L_n(\{z_{n,k}\}, \cdot) \) be defined as in (1.3). Then the following statements are equivalent.

(i) The family \( \{W_n\} \) as defined in (1.17) provides uniform \( A_p \)-weights, and the family of sample points \( \{z_{n,k}\} \) satisfies

\[
\min_{0 \leq j < k \leq n} |z_{n,j} - z_{n,k}| \geq \frac{c}{n}, \quad n = 1, 2, \ldots,
\]

(1.20)
for some positive constant \( c \).

(ii) The Marcinkiewicz-Zygmund inequalities (1.15) are valid for the family \( \{z_{n,k}\} \).
(iii) The Lagrange polynomial interpolation operators $L_n = L_n(\{z_{n,k}\}, \cdot)$ have the approximation property

$$
\| L_n f - f \|_p \leq c_p \left( f; \frac{1}{n + 1} \right)_p, \quad f \in A, \quad n = 1, 2, \ldots, \quad (1.21)
$$

where $c_p > 0$ depends only on $p$.

The impact of this theorem to complex approximation theory is that both the Marcinkiewicz-Zygmund inequalities and the property of uniform $A_p$-weights can be used to study the rate of $L^p$-convergence of the Lagrange polynomial interpolation operators with sample points $\{z_{n,k}\} \subset T$ which may be different from the $(n + 1)$st roots of unity $\{\omega_{n,k}\}$. For instance, as mentioned above, since the $O(\frac{1}{n})$ convergence of (1.13) implies the validity of the Marcinkiewicz-Zygmund inequalities (1.15), the following result is a consequence of Theorem 1.

**Corollary 1.** Let $1 < p < \infty$ and suppose that the family $(t_{n,k})$ in (1.1) that defines the family $\{z_{n,k}\}$ of sample points as in (1.2) satisfies

$$
\max_{0 \leq k \leq n} \left| t_{n,k} - \frac{2\pi k}{n + 1} \right| \leq \frac{\delta_p}{n + 1}, \quad n = 1, 2, \ldots, \quad (1.22)
$$

for some positive number $\delta_p > 0$, then the Lagrange polynomial interpolation operators $L_n = L_n(\{z_{n,k}\}, \cdot)$ have the approximation property (1.21).

What is perhaps more interesting is that the “continuity” in $p$ of the property of uniform $A_p$-weights implies the continuity in $p$ of the rate of $L^p$-convergence of $\| L_n f - f \|_p$, $f \in A$, and of the validity of the Marcinkiewicz-Zygmund inequalities, to be stated below. For this and other nice properties of $A_p$-weights, we refer the reader to the important work [6] of Muckenhoupt.

**Corollary 2.** Suppose that the Marcinkiewicz-Zygmund inequalities (1.15) are valid for some $p_0$, $1 < p_0 < \infty$, then they are also valid for all $p$, $p_0 - \delta < p < p_0 + \delta$, for some $\delta$, where $0 < \delta < p_0 - 1$.

**Corollary 3.** Suppose that the Lagrange polynomial interpolation operators $L_n = L_n(\{z_{n,k}\}, \cdot)$ have the approximation property (1.21) for some $p_0$, $1 < p_0 < \infty$, then they have the same approximation property for all $p$, $p_0 - \delta < p < p_0 + \delta$, for some $\delta$, where $0 < \delta < p_0 - 1$.

We remark that condition (1.22) on $\{z_{n,k}\}$ is sharp in the sense that there exists some $\delta_2 > 0$, such that the operators $L_n = L_n(\{e^{i\theta_{n,k}}\}, \cdot)$. 

where

\[
\begin{align*}
\hat{t}_{n,k} := \begin{cases} 
\frac{2\pi k}{n+1} + \frac{\delta_2}{n+1}, & \text{for } 0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor; \\
\frac{2\pi k}{n+1}, & \text{for } \left\lfloor \frac{n}{2} \right\rfloor < k \leq n,
\end{cases}
\end{align*}
\]  

(1.23)

satisfy (1.21) for \( p = 2 \), but not for sufficiently large \( p \) or sufficiently small \( p - 1 > 0 \). In fact, we will show in the final section of this paper that \( \delta_2 > 0 \) can be so chosen that we even have \( \|\hat{L}_n\|_p \to \infty \) for all sufficiently large \( p \), and (1.21) fails for all sufficiently small \( p - 1 \), although it is obvious that

\[
\|\hat{L}_n f - f\|_p \leq \|\hat{L}_n f - f\|_2 \to 0, \quad f \in A,
\]

(1.24)

for all \( p, 1 \leq p \leq 2 \). This example also shows that the bounds \( \delta_p \) in (1.22) necessarily tend to 0 as \( p \to 1 \) or \( \infty \).

2. PRELIMINARY LEMMAS

We first derive a necessary condition on the distribution of the sample points \( \{z_{n,k}\} \subset T \) for the validity of the order of approximation (1.21) and that of the Marcinkiewicz-Zygmund inequalities (1.15).

**Lemma 1.** Let \( 1 < p < \infty \). Then for either (1.21) or (1.15) to be valid, the family of sample points \( \{z_{n,k}\} \) must satisfy

\[
\min_{0 \leq k \leq n} |z_{n,k+1} - z_{n,k}| \geq \frac{c_p}{n}, \quad n = 1, 2, \ldots,
\]

(2.1)

for some positive constant \( c_p \), where \( z_{n,n+1} := z_{n,0} \).

**Proof.** Without loss of generality, we may assume that \( t_{n,0} = 0 \),

\[
\min_{0 \leq k \leq n} (t_{n,k+1} - t_{n,k}) = t_{n,1} - t_{n,0} = t_{n,1},
\]

(2.2)

and

\[
N := \left\lfloor \frac{2\pi}{t_{n,1}} \right\rfloor - 1 > n,
\]

(2.3)

where \( t_{n,n+1} := 2\pi \) and \( \lfloor x \rfloor \) denotes, as usual, the largest integer not exceeding \( x \). Indeed, if \( N \leq n \), then (2.1) is obvious. For each \( N \) satisfying (2.3), set

\[
P_N(z) := \frac{(z^{N+1} - 1)(z - z_{n,1})}{(N+1)(z - 1)(z - \omega_{N,1})},
\]

(2.4)
It is clear that \( P_N \in \pi_N \) and satisfies

\[
\begin{cases}
|P_N(1)| = \frac{\omega_{n,1} - 1}{\omega_{n,1} - 1}, \\
|P_N(\omega_{N,k})| \leq 1, & k = 0, 1; \\
|P_N(\omega_{N,k})| = 0, & k = 2, \ldots, N,
\end{cases}
\]

as well as

\( P_N(z_{n,1}) = 0 \).

As a consequence of the second and third lines in (2.5), it follows from the Bernstein inequality and the second inequality in (1.12), that

\[ \|P_N\|_p \leq d_p N^{1-(1/p)} . \]

Hence, by the assumption (1.21) and the estimate (1.8), we have

\[ \|L_n^{-1} P_N\|_p \leq \|P_N\|_p + \frac{c_p}{n} \|P_N\|_p \leq \frac{c}{n} N^{1-(1/p)} , \]

where \( L_n = L_n((z_{n,k}) \cdot) \).

On the other hand, by using the notation

\[ \rho_n := \frac{n + 1}{n} \]

we have, from the first line in (2.5) and by applying Schwarz's lemma,

\[
\left| \frac{\rho_n^2 - z_{n,1}}{\rho_n(\omega_{N,1} - 1)} \right| = \left| \frac{(L_n P_N)(z_{n,1}) - (L_n P_N)(1)}{\rho_n(z_{n,1} - 1)/(\rho_n^2 - z_{n,1})} \right| \\
\leq 2 \max_{|z| = \rho_n} |(L_n P_N)(z)| .
\]

Hence, by introducing the notation

\[ q_n^\#(z) := z^n q_n \left( \frac{1}{z} \right) \]

for any \( q_n \in \pi_n \), we have

\[
\|L_n P_N\|_\infty = \|(L_n P_N)^*\|_\infty \\
\geq \left\| \left( L_n P_N \right)^* \left( \frac{1}{\rho_n} \right) \right\|_\infty = \max_{|z| = 1/\rho_n} |(L_n P_N)^*(z)| \\
= \max_{|z| = 1/\rho_n} \left| z^n (L_n P_N) \left( \frac{1}{z} \right) \right| = \rho_n^{-n} \max_{|z| = 1/\rho_n} |(L_n P_N)(z)| \\
\geq \frac{1}{2e} \frac{\rho_n^2 - 1}{\rho_n(\omega_{N,1} - 1)} \geq \frac{N + 1}{2e \pi (n + 1)} ,
\]

(2.9)
so that by a well-known result in $H^p$ estimates (see [3, Theorem 5.9]), we obtain

$$
\| L_n P_N \|_p \geq c \left( 1 - \frac{1}{\rho_n} \right)^{1/p} \max_{|z|=1/\rho_n} |(L_n P_N)(z)|
\geq c n^{-1/p} \rho_n^{-n} \| L_n P_N \|_\infty \geq \frac{c n}{n^{1+1/p}}. \tag{2.10}
$$

Here and throughout, all constants, independent of $n$ and $N$, are denoted by the same symbol $c$. Hence, by applying (2.3), (2.7), and (2.10), we have

$$
\left| \frac{2\pi}{\tau_{n,1}} \right| = N + 1 \leq cn.
$$

In view of (2.2), this yields (2.1).

Next, suppose that the Marcinkiewicz-Zygmund inequalities (1.15) are valid for the family $(z_{n,k})$ of sample points on $T$. In particular, (1.15) is valid for the polynomial $P^*_n \in \pi_n$ uniquely determined by the interpolating property

$$
P^*_n(z_{n,k}) = \begin{cases} 
1 & \text{for } k = 0 \\
0 & \text{for } k = 1, \ldots, n.
\end{cases} \tag{2.11}
$$

Then by the second inequality in (1.15), we have

$$
\| P^*_n \|_p \leq cn^{-1/p}; \tag{2.12}
$$

and by applying Schwarz’s lemma, we also have

$$
\max_{|z|=1} | P^*_n(\rho_n z) | \geq \frac{| P^*_n(z_{n,1}) - P^*_n(z_{n,0}) |}{2 | \rho_n(z_{n,1} - z_{n,0})/(\rho^2 - z_{n,0} z_{n,1}) |} \geq \frac{\rho_n^2 - 1}{2 \rho_n | e^{i\tau_{n,1}} - 1 |} \geq \frac{1}{2(n + 1)\tau_{n,1}}.
$$

Hence, it follows from a similar estimation as in (2.10) that

$$
\| P^*_n(\rho_n^2 z) \|_p \geq c \left( 1 - \frac{1}{\rho_n} \right)^{1/p} \max_{|z|=1/\rho_n} | P^*_n(\rho_n^2(z)) |
\geq \frac{c}{n^{(1+1/p)\tau_{n,1}}} \tag{2.13}
$$
On the other hand, using the same estimate as in (2.9), we have
\[ \|P_n^* (\rho_n z \cdot )\| \leq e^2 \|P_n^*\|_p. \]
Hence, combining this with the estimates (2.12) and (2.13), we obtain
\[ t_{n,1} \geq \frac{c}{n}, \]
which, in view of (2.2), yields (2.1).

We will next show that the condition (2.1), on the distribution of the sample points \( \{z_{n,k}\} \subset T \), already gives rise to certain inequalities.

**Lemma 2.** Let \( 1 < p < \infty \), and suppose that the family \( \{z_{n,k}\} \subset T \) of sample points satisfies (2.1). Then the following two statements hold.

(i) For any complex constants \( a_0, \ldots, a_n \), there exists some \( f^* \in H^p \), such that
\[ f^* \left( \frac{1}{\rho_n} z_{n,k} \right) = a_k, \quad k = 0, \ldots, n, \]
and
\[ \|f^*\|_p \leq \frac{c}{n + 1} \sum_{k=0}^{n} |a_k|^p. \]

(ii) There exists some positive constant \( c_p \) such that
\[ \frac{1}{n + 1} \sum_{k=0}^{n} \left| f \left( \frac{1}{\rho_n} z_{n,k} \right) \right|^p \leq c_p \|f\|_p^p, \quad f \in H^p. \]

**Proof.** Since \( 1 - \|1/\rho_n z_{n,k}\| = \frac{1}{n + 1} \), it follows from Carleson's interpolation theory (see [3, Chapter 9]) that it is sufficient to verify that the family \( \{(1/\rho_n) z_{n,k}\} \), when \( \{z_{n,k}\} \) satisfies (2.1), is uniformly separated. Furthermore, by a result in [4, p. 300], to show that \( \{(1/\rho_n) z_{n,k}\} \) is uniformly separated, it is sufficient to show that it satisfies
\[ \min_{j \neq k} \left| \frac{1}{\rho_n} z_{n,j} - \frac{1}{\rho_n} z_{n,k} \right| \geq c \left( 1 - \frac{1}{\rho_n} z_{n,k} \right), \quad n = 1, 2, \ldots, \quad (2.14) \]
as well as
\[ \sum_{j, \|(1/\rho_n) z_{n,j} - z\| < r} \left( 1 - \frac{1}{\rho_n} z_{n,j} \right) \leq cr, \quad 0 < r < 1 \text{ and } |z| = 1, \quad (2.15) \]
for some positive constant $c$. The reason is that (2.15) is a condition for the distribution

$$
\sum_{j=0}^{n} \left( 1 - \left| \frac{1}{\rho_{n,j}} z_{n,j} \right| \right) \delta_{z_{n,j}},
$$

where $\delta_w$ denotes the Dirac delta at $w$, to be a Carleson measure (cf. [4]). It is clear that (2.14) is a trivial consequence of the hypothesis (2.1). To prove that (2.15) holds for all $r, 0 < r < 1$, and all $z \in T$, we need the notation

$$
\Delta(z, r) := \{ \xi : |\xi - z| < r \}.
$$

If $r < \frac{1}{n + 1}$, then $(1/\rho_n) z_{n,k} \in \Delta(z, r)$, so that

$$
\sum_{(1/\rho_n) z_{n,k} \in \Delta(z, r)} \left( 1 - \left| \frac{1}{\rho_n} z_{n,k} \right| \right) = 0.
$$

So, we only need to consider $r \geq \frac{1}{n + 1}$. Let $\theta_1$ and $\theta_2$ denote, respectively, the minimum and maximum arguments (mod $2\pi$) of the points on $\Delta(z, r) \cap \{ z : |z| = 1/\rho_n \}$. Then it is obvious that $(\theta_2 - \theta_1) \leq \pi r$. In addition, by the hypothesis (2.1), it follows that there are at most $\max \frac{cnr + 1}{(n + 1)r}$ number of $z_{n,k}$ with argument in $[\theta_1, \theta_2]$. Thus, we have

$$
\sup_{r > 0} \frac{1}{r} \sum_{j, k : (1/\rho_n) z_{n,j} - z < r} \left( 1 - \left| \frac{1}{\rho_n} z_{n,j} \right| \right)
$$

$$
= \max_{r \geq \frac{1}{n + 1}} \frac{1}{r} \sum_{(1/\rho_n) z_{n,k} \in \Delta(z, r)} \left( 1 - \left| \frac{1}{\rho_n} z_{n,k} \right| \right)
$$

$$
\leq \max_{r \geq \frac{1}{n + 1}} \frac{cnr + 1}{(n + 1)r} \leq c,
$$

and this holds for all $z \in T$. This completes the proof of the lemma.

3. PROOF OF THEOREM 1

We are now ready to establish Theorem 1. We will first prove the equivalence of (ii) and (iii), and next the equivalence of (i) and (ii).

Proof of (iii) $\Rightarrow$ (ii). Suppose that the order of approximation (1.21) is valid for all $f \in \mathcal{A}$. Let $P_n$ be an arbitrary polynomial in $\pi_n$, and consider
the functions \( f_n \in \mathcal{A} \) defined by

\[
    f_n(z) := f^* \left( \frac{1}{\rho_n} z \right),
\]

where \( f^* \in H^p \) is given by (i) in Lemma 2 with \( a_k = P_n(z_{n,k}) \). Hence, we have

\[
    \begin{cases}
        f_n(z_{n,k}) = P_n(z_{n,k}), \\
        \|f^*\|_p^p \leq \frac{c}{n + 1} \sum_{k=0}^{n} |P_n(z_{n,k})|^p.
    \end{cases}
\]

By the uniqueness of polynomial interpolation, we also have

\[
    L_n f_n = P_n,
\]

where \( L_n = L_n((z_{n,k}), \cdot) \). Hence, by (1.21) with \( f = f_n \) and applying (1.8), we have

\[
    \|P_n\|_p = \|L_n f_n\|_p \leq \|f_n\|_p + \frac{c_p}{n} \|f_n\|_p.
\]

But since \( \|f_n\|_p \leq \|f^*\|_p \) and

\[
    \|f_n\|_p \leq c \left( 1 - \frac{1}{\rho_n} \right)^{-1} \|f^*\|_p = c(n + 1)\|f^*\|_p
\]

(see [3, p. 80]), we have, from (3.3),

\[
    \|P_n\|_p \leq c\|f^*\|_p;
\]

and it follows from the second statement in (3.2) that

\[
    \|P_n\|_p^p \leq \frac{c}{n + 1} \sum_{k=0}^{n} |P_n(z_{n,k})|^p.
\]

This is the first inequality in (1.15) with \( c_p = c^{-1} \). The second inequality in (1.15) is a simple consequence of (ii) in Lemma 2 with \( f(z) = P_n(\rho_n z) \), by observing that

\[
    \|P_n(\rho_n \cdot)\|_p = \rho_n^n \left\| P_n^* \left( \frac{\cdot}{\rho_n} \right) \right\|_p \\
    \leq \rho_n^n \|P_n^*\|_p = \rho_n^n \|P_n\|_p \leq c \|P_n\|_p.
\]
where the notation of \( P_n^* \) introduced in (2.8) is used and the same estimate as (2.9) is followed.

**Proof of (ii) \Rightarrow (iii).** We first recall that the Jackson kernels

\[
K_n(t) := \left( \frac{\sin \left( \frac{1}{2} n/4 \right)t}{\left| n/4 \right| \sin (t/2)} \right)^4
\]

provide “optimal-order” approximants \( P_n f \in \pi_n \) of any \( f \in A \) from \( \pi_n \), given by

\[
(P_n f)(z) := \int_{0}^{2\pi} f(ze^{it}) K_n(t) \, dt / \int_{0}^{2\pi} K_n(t) \, dt. \tag{3.4}
\]

For each \( k, 0 \leq k \leq n \), let \( j_k \) be the integer between 0 and \( n \) so chosen that \( 2\pi j_k/(n + 1) \) is closest to \( t_{n,k} \). Hence, we have

\[
\left| t_{n,k} - \frac{2\pi j_k}{n + 1} \right| \leq \frac{\pi}{n + 1}.
\]

Let

\[
M_n := \max_{0 \leq l \leq n} \# \{ k : j(k) = l \}.
\]

Then for any \( f \in A \), we have, from (1.4),

\[
\left( \frac{1}{n + 1} \sum_{k=0}^{n} |f(z_{n,k}) - f(\omega_{n,j_k})|^p \right)^{1/p} 
\leq \left( \frac{M_n}{n + 1} \sum_{l=0}^{n} \left( \Omega \left( f, \frac{2\pi l}{n + 1}, \frac{\pi}{n + 1} \right) \right)^p \right)^{1/p} 
\leq c \tau \left( f, \frac{\pi}{n + 1} \right)^p. \tag{3.5}
\]

To obtain an analogous estimate for \( P_n f \), we first observe that the mean module of continuity is rotational invariant. More precisely, by setting

\[
f_i(z) := f(ze^{-it}),
\]

we have \( \tau(f_i; \delta)_p = \tau(f; \delta)_p \) for all \( t \). Hence, from (3.4), by applying Minkowski’s inequality and Lemma 4 in [2], we have

\[
\left( \frac{1}{n + 1} \sum_{k=0}^{n} |(P_n f)(z_{n,k}) - (P_n f)(\omega_{n,j_k})|^p \right)^{1/p} 
\leq \int_{0}^{2\pi} \left( \frac{1}{n + 1} \sum_{k=0}^{n} |f_i(z_{n,k}) - f_i(\omega_{n,j_k})|^p \right)^{1/p} K_n(t) \, dt / \int_{0}^{2\pi} K_n(t) \, dt.
\]
\[
\begin{align*}
\leq & \sup_i \left\{ \frac{M_n}{n+1} \sum_{l=0}^n \left( \Omega \left( f; \frac{2\pi l}{n+1} + t, \frac{\pi}{n+1} \right) \right)^p \right\}^{1/p} \\
\leq & c \sup_i \tau \left( f, \frac{\pi}{n+1} \right)_p = c \tau \left( f, \frac{\pi}{n+1} \right)_p. \quad (3.6)
\end{align*}
\]

On the other hand, we also have, from our earlier work (see (2.12) in [2]), that
\[
\begin{align*}
\left( \frac{1}{n+1} \sum_{k=0}^n \left| (P_n f - f)(\omega_{n,k}) \right|^p \right)^{1/p} \\
\leq & \left( \frac{M_n}{n+1} \sum_{k=0}^n \left| (P_n f - f)(\omega_{n,k}) \right|^p \right)^{1/p} \\
\leq & c \tau \left( f, \frac{\pi}{n+1} \right)_p. \quad (3.7)
\end{align*}
\]

Therefore, summing (3.5)–(3.7), we obtain
\[
\left( \frac{1}{n+1} \sum_{k=0}^n \left| (P_n f - f)(z_{n,k}) \right|^p \right)^{1/p} \leq c \tau \left( f, \frac{\pi}{n+1} \right)_p, \quad (3.8)
\]
where \( c \) in (3.8) is the sum of the constants in (3.5)–(3.7) that are also denoted by \( c \).

Now, suppose that the first inequality of the Marcinkiewicz-Zygmund inequalities (1.15) holds for the sample points \( \{z_{n,k}\} \) and \( L_n = L_n(\{z_{n,k}\}, \cdot) \). Then it follows from (3.8) that
\[
\|L_n(P_n f - f)\|_p \leq \left\{ \frac{1}{c_p} \frac{1}{n+1} \sum_{k=0}^n \left| (L_n(P_n f - f))(z_{n,k}) \right|^p \right\}^{1/p} \\
= \left\{ \frac{1}{c_p} \frac{1}{n+1} \sum_{k=0}^n \left| (P_n f - f)(z_{n,k}) \right|^p \right\}^{1/p} \\
< c \tau \left( f, \frac{\pi}{n+1} \right)_p. \quad (3.9)
\]
Hence, since it is clear from the definition of \( P_n \) in (3.4) that
\[
\|P_n f - f\|_p \leq c \sup_{0 < t \leq 1/n} \|f - f\|_p \leq c \tau \left( f, \frac{\pi}{n+1} \right)_p,
\]
we conclude, by applying (3.9), that
\[
\| L_n f - f \|_p \leq \| P_n f - f \|_p + \| P_n f - L_n f \|_p \\
= \| P_n f - f \|_p + \| L_n (P_n f - f) \|_p \\
\leq c\tau \left( f, \frac{\pi}{n + 1} \right)_p.
\]

Proof of (ii) => (i). Suppose that the Marcinkiewicz-Zygmund inequalities (1.15) are valid for the family \( \{ z_n \} \). Then assertion (1.20) follows from Lemma 1. Therefore, it remains to verify that the family \( \{ W_n \} \) in (1.17) provides uniform \( A_p \)-weights.

For this purpose, we consider the space \( L^p(T, w) \) of functions \( f \) defined a.e. on \( T \) with finite weighted norm

\[
\| f \|_{p, w} := \left( \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^p w(e^{i\theta}) \, d\theta \right)^{1/p} < \infty,
\]

where \( w > 0 \) on \( T \) is the weight function for this space. Hence, for the constant weight \( w = 1 \), we have

\[
\| f \|_p = \| f \|_{p, 1}.
\]

It is well-known that the Cauchy integral operator

\[
(Cf)(z) := \frac{1}{2\pi} \int_{|\xi| = 1} \frac{f(\xi)}{\xi - z} \, d\xi, \quad z \in U,
\]

is of strong type \( p - p \); i.e. there exists some positive constant \( c_p \) such that

\[
\| Cf \|_p \leq c_p \| g \|_p, \quad g \in L^p(T),
\]

where \( L^p(T) := L^p(T, 1) \). To show that the family \( \{ W_n \} \) in (1.17) provides uniform \( A_p \)-weights, we recall from \( [4, 6] \) that it is sufficient to show that, in analogy to (3.10),

\[
\| Cf \|_{p, W_n^p} \leq d_p \| f \|_{p, W_n^p}, \quad f \in \L^p(T, |W_n|),
\]

for some positive constant \( d_p \), independent of \( n \). To this end, we consider the Lagrange polynomial interpolation operators

\[
L_n^* = L_n \left( \left( \frac{1}{P_n} z_n, k \right) \right)
\]

(3.12)
with sample points \((1/\rho_n)z_{n,k}\), and observe that
\[
(C^*)_n(z) = \frac{W_n(z)}{2\pi i} \int_{|\xi| = 1} \frac{g(\xi)}{W_n(\xi) - z} d\xi + (L^*_n(C^*))_n(z), \quad z \in U,
\]
(3.13)
for all \(g \in L^p(T)\). Hence, for any \(f \in L^p(T, |W_n|^p)\) and setting \(g = fW_n\), we have, by applying (3.13),
\[
\|Cf\|_{p, |W_n|^p} = \lim_{r \uparrow 1} \left(\frac{1}{2\pi r} \int_{|z| = r} |(C^*)_n(z) - (L^*_n(C^*)_n(z)|^p |dz|\right)^{1/p} \\
\leq \|Cg\|_p + \|L^*_n(C^*)_n\|_p.
\]
(3.14)
Now, by using the same estimate as in (2.9), and then applying the first of the Marcinkiewicz-Zygmund inequalities and statement (ii) in Lemma 2, we obtain
\[
\|L^*_n(C^*)_n\|_p \leq e \left\| \left( L^*_n(C^*)_n \left( \frac{1}{\rho_n} \right) \right) \right\|_p \\
\leq e \left\{ \frac{1}{c_p} \frac{1}{n+1} \sum_{k=0}^n \left( L^*_n(C^*)_n \left( \frac{1}{\rho_n} z_{n,k} \right) \right)^p \right\}^{1/p} \\
= e \left\{ \frac{1}{c_p} \frac{1}{n+1} \sum_{k=0}^n \left( C^*_n \left( \frac{1}{\rho_n} z_{n,k} \right) \right)^p \right\}^{1/p} \\
\leq e \|Cg\|_p.
\]
(3.15)
On the other hand, it follows from (3.10) that
\[
\|Cg\|_p \leq c_p \|g\|_p = c_p \|f\|_{p, |W_n|^p}.
\]
(3.16)
Hence, (3.11) follows from (3.14)–(3.16).

The proof of (i) \(\Rightarrow\) (ii) is similar to the derivation in [1, Section 3]. This completes the proof of the theorem. 

4. LOCALNESS IN \(p\)

In this section, we will show that the results stated in Corollaries 2 and 3 are local in the sense that the restriction governed by \(\delta > 0\) cannot be removed.
Let \( \hat{L}_n = L_n(e^{i\delta}, \cdot) \) be the Lagrange polynomial interpolation operators with sample points \( \{e^{i\delta}\} \) defined in (1.23). In our earlier work [1], we have already shown that the Marcinkiewicz-Zygmund inequalities (1.15) hold for \( p = 2 \), but \( \|\hat{L}_n f\|_p \to \infty \) for all sufficiently large values of \( p \) and some \( f \in A \) (see [1, Section 4]). Hence, in view of Theorem 1, it is sufficient to show that the family \( \{W_n\} \), with \( z_{n,k} = e^{i\delta} \), does not provide uniform \( A_p \)-weights for all sufficiently small \( p - 1 > 0 \), although (1.24) holds for all \( p, 1 \leq p < 2 \).

To establish this claim, we recall from (4.9) in [1] that

\[
|W_n(e^{it})| \leq \varepsilon_0(n + 1)^{-\varepsilon_0}, \quad \frac{n}{n + 1} \pi \leq t \leq \pi, \tag{4.1}
\]

for some positive constants \( \varepsilon_0 \) and \( \varepsilon_1 \) with \( 0 < \varepsilon_0 < 1 \), independent of \( n \). In addition, we also have, for \( \frac{5}{4} \pi \leq t \leq \frac{7}{4} \pi \),

\[
\left| \frac{W_n(e^{it})}{e^{i(n+1)t} - \rho_n^{-n-1}} \right| = \left| \prod_{k=0}^{\lfloor \frac{t}{\pi} \rfloor} \frac{e^{it} - \rho_n^{-1} \omega_{n,k} e^{i(\delta/n+1)}}{e^{it} - \rho_n^{-1} \omega_{n,k}} \right| \\
= \left| \prod_{k=0}^{\lfloor \frac{t}{\pi} \rfloor} 1 - \frac{\rho_n^{-1} \omega_{n,k} (e^{i(\delta/n+1)} - 1)}{e^{it} - \rho_n^{-1} \omega_{n,k}} \right| \\
\geq \left| \prod_{k=0}^{\lfloor \frac{t}{\pi} \rfloor} 1 - \sqrt{2} \sin \left( \frac{\delta}{2n + 2} \right) \right| \geq \varepsilon_2 > 0
\]

where \( \varepsilon_2 \) is independent of \( n \); so that

\[
|W_n(e^{it})| \geq \varepsilon_2 |1 - \rho_n^{-n-1}| \geq \frac{\varepsilon_2}{2}, \quad \frac{5}{4} \pi \leq t \leq \frac{7}{4} \pi. \tag{4.2}
\]

Hence, setting

\[
I_n := \left[ \frac{n}{n + 1} \pi, \frac{7}{4} \pi \right],
\]

we have, from (4.2),

\[
\frac{1}{|I_n|} \int_{I_n} |W_n(e^{it})|^\prime \, dt \geq \frac{1}{\pi} \int_{\frac{5}{4} \pi}^{\frac{7}{4} \pi} |W_n(e^{it})| \, dt \geq \frac{\varepsilon_2}{4}, \tag{4.3}
\]
for all \( p > 1 \). On the other hand, it follows from (4.1) that

\[
\frac{1}{|I_n|} \int_{I_n} |W_n(e^{it})|^{-q} \, dt \geq \frac{1}{\pi} \int_{n+1}^{\pi} |W_n(e^{it})|^{-q} \, dt \\
\geq \varepsilon_1^{-q}(n + 1)^{q\varepsilon_0 - 1},
\]

where \( q = p/(p - 1) \). Therefore, for all \( p \), with

\[
1 < p < \frac{1}{1 - \varepsilon_0}
\]

so that \( q\varepsilon_0 - 1 > 0 \), we conclude from (4.3)–(4.4) that

\[
\left\{ \frac{1}{|I_n|} \int_{I_n} |W_n(e^{it})|^p \, dt \right\}^{\frac{1}{p}} \left\{ \frac{1}{|I_n|} \int_{I_n} |W_n(e^{it})|^{-1/q} \, dt \right\}^{\frac{1}{q}} \to \infty
\]

as \( n \to \infty \). \( \square \)

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**REFERENCES**


