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# The Spectrum for 2-Perfect 6-Cycle Systems

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Recently, the spectrum problem for 2-perfect $m$-cycle systems has been studied by several authors. In this paper we find the spectrum for 2-perfect 6-cycle systems with two possible exceptions. The connection between these systems and quasigroups satisfying some 2 variable identities is discussed. 

## 1. INTRODUCTION

In the last 30 years, there has been much interest in decomposing a complete graph $K_v$ into edge-disjoint copies of a graph $G$. The most popular choices for $G$ have been a complete graph (block designs) and a cycle. When $G$ is a cycle of length $m$, such a decomposition is called an *$m$-cycle system of $K_v$*.

More recently, additional structure has been asked of the decomposition: a graph $H$ that is closely related to $G$ is defined; then the decomposition into copies of $G$ is constructed in such a way that replacing each copy of $G$ by a corresponding copy of $H$ results in a decomposition of the complete graph into edge-disjoint copies of $H$. The problem of most interest then is to find the *spectrum* of such a decomposition; that is, the values of $v$ for which there exists such a decomposition of $K_v$. For example, the following are such problems that have been solved.

The nesting of a Steiner triple system is such a problem, where $G$ is $K_3$ and $H$ is the complement of $G$ in $K_4$ (so $H$ is a star, joining a fourth vertex to each of the three vertices in $G$). The problem of finding the spectrum of Steiner triple systems that have a nesting has been completely solved [3, 8, 16]. A generalization of this problem is the nesting of a cycle system: in this case $G$ is a cycle of length $m$ and $H$ is the star consisting of $m+1$ vertices, one vertex being joined to each of the vertices in $G$. Again, the



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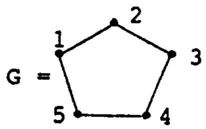
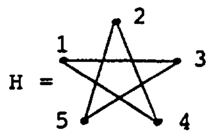
spectrum for which there exists such a decomposition of $K_v$ has been found [7, 9], with a few exceptions for each value of $m$.

If $G$ consists of a path containing 3 edges and $H$ is the complement of $G$ (so is a path of length 3) then the spectrum for this decomposition has been settled [5] (see also [14]).

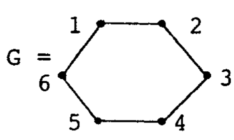
A third setting for this problem has been considered, where again $G$ is a cycle of length $m$, but now $H$ is formed by joining two vertices if they are distance $i$ apart in $G$, for some $1 < i < m/2$. So for $m = 5$, $i$ must be 2 and so $G = (1, 2, 3, 4, 5)$ and $H = (1, 3, 5, 2, 4)$.

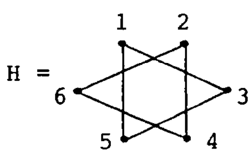



Such a decomposition of $K_v$ when $m = 5$ is called a Steiner pentagon system and its spectrum has been found [10]. The problem when $m = 7$ and $i = 2$ has also been essentially settled [11]. This problem has also been studied in the context of directed graphs [1, 6, 12]. Following the notation used in that setting, if $G$ is an $m$-cycle and $H$ is formed from $G$ by joining two vertices if they are distance $i$ apart in $G$, then such a decomposition of $K_v$ we define to be an *$i$-perfect $m$-cycle system of $K_v$*.

In this paper the spectrum for this problem is settled in the case where $m = 6$ (and so $i = 2$) except for two values of $v$. Note that $G$ is a 6-cycle and so $H$ is the union of two 3-cycles.




Therefore, when $m = 6$ the decomposition of $K_v$ into edge-disjoint copies of $H$ is a Steiner triple system that contains an even number of $K_3$s. Clearly a necessary condition for the existence of such a decomposition of $K_v$ is that $v \equiv 1$ or $9 \pmod{12}$ (since the number of edges in $K_v$ must be divisible by 6 and each vertex must have even degree). The rest of this paper is devoted to showing that this condition is also sufficient for the existence of a 2-perfect 6-cycle system, except for $v = 9$ and possibly except for $v \in \{45, 57\}$.

One of the main tools used in proving this result is to find an edge-disjoint decomposition of the complete tripartite graph $K_{2x,2x,2x}$ into 6-cycles so that the triangles formed by joining vertices distance 2 apart in the

6-cycles also form an edge-disjoint decomposition of $K_{2x,2x,2x}$. Naturally we define such a decomposition to be a *2-perfect 6-cycle decomposition of* $K_{2x,2x,2x}$.

Finally, it is worth noting that a 2-perfect 6-cycle system can be used to define a quasigroup as follows: Let $C$ be a 2-perfect 6-cycle system of $K_v$ defined on the vertex set $Q$. Define a binary operation "$\circ$" on $Q$ by: $x \circ x = x$ for all $x \in Q$, and $x \circ y = z$ if and only if $(x, y, z, a, b, c) \in C$. Then $(Q, \circ)$ is a quasigroup. A little reflection (but not too much) shows that $(Q, \circ)$ satisfies the 2-variable identities $(yx)x = y$ and $(xy)(y(xy)) = x(yx)$. On the other hand, a quasigroup $(Q, \circ)$ satisfying the above 2-variable identities which is *also* antisymmetric defines a 2-perfect 6-cycle system $C$ of $K_v$, where $C = \{(x, y, x \circ y, y \circ (x \circ y), x \circ (y \circ x), y \circ x) \mid x, y \in Q, x \neq y\}$. So a 2-perfect 6-cycle system is equivalent to an anti-symmetric quasigroup satisfying the three 2-variable identities in $I = \{x^2 = x, (yx)x = y, (xy)(y(xy)) = x(yx)\}$. Whether or not there exists a finite collection of 2-variable quasigroup identities $K$ so that a 2-perfect 6-cycle system is equivalent to a quasigroup satisfying $I \cup K$ is an open and (so it seems to the authors) interesting problem.

## 2. Preliminary Results

We begin with some notation. Let $(v_0, v_1, ..., v_{m-1})$ denote the $m$-cycle consisting of the edges $v_i v_{i+1}$ for $0 \leqslant i \leqslant m-1$, reducing the subscript modulo $m$. Corresponding to the 6-cycle $G = (1, 2, 3, 4, 5, 6)$ is the subgraph $H$ consisting of the two 3-cycles $(1, 3, 5)$ and $(2, 4, 6)$. Also, define $Z_n = \{0, 1, ..., n-1\}$.

We need to know that 2-perfect 6-cycle systems exist for some small values.

Example 2.1. For $v = 13$ define

$$C_{13} = \{(0, 5, 2, 8, 7, 9) + i \mid 0 \leqslant i \leqslant 12\},$$

where $(v_0, v_1, v_2, v_3, v_4, v_5) + i = (v_0 + i, v_1 + i, v_2 + i, v_3 + i, v_4 + i, v_5 + i)$, reducing each component modulo 13. Then $C_{13}$ is a 2-perfect 6-cycle system of $K_{13}$ defined on the vertex set $Z_{13}$.

For $v = 21$ define

$$\begin{aligned} C_{21} = \{&((0, 0), (0, 6), (0, 1), (2, 1), (0, 3), (1, 0)) + (0, i), \\ &((0, 0), (1, 2), (1, 0), (0, 4), (2, 0), (1, 1)) + (0, i), \\ &((0, 0), (0, 3), (1, 2), (2, 6), (2, 1), (2, 2)) + (0, i), \\ &((0, 0), (2, 6), (1, 3), (1, 2), (1, 6), (2, 4)) + (0, i), \\ &((0, 0), (2, 1), (2, 4), (1, 3), (2, 5), (1, 5)) + (0, i) \mid 0 \leqslant i \leqslant 6\} \end{aligned}$$

reducing the second component modulo 7. Then $C_{21}$ is a 2-perfect 6-cycle system of $K_{21}$ defined on the vertex set $Z_3 \times Z_7$.

For $v = 25$ define

$$C_{25} = \{(0, 8, 2, 24, 12, 7) + i, (0, 4, 3, 18, 7, 23) + i \mid 0 \leqslant i \leqslant 24\},$$

reducing each component modulo 25. Then $C_{25}$ is a 2-perfect 6-cycle system of $K_{25}$ defined on the vertex set $Z_{25}$.

For $v = 33$ define

$$\begin{aligned} C_{33} = \{&((0, 0), (0, 2), (0, 1), (0, 5), (1, 6), (1, 9)) + (0, i),\\ &((1, 0), (0, 4), (1, 1), (2, 3), (2, 6), (2, 5)) + (0, i),\\ &((1, 0), (1, 2), (1, 3), (2, 3), (2, 7), (1, 4)) + (0, i),\\ &((0, 6), (1, 1), (2, 0), (1, 5), (2, 1), (2, 3)) + (0, i),\\ &((0, 7), (1, 7), (2, 0), (0, 4), (2, 3), (0, 10)) + (0, i),\\ &((0, 0), (2, 5), (0, 2), (0, 7), (1, 1), (2, 9)) + (0, i),\\ &((0, 0), (1, 3), (0, 4), (1, 8), (1, 2), (2, 0)) + (0, i),\\ &((0, 8), (1, 10), (2, 0), (0, 10), (2, 5), (2, 10))\\ &+ (0, i)) \mid 0 \leqslant i \leqslant 10\}, \end{aligned}$$

reducing the second component modulo 11. Then $C_{33}$ is a 2-perfect 6-cycle system of $K_{33}$ defined on the vertex set $Z_3 \times Z_{11}\}$.

Before moving on to the main construction, we need some preliminary results.

LEMMA 2.2. *For any integer $x \geqslant 2$, there exists a 2-perfect 6-cycle system of $K_{2x, 2x, 2x}$.*

*Proof.* Let $L_1$ be a latin square of order $x$ on the symbols $\{0, ..., x-1\}$ and define a latin square $L$ of order $2x$ on the symbols $\{0, ..., 2x-1\}$ as follows: if cell $(i, j)$ of $L_1$ contains symbol $k$ then we have cells $(2i, 2j)$, $(2i, 2j+1)$, $(2i+1, 2j)$ and $(2i+1, 2j+1)$ of $L$ containing symbols $2k$, $2k+1$, $2k+3$ and $2k+2$, respectively. Let $(\{0, ..., 2x-1\}, \cdot)$ be the quasigroup corresponding to $L$.

Let the vertex set of $K_{2x, 2x, 2x}$ be $\{(i, j) \mid 0 \leqslant i \leqslant 2x-1, 0 \leqslant j \leqslant 2\}$. Define the 2-perfect 6-cycle system of $K_{2x, 2x, 2x}$ by $C = \{((2i, 0), (2j, 1), (2i \cdot (2j+1), 2), (2i+1, 0), (2j+1, 1), ((2i+1) \cdot 2j), 2)), ((2i, 0), (2j+1, 1), (2i \cdot 2j, 2), (2i+1, 0), (2j, 1), ((2i+1) \cdot (2j+1), 2)) \mid 0 \leqslant i \leqslant x-1, 0 \leqslant j \leqslant x-1\}$. It is easy to check this is a 2-perfect 6-cycle system, though it may help to note that by the construction of $L$, $2i\cdot(2j+1)=2k+1\neq 2k+3=(2i+1)\cdot 2j$ and $2i\cdot 2j=2k\neq 2k+2=(2i+1)\cdot(2j+1)$, so the 6-cycles are defined on 6 distinct vertices. ▮

LEMMA 2.3. *There exists a pairwise balanced design (PBD) of order $v$ with block sizes* 3 *and* 4 *that has a parallel class for all* $v\equiv 1 \pmod 3$ *except for* $v\in\{7, 10\}$.

*Proof.* Brouwer [2] has shown that for all $u\equiv 0$ or $1 \pmod 3$ there exists a PBD of order $u$ with block sizes 3 and 4. Furthermore, when $u\equiv 0 \pmod 3$ his construction yields a design which contains at least one block of size 4 unless $u\in\{3, 6, 9\}$. When $u\in\{3, 6, 9\}$ such a design with a block of size four does not exist.

We use Brouwer's result to make sure that when $v\equiv 1 \pmod 3$ the PBD's contain a parallel class.

If $v=9x+1$ and $x\geqslant 2$, then construct a PBD with the set of blocks being $B$ and on the set of symbols $\{\infty\}\cup\{(i,j)\mid 1\leqslant i\leqslant 3,\ 1\leqslant j\leqslant 3x\}$ as follows. Place the blocks from a PBD of order $3x$ with block sizes 3 and 4 on the vertex set $\{(i,j)\mid 1\leqslant j\leqslant 3x\}$ in $B$ for each $i\in\{1,2,3\}$. Let $(Q,\cdot)$ be an idempotent quasigroup of order $3x$ with a transversal $T$ intersecting the diagonal in just the cell $(1,1)$ (a pair of idempotent orthogonal latin squares of order $3x$ is sufficient, (see [17], for example) and it is easy to construct one of order 6, so such a latin square exists for all $x\geqslant 2$). Let $B$ also contain the blocks $\{\infty, (1,i), (2,i), (3,i)\}$ for $1\leqslant i\leqslant 3x$ and, for $i\neq j$, the blocks $\{(1,i), (2,j), (3, i\cdot j)\}$. This defines a PBD with block sizes 3 and 4 with a parallel class consisting of the block $\{\infty, (1,1), (2,1), (3,1)\}$ and the blocks $\{(1,i), (2,j), (3, i\cdot j)\}$ for each cell $(i,j)$ in $T$ except for $(1,1)$.

If $v=9x+4$ and $x\geqslant 1$, then the same construction as in the case when $v=9x+1$ will work if a PBD of order $3x+1$ and an idempotent quasigroup of order $3x+1$ with a transversal intersecting the diagonal in just the cell $(1,1)$ is used.

If $v=9x+7$ and $x\geqslant 3$, then define a PBD with set of blocks $B$ on the symbols $\{\infty\}\cup\{(i,j)\mid 1\leqslant i\leqslant 3,\ 1\leqslant j\leqslant 3x+2\}$ as follows. Place the blocks of a PBD of order $3x+3$ with block sizes 3 and 4, at least one block having size 4 (so $x\geqslant 3$), and on the vertex set $\{\infty\}\cup\{(i,j)\mid 1\leqslant j\leqslant 3x+2\}$ in $B$ for each $i\in\{1,2,3\}$, except that the block $\{\infty, (i,1), (i,2), (i,3)\}$ is omitted. Let $L$ be a latin square of order $3x+2$ with a subsquare of order 3 (on rows, columns and symbols 1, 2 and 3) which contains a transversal $T$ that intersects the subsquare in precisely the cell $(1,1)$ and let $(\{1, ..., 3x+2\}, \cdot)$ be the corresponding quasigroup. (Such a latin square exists of order 8 and for all other orders, the incomplete latin square

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 7 | 6 |
| 3 | 1 | 2 | 4 | 8 |
| 2 | 3 | 1 | 8 | 5 |
| 8 | 6 | 4 | 5 | 7 |
| 5 | 8 | 7 | 6 | 4 |

can be embedded in a latin square of any order $n \geqslant 11$ in which cells (6, 6), (7, 7) and $(i, i)$ for $8 \leqslant i \leqslant n$ contain symbols 2, 3 and $i$, respectively [15]; then select $T$ to consist of cells (1, 1), (2, 4), (3, 5), (4, 2), (5, 3) and $(i, i)$ for $6 \leqslant i \leqslant n$). Place the blocks of a PBD with block sizes 3 and 4 of order 10 on the vertex set $\{\infty\} \cup \{(i, j) \mid 1 \leqslant i \leqslant 3, 1 \leqslant j \leqslant 3\}$ in $B$, making sure that $b = \{\infty, (1, 1), (2, 1), (3, 1)\}$ is a block. Finally, for $1 \leqslant i \leqslant 3x+2$, $1 \leqslant j \leqslant 3x+2$ but not both $i$ and $j$ in $\{1, 2, 3\}$, add the block $\{(1, i), (2, j), (3, i \cdot j)\}$ to $B$. Then the blocks of $B$ form a PBD of block sizes 3 and 4 and the blocks arising from the cells in $T$ without cell (1, 1) together with $b$ form a parallel class.

Suppose $v = 9x+7$ and $x \in \{1, 2\}$. For $v = 16$ there exists a resolvable BIBD with block size 4. For $v = 25$, introduce four new points $\infty_1$, $\infty_2$, $\infty_3$, and $\infty_4$ to a Kirkman triple system of order 21 (see [17], for example), add $\infty_i$ to the blocks in parallel class $i$ for $1 \leqslant i \leqslant 4$ and add the block $b = \{\infty_1, \infty_2, \infty_3, \infty_4\}$. Then parallel class 5 of the Kirkman triple system together with $b$ forms a parallel class in the PBD of order 25 constructed in this way. ▌

Of course, the PBD's constructed in Lemma 2.3 are group divisible designs (GDD's) with groups and blocks of sizes 3 and 4. We now obtain some more GDD's.

LEMMA 2.4. *For all $x \geqslant 2$ there exists a GDD of order $6x+5$ with blocks of size 3, one group of size 5 and the remaining groups of size 3.*

*Proof.* We make use of Wilson's construction for (partial) Steiner triple systems [18]. Let $G$ be the graph with $V(G) = \{0, 1, ..., 6x+2\} - \{0, 2x+1, 4x+2\}$, and $E(G) = \{\{y, -y\}, \{y, 2y\} \mid y \in V(G)\}$, all elements being reduced modulo $6x+3$. Wilson shows that this graph has a one-factorization with 1-factors $F_1$, $F_2$, and $F_3$, and so the following define a PBD with one block of size 5 and the rest of size 3:

(a) $\{\infty_1, \infty_2, 0, 2x+1, 4x+2\}$ is a block,

(b) if $\{u, v\} \in F_1, F_2$ or $F_3$ then $\{\infty_1, u, v\}$, $\{\infty_2, u, v\}$ or $\{0, u, v\}$ is a block respectively, and

(c) if $u \neq v \neq w \neq u$, $u+v+w \equiv 0 \pmod{6x+3}$, $0 \notin \{u, v, w\}$ and $\{y, 2y) \not\subseteq \{u, v, w\}$ for any $y \in \{1, ..., 6x+2\}$, then $\{u, v, w\}$ is a block.

Therefore the result follows if $\{1, ..., 6x+2\} - \{2x+1, 4x+2\}$ can be partitioned into sets $\{u, v, w\}$ such that $u+v+w \equiv 0 \pmod{6x+3}$, for then these blocks together with the block of size 5 form a parallel class. This set can be so partitioned. Cyclic triple systems of all orders $6x+3 \neq 9$ have been formed [13] (or see [4]) by partitioning $\{1, ..., 3x+1\} - \{2x+1\}$ into sets $\{u, v, w\}$ satisfying either $u+v+w \equiv 0 \pmod{6x+3}$ or $u+v=w$. For each triple $\{u, v, w\}$ with $u+v+w \equiv 0 \pmod{6x+3}$ take the triples $\{u, v, w\}$ and $\{-u, -v, -w\}$ and for each triple with $u+v=w$ take the triples $\{u, v, -w\}$ and $\{-u, -v, w\}$ (reducing elements modulo $6x+3$) to produce the desired partition. ▮

The following result is the main construction for 2-perfect 6-cycle systems.

LEMMA 2.5. *If there exists a GDD of order t with blocks of size* 3 *and groups h in the set H, and if for* $s \geq 2$ *there exists a 2-perfect 6-cycle system of* $K_{2s|h|+1}$ *for all* $h \in H$, *then there exists a 2-perfect 6-cycle system of* $K_{2st+1}$.

*Proof.* Define a 2-perfect 6-cycle system of $K_{2st+1}$ on the vertex set $\{\infty\} \cup (Z_{2s} \times Z_t)$ as follows:

(1) for each block $\{u, v, w\}$ define a 2-perfect 6-cycle system of $K_{2s,2s,2s}$ on the vertex set $Z_{2s} \times \{u, v, w\}$ (this exists by Lemma 2.2), and

(2) for each group $h \in H$ define a 2-perfect 6-cycle system of order $2s|h|+1$ on the vertex set $\{\infty\} \cup (Z_{2s} \times h)$. ▮

## 3. CONSTRUCTIONS OF 2-PERFECT 6-CYCLE SYSTEMS

THEOREM 3.1. *For all* $v \equiv 1$ *or* $9 \pmod{12}$ *except for* $v=9$ *and possibly* $v \in \{45, 57\}$, *there exists a 2-perfect 6-cycle system of* $K_v$.

*Proof.* We consider the cases $v \equiv 1, 9, 13$, and $21 \pmod{24}$ in turn.

*Case* 1. $v \equiv 1 \pmod{24}$.

Let $v = 24x+1$. By removing one symbol from a Steiner triple system of order $24x+1$ (see [17], for example), a GDD with blocks of size 3 and groups of size 2 results. We shall use such a design many times!

If $2x = 6y$ then $v = 12(6y)+1$ and there exists a GDD of order $6y$ with blocks of size 3 and groups of size 2. Apply Lemma 2.5 with $t = 6y$ and $s = 6$. Since $|h| = 2$ for all $h \in H$ and since a 2-perfect 6-cycle system of $K_{2s|h|+1}$ is constructed in Example 2.1, the result follows.

If $2x = 6y + 2$ then $v = 12(6y + 2) + 1$ and there exists a GDD of order $6y + 2$ with blocks of size 3 and groups of size 2. Apply Lemma 2.5 with $t = 6y + 2$ and $s = 6$. A 2-perfect 6-cycle system of $K_{25}$ is constructed in Example 2.1.

If $2x = 6y + 4$ then $v = 6(12y + 8) + 1$ and there exists a GDD of order $12y + 8$ with blocks of size 3 and groups of size 2. Apply Lemma 2.5 with $t = 12y + 8$ and $s = 3$. A 2-perfect 6-cycle system of $K_{13}$ is constructed in Example 2.1.

*Case* 2. $v \equiv 9 \pmod{24}$.

Let $v = 24x + 9 = 4(6x + 2) + 1$. For $x \neq \{2, 3\}$, from Lemma 2.3, there exists a PBD with set of blocks $B$ on the set of symbols $Z_{3x+1}$ with block sizes 3 and 4 which has a parallel class. Define a 2-perfect 6-cycle system of $K_v$ on the vertex set $\{\infty\} \cup (Z_4 \times (Z_{3x+1} \times Z_2))$ as follows:

(1) for each block $\{u, v, w\}$ of $B$ which does not lie in the parallel class, place four 2-perfect 6-cycle systems of $K_{4,4,4}$, one on each of the following vertex sets: $Z_4 \times \{(u, 0), (v, 0), (w, 0)\}$, $Z_4 \times \{(u, 0), (v, 1), (w, 1)\}$, $Z_4 \times \{(u, 1), (v, 0), (w, 1)\}$ and $Z_4 \times \{(u, 1), (v, 1), (w, 0)\}$;

(2) for each block $(t, u, v, w\}$ of $B$ which does not lie in the parallel class, place eight 2-perfect 6-cycle systems of $K_{4,4,4}$, one one each of the following vertex sets:

$$Z_4 \times \{(t, 0), (u, 0), (v, 0)\}, Z_4 \times \{(t, 0), (u, 1), (w, 0)\},$$
$$Z_4 \times \{(t, 0), (v, 1), (w, 1)\}, Z_4 \times \{(u, 0), (v, 1), (w, 0)\}$$
$$Z_4 \times \{(t, 1), (u, 1), (v, 1)\}, Z_4 \times \{(t, 1), (u, 0), (w, 1)\},$$
$$Z_4 \times \{(t, 1), (v, 0), (w, 0)\}, Z_4 \times \{(u, 1), (v, 0), (w, 1)\};$$

and

(3) for each block $b$ in the parallel class of $B$, place a 2-perfect 6-cycle system of order $8|b| + 1$ on the vertex set $\{\infty\} \cup (Z_4 \times (b \times Z_2))$ (2-perfect 6-cycle systems of $K_{25}$ and of $K_{33}$ are constructed in Example 2.1).

If $x = 3$ then $v = 81$ and so the construction used in Case 1 starting with a GDD of order 8 with blocks of size 3 and groups of size 2 and then using 2-perfect 6-cycle systems of $K_{10,10,10}$, and finally placing a 2-perfect 6-cycle system of $K_{21}$ (see Example 2.1) on the vertex set $\{\infty\} \cup (Z_{10} \times h)$ for each group $h$ gives a 2-perfect 6-cycle systems of $K_{81}$.

*Case* 3. $v \equiv 13 \pmod{24}$.

Let $v = 24x + 13$. Then $v = 4(6x + 3) + 1$ and there exists a GDD of order $6x + 3$ with groups and blocks of size 3 (just remove the blocks in a parallel class from a Kirkman triple system of order $6x+3$ to form the groups). Apply Lemma 2.5 with $t=6x+3$ and $s=2$. A 2-perfect 6-cycle system of $K_{13}$ is constructed in Example 2.1.

*Case* 4. $v \equiv 21 \pmod{24}$.

Let $v=24x+21$. Then $v=4(6x+5)+1$ and for $x \geqslant 2$ there exists a GDD with blocks and groups of size 3 except for one group of size 5 (this is constructed in Lemma 2.4). Apply Lemma 2.5 with $t=6x+5$ and $s=2$. Examples of 2-perfect 6-cycle systems of $K_{13}$ and $K_{21}$ are constructed in Example 2.1.

## 4. Open Problems

Clearly many problems related to Theorem 3.1 still remain. To find the spectrum of $i$-perfect $m$-cycle systems in general is an extremely difficult problem. At this stage, the spectrum for $m$-cycle systems has not yet been found, and it is likely that this spectrum will be essentially the same as the spectrum for $i$-perfect $m$-cycle systems. Then of course there is the more difficult problem of constructing $I$-perfect $m$-cycle systems; that is, an $m$-cycle system that is $i$-perfect for all $i \in I$. In the case where $I = \{1, ..., \lfloor m/2 \rfloor\}$ such a system is a Steiner $m$-cycle system. Finally an easier problem: find 2-perfect 6-cycle systems of $K_{45}$ and $K_{57}$.

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*Note added in proof.* Recently Elizabeth Billington, University of Queensland, has found 2-perfect 6-cycle systems of orders 45 and 57, thereby removing the possible exceptions to Theorem 3.1.

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