Physics Letters B 731 (2014) 131–135

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

New analytical solutions for bosonic field trapping in thick branes

R.R. Landim [a], G. Alencar [a,*], M.O. Tahim [b], R.N. Costa Filho [a]

[a] Departamento de Física, Universidade Federal do Ceará, Caixa Postal 6030, Campus do Pici, 60455-760 Fortaleza, Ceará, Brazil
[b] Universidade Estadual do Ceará, Faculdade de Educação, Ciências e Letras do Sertão Central, R. Epitácio Pessoa, 2554, 63.900-000 Quixadá, Ceará, Brazil

ARTICLE INFO

Article history:
Received 26 November 2013
Received in revised form 3 February 2014
Accepted 4 February 2014
Available online 14 February 2014
Editor: M. Cvetič

ABSTRACT

New analytical solutions for gravity, scalar and vector field localization in Randall–Sundrum (RS) models are found. A smooth version of the warp factor with an associated function $f(z) = \exp(3A(z)/2)$ inside the walls ($|z| < d$) is defined, leading to an associated equation and physical constraints on the continuity and smoothness of the background resulting in a new space of analytical solutions. We solve this associated equation analytically for the parabolic and Pöschl–Teller potentials and analyze the spectrum of resonances for these fields. By using the boundary conditions we are able to show that, for any of these solutions, the density probability for finding a massive mode in the membrane has a universal behavior for small values of mass given by $|\psi_m(0)|^2 = \beta_1 m + \beta_3 m^3 + \beta_L m^3 \log(m) + \cdots$. As a consequence, the form of the leading order correction, for example, to the Newton's law is general and does not depend on the potential used. At the end we also discuss why complications arise when we use the method to find analytical solutions to the fermion case.

1. Introduction

After the seminal work of Randall and Sundrum (RS) [1] several other results have been developed based on the idea of membranes as topological defects and its implications for brane world physics [2–7]. In these models one must determine the space of solutions of a Schrödinger equation with a specific potential which depends on the warp factor. That is, one needs to solve a Sturm–Liouville problem to find eigensolutions with eigenvalues. A particular application of this kind of model is in the study of gravity trapping in a finite thickness domain wall [8], where a constant potential in the region near/over the membrane is chosen in order to find analytical solutions. The benefits of such analytical solutions are worth because allow explicit analysis of the Kaluza–Klein masses and opens up possibilities for analytical studies of fermionic modes. These analysis and possibilities can be extended even more if different potentials for the graviton wave function modes can be solved analytically.

In this manuscript, we present a new explicit integrable Schrödinger potentials for the graviton wave function modes parameterized by the thickness of the wall. The warp factor is chosen in order to be continuous in the boundary of two regions of the space–time. These two regions basically describe the interaction right near/over the membrane location and interactions far from the membrane. We find an equation that drives the profile of the brane. With this we show that the function used in [8] is just a particular solution of the equation presented here. This reveals a new space of analytical solutions and, as direct consequences, new zero modes, Kaluza–Klein modes, new resonance behavior, and so on. The new analytical solutions are encoded in a Schrödinger-like differential equation with zero eigenvalue. With this it is possible to show that, for small values of m, the probability density for finding a mass mode in the membrane does not depend on the chosen potential. In this way, the leading order correction of the four dimensional Newton's law, for example, has a general expression that does not depend on the potential used. In the following lines we discuss how to apply this method to study the physics of gravitational fields, scalar fields and gauge vector fields in the RS scenario.

2. The associated equation

To start our reasoning we remember the fact that the mass spectrum of the gravity field is driven by a Schrödinger like equation [9]

$$-\psi_m''(z) + U(z)\psi_m(z) = m^2\psi(z), \tag{1}$$

where the effective potential depends on the warp factor, $A(z)$, as below

$$U(z) = \frac{3}{2}A''(z) + \frac{9}{4}A'(z)^2, \tag{2}$$

* Corresponding author.
E-mail address: geovamaciel@gmail.com (G. Alencar).

http://dx.doi.org/10.1016/j.physletb.2014.02.004
0370-2693

and the metric in the conformal coordinate is given by $ds^2 = e^{2A(z)}(dx^2 + dz^2)$. By analyzing the above equation it has been shown that the zero mode ($m = 0$) is trapped in the membrane. However as $\lim_{z\to\infty} U(z) = 0$, the massive modes are not localized. An important phenomenological aspect related to this is the appearance of resonances. This allows for the possibility of unstable massive modes that could be seen in the membrane. Most of these studies have been performed numerically by considering smooth versions of the RS model. These smooth versions can be obtained, for instance, by considering the membrane as a topological defect generated by a scalar field. In this scenario the condition imposed is that for large z the RS warp factor is recovered. These models have many interesting properties and have been widely studied over the last decade [2–7].

Another way to get a smooth version of the RS model is by considering the brane as a thick domain wall [8,10]. In these papers the potential used depends on one parameter $0 \leqslant x \leqslant \pi/2$ which determines the thickness of the membrane. In order to get the desired smooth version, A and A' must be continuous and this imposes some restrictions on the form of $A(z)$ in the membrane. The choice of [8] was $A(z) = \frac{2}{3}\ln\cos(\sqrt{V_0}|z|)$ and this give the effective potential $U = -V_0$. With this, the resonances of the model were studied analytically in detail.

In order to obtain a wider class of new exact solutions, we divide the warp factor $A(z)$ in two regions, $|z| \leqslant d$ and $|z| \geqslant d$:

$$A(z) = \begin{cases} \frac{2}{3}\ln[f_0(z)], & |z| \leqslant d, \\ \ln(\frac{1}{|z|+\beta}), & |z| \geqslant d, \end{cases} \tag{3}$$

where we defined $f_0(z)$ as the associated function. The analytical solution for $|z| \geqslant d$ is already known [8,10], then we focus in analytical solutions for $|z| \leqslant d$. With Eqs. (2) and (3) we get that $f_0(z)$ satisfies the associated equation

$$-f_0''(z) + U(z)f_0(z) = 0, \tag{4}$$

this is exactly the effective Schrödinger equation (1) for $m^2 = 0$. In order to implement the boundary conditions, we restrict to even functions $f_0(z) = f_0(-z) = g_0(z)$ with $U(z) = U(-z)$ in (4), guarantying that the boundary conditions are satisfied in both edges of the brane, $z = d$ and $z = -d$. The condition $A(0) = 0$ implies that $g_0(0) = 1$. Since $g_0(z)$ is an even function we also have $A'(0) = 0$. This conditions fix completely the solution of (4). With the above considerations and by imposing continuity of $A(z)$ and $A'(z)$ at $z = \pm d$ we obtain

$$\frac{2}{3}g_0'(d) = -g_0(d)^{1+2/3}, \tag{5}$$

$$\beta = -d + \frac{1}{g_0(d)^{2/3}}, \tag{6}$$

with the conditions $g_0'(d) < 0$ and $g_0(z)$ positive and limited in $|z| \leqslant d$. In the last section it will become clear why we have written $5/3 = 1 + 2/3$.

3. Analytical solutions

As a first example, let us consider a constant potential $-|V_0|$ in the region $|z| \leqslant d$ [8]. We know that the solution for the Schrödinger equation (1) is a linear combination of $\cos(\sqrt{m^2 + |V_0|}z)$ and $\sin(\sqrt{m^2 + |V_0|}z)$. The even solution for $m = 0$ that satisfies $g_0(0) = 1$ is $\cos(\sqrt{|V_0|}z)$. The conditions $g_0'(d) < 0$ and $g_0(z) > 0$ implies that $0 < \sqrt{|V_0|}d < \pi/2$. This is the solution introduced in [8,10]. As a straightforward application of the method developed here lets examine the harmonic oscillator with Schrödinger equation

$$-\psi_m''(z) + z^2\psi_m(z) = m^2\psi_m(z). \tag{7}$$

The above equation can be cast in the form of a Kummer equation [11], by writing $\psi_m(z) = e^{-z^2/2}w_m(z)$ and next using the transformation $u = z^2$, we obtain

$$uF''(u) + (b - u)F'(u) - aF(u) = 0, \tag{8}$$

with $b = 1/2$ and $a = (1 - m^2)/4$. Then, solutions of (7) are linear combinations of $g_m(z) = e^{-z^2/2}F_1(a; \frac{1}{2}; z^2)$ and $h_m(z) = ze^{-z^2/2}F_1(a + \frac{1}{2}; \frac{3}{2}; z^2)$, where $F_1(a; b; z)$ is the Kummer confluent hypergeometric function. From now on we will use for the even (odd) solution in $|z| \leqslant d$ the notation $g_m(z)$ ($h_m(z)$). The even solution for $m = 0$ with $g_0(0) = 1$ is $g_0(z) = e^{-z^2/2}F_1(\frac{1}{4}; \frac{1}{2}; z^2)$.

In fact, we can find a large new class of solutions simply considering $U(z) = az^2 + b$, with $a > 0$. Using the above-mentioned steps we find the pair of solutions

$$g_m(z) = e^{-\sqrt{a}z^2/2}F_1\left(\frac{b}{4\sqrt{a}} - \frac{m^2}{4\sqrt{a}} + \frac{1}{4}; \frac{1}{2}; \sqrt{a}z^2\right), \tag{9}$$

$$h_m(z) = ze^{-\sqrt{a}z^2/2}F_1\left(\frac{b}{4\sqrt{a}} - \frac{m^2}{4\sqrt{a}} + \frac{3}{4}; \frac{3}{2}; \sqrt{a}z^2\right), \tag{10}$$

and $W(g, h)(z) = 1$, where $W(f_1, f_2)(x) = f_1(x)f_2'(x) - f_1'(x)f_2(x)$ is the Wronskian of f_1, f_2. It is worthwhile to mention that the Wronskian is constant for Schrödinger-like equations. The even solution satisfies $g_0(0) = 1$ and the value of the constants a, b and d are related in order to give $g_0'(d) < 0$ and $g_0(z) > 0$. As an example, for $b = -5$ and $a = 1$, $g_0(z)$ is positive defined for $|z| < 0.707107$ and $d = 0.243928$.

The method described here can be applied to many other cases known in physics. One possibility is the problem for a particle in a box subject to a constant field. This is described by a linear potential $U(z) = az$ giving rise to the Airy functions with solutions $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$. However, this potential do not satisfies the condition of being even. Another class of analytical solutions can be found by considering exponential potentials. From these the only even one is the Pöschl–Teller potential, where the Schrödinger-like equation is

$$\psi_m''(z) + \left(m^2 + a^2b(b+1)\,\mathrm{sech}^2(az)\right)\psi_m(z) = 0. \tag{11}$$

Rewriting $\psi(z) = w(z)/\cosh^b(az)$ and next using the transformation $u = -\sinh^2(az)$, we can write (11) as a hypergeometric differential equation

$$u(1-u)F''(u) + \left(\gamma - (\alpha + \beta + 1)u\right)F'(u) - \alpha\beta F(u) = 0,$$

where $\gamma = 1/2$, $\alpha = (-b + im/a)/2$ and $\beta = -(b + im/a)/2$. Therefore the linearly independent solutions of (11) are $g_m(z) = F(\alpha, \beta; 1/2; -\sinh^2(az))/\cosh^b(az)$ and $h_m(z) = \sinh(az)F(\alpha + 1/2, \beta + 1/2; 3/2; -\sinh^2(az))/\cosh^b(az)$ with $W(g_m, h_m)(z) = a$ and $g_0(0) = g_m(0) = 1$.

After fixing the background with the above method we turn our attention to the gravity resonances. The interesting fact about this background is that we automatically have exact solutions to Eq. (1) just by not fixing $m = 0$. With this we get an analytical expression for our resonances. As we are interested in resonances we must consider a plane wave coming from $-\infty$. This plane wave will collide with the membrane and will generate a reflected and a transmitted wave. Therefore, for $z < -d$ we must have a linear combination of waves moving to the left and to the right. For $z > d$ we must have only one wave moving to the right. In order to analyze the resonances we fix the coefficient of the incoming wave equal to one. In this way, the Schrödinger-like equation (1) has the solution

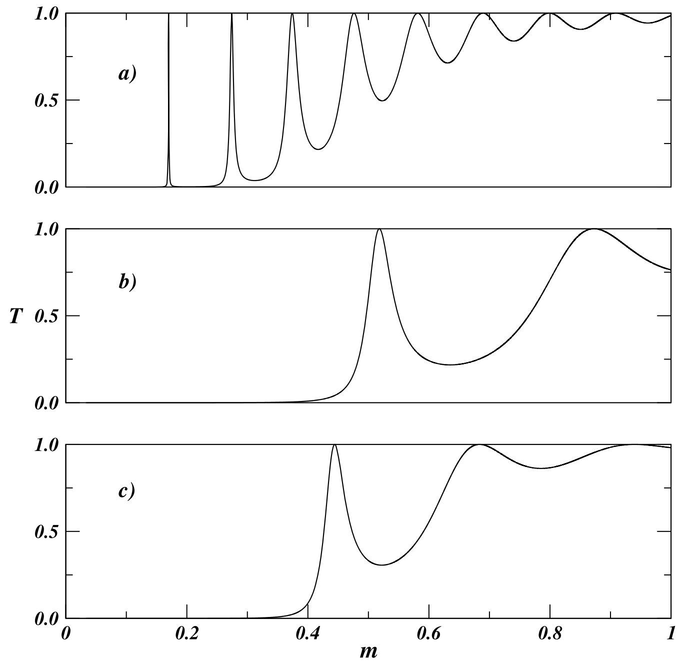

Fig. 1. The transmission coefficient for parabolic potential with: (a) $a = 0.001$, $b = -0.1$ ($d = 3.8122$); (b) $a = 0.01$, $b = -0.1$ ($d = 5.49082$) and (c) $a = 0.00001$, $b = -0.01$ ($d = 13.9833$).

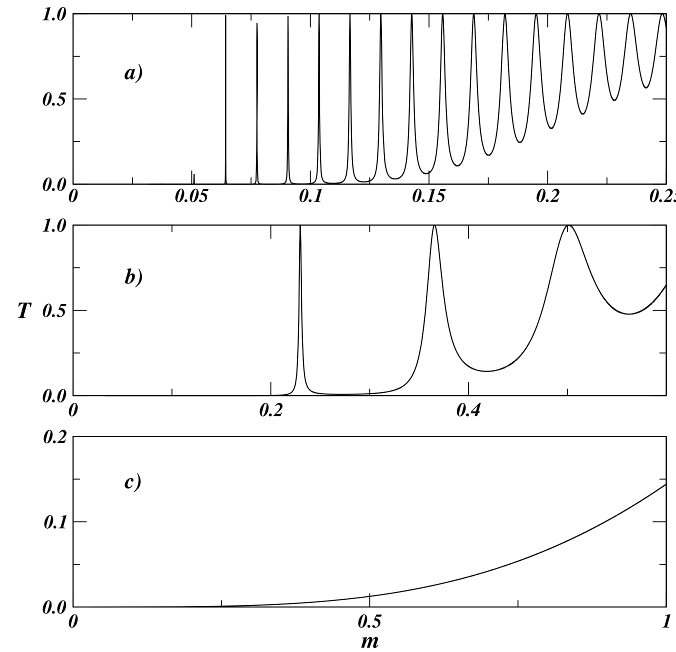

Fig. 2. The transmission coefficient for the Pöschl–Teller potential with (a) $a = 0.01$, $b = 1$ ($d = 115.369$); (b) $a = 0.1$, $b = 1$ ($d = 10.1721$), and (c) $a = 1$, $b = 1$ ($d = 0.537862$).

$$\psi_m(z) = \begin{cases} A_m g_m(z) + B_m h_m(z), & |z| \leqslant d, \\ F_m(z) + C_m E_m(z), & z \leqslant -d, \\ D_m F_m(z), & z \geqslant d \end{cases} \tag{12}$$

where $E_m(z) = \sqrt{\frac{\pi u}{2}} H_2^{(2)}(u)$, $F_m(z) = \sqrt{\frac{\pi u}{2}} H_2^{(1)}(u)$, with $u = m(|z| + \beta)$. We shall use the convention that $g_m(0) = 1$. Taking the continuity of the wave function and its derivative at $z = \pm d$ and using the fact that $g_m(z)$, $E_m(z)$, $F_m(z)$ are even and $h_m(z)$ are odd we finally obtain the transmission coefficient

$$T(m) = \frac{m^2 |W(g_m, h_m)(d)|^2}{|W(F_m, g_m)(d) W(F_m, h_m)(d)|^2}. \tag{13}$$

With this expression and the solutions found before we can easily obtain graphics for the transmission coefficients. In Fig. 1 we show the transmission for the parabolic potential. As expected, depending on the parameters we can have different peaks of resonances. It is important to point that as narrow the resonance peak the stronger is the signal of the mass mode. In Fig. 1 we see that happening for the first peaks. For the Pöschl–Teller potential Fig. 2 shows resonances depending on the parameters. For the parameters used the resonances are more peaked, what is phenomenologically more interesting. It is clear that the potentials present resonances with different characteristics.

4. Correction to the Newton's law

In order to analyze the deviation of Newton's law, we must calculate the probability density of the wave function at $z = 0$:

$$\left|\psi_m(0)\right|^2 = |A_m|^2 = \frac{m^2}{|W(F_m, g_m)(d)|^2}. \tag{14}$$

The above expression is plotted in Fig. 3 for the parabolic potential with $a = 0.00001$ and $b = -0.01$.

For small values of m, $F_m(z)$ has the following form for $z > 0$:

$$F_m(z) \approx a_{-3/2}(z) m^{-3/2} + a_{1/2}(z) m^{1/2} + a_{5/2}(z) m^{5/2} + a_L(z) m^{5/2} \log(m), \tag{15}$$

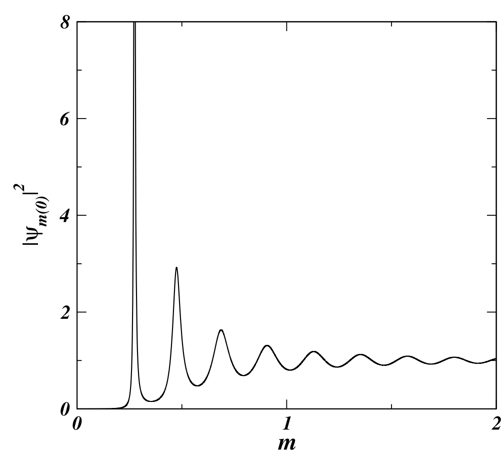

Fig. 3. The amplitude probability as function of m.

where

$$a_{-3/2}(z) = -2i\sqrt{\frac{2}{\pi}} (z + \beta)^{-3/2}, \tag{16}$$

$$a_{1/2}(z) = -i \frac{(z + \beta)^{1/2}}{\sqrt{2\pi}}, \tag{17}$$

$$a_{5/2}(z) = \left(\gamma_0 + 4i \log(z + \beta)\right) \frac{(z + \beta)^{5/2}}{16\sqrt{2\pi}}, \tag{18}$$

$$a_L(z) = 4i \frac{(z + \beta)^{5/2}}{16\sqrt{2\pi}}, \tag{19}$$

with $\gamma_0 = 2\pi - 3i + 4i\gamma - 4i \log(2)$ and γ is the Euler–Mascheroni constant. The function $g_m(z)$ is well defined for $m = 0$, i.e., $g_m(z)$ do not have poles at $m = 0$. Therefore, for small m we can expand $g_m(z)$ in a power series in m^2: $g_m(z) = g_0(z) + b_1(z) m^2 + b_2(z) m^4 + \cdots$. The Wronskian $W(a_{-3/2}, g_0)(d)$ is zero due to the boundary conditions (5). Then we have $W(F_m, g_m)(d) = \sqrt{m}(\alpha_0 + \alpha_2 m^2 + \alpha_L m^2 \log(m) + \cdots)$. With this we arrive at the following expansion for $m \ll 1$:

$$\left|\psi_m(0)\right|^2 = \beta_1 m + \beta_3 m^3 + \beta_L m^3 \log(m) + \cdots, \tag{20}$$

where β_1, β_3, β_L are coefficients that depend on the potential used through $a_i(d)$, $b_i(d)$. For large m, $F_m(z) \sim e^{imz}$ and $g_m(z) \sim \cos mz$ consequently $|A_m|^2 \to 1$ when $m \to \infty$.

To calculate the deviation of Newton's law we have to plug the expansion (20) in the expression

$$\delta(r) = \int_0^\infty \left|\psi_m(0)\right|^2 e^{-mr}\, dm, \tag{21}$$

and integrate from $(0, M_c)$ for the small M expansion and (M_c, ∞) with $M_c < 1$. The integration gives us

$$\begin{aligned}\delta(r) = {} & \frac{\beta_1}{r^2}\big(1 - e^{-M_c r}(1 + M_c r)\big) \\ & + \frac{\beta_3}{r^4}\big(6 - e^{-M_c r}\big((M_c r)^3 + 3(M_c r)^2 + 6 M_c r + 6\big)\big) \\ & \times \frac{\beta_L}{r^4}\big(11 - 6\gamma + 6\log(M_c) - \Gamma(4, M_c r)\log(M_c) \\ & - 6\log(M_c r) - G^{3,0}_{2,0}(1, 1, 0, 0, 4 \mid M_c r)\big) + \frac{e^{-M_c r}}{r},\end{aligned} \tag{22}$$

where $\Gamma(y, x)$ is the complete gamma function and $G^{3,0}_{2,0}(1, 1, 0, 0, 4|x)$ is the Meijer G-function.

5. Scalar, vector and fermion fields

In a previous work the present authors have found analytical solutions for other bosonic fields in the potential well case $U = -V_0$ [10]. We can use this and the above method to find new analytical solutions to the scalar and vector fields. What was pointed out in that work is that for any effective potential which takes the form $U(z) = cA'' + c^2 A'^2$, an analytical solution can be found easily just by changing some parameters of the theory. We have also shown that for the scalar and vector fields $c = 3/2$ and $c = 1/2$ respectively. We also pointed that the parameter for the scalar field is the same of the gravity field, therefore all results are identical and we just need to consider the vector case here. The only thing we need to do is to change $2/3 \to 1/c$ in Eqs. (3) and (5). The effect of this for the exterior solution is to change the order of the Hankel function $2 \to 1/2 + c$. The interior region is changed throughout the contour conditions (5). Therefore all the main results, namely Eqs. (12), (13), (14) and (20) are kept unchanged up to the above change in the parameters. Therefore, the expansion for small masses have the same general behavior also for the scalar and vector fields. In Fig. 4 we show the transmission coefficients for the vector case. Note that the change of the potential can give very different phenomenological results for a observer in the brane.

At this point some discussion about the fermion case is important. The main difference between this and the bosonic case is the fact that the way to get a nontrivial potential is through the addition of a coupling with a scalar field. This is described in very well in Ref. [12], where it is used the constant potential of [8]. The fact is that when this coupling is considered two additional complications arise. One is the fact that the following relation $\phi'(z)^2 = 3(A'(z)^2 - A''(z))$ must be satisfied [12]. Therefore we must impose the continuity of the second derivative of the warp factor if we wants a continuous first derivative of the scalar field. Another and more difficult problem is that the effective potential is given by

$$U_\pm(z) = \big(\eta F(\phi(z)) e^{A(z)}\big)^2 \pm \frac{d}{dz}\big(\eta F(\phi(z)) e^{A(z)}\big), \tag{23}$$

which do not posses a general analytical solution even for the simplest case of the constant potential [12].

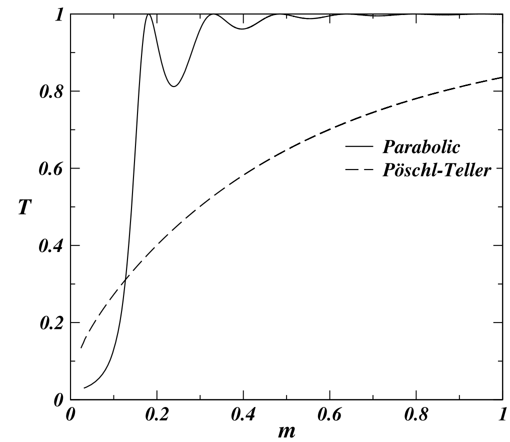

Fig. 4. The transmission coefficient for the parabolic ($d = 9.9967$, $a = 0.00001$; $b = -0.01$) and Pöschl–Teller potentials ($d = 0.222377$, $a = 1$, $b = 1$).

6. Summary and conclusions

In summary we have described a prescription to find new analytical solutions in thick brane models. Two cases are presented as examples: we have shown results for the parabolic and the Pöschl–Teller potentials. We have studied the problem of resonances in co-dimension one brane world, finding the mass spectra of the new profiles for the thick branes proposed. For these potentials, the transmission coefficients for the masses are plotted expressing the different peculiarities of each potential. It is important to stress that this work is a generalization of the article [8]. We have shown that the solution used by Mirjam et al. is a particular solution of a wider class described here, including the cases of scalar and vector fields. We also have shown that the correction in the Newton's law found by them is valid for any of these wider class of solutions. The results found here opens up the possibility to explore different backgrounds to study gravity, scalar and vector field resonances in the Randall–Sundrum model. For the fermionic case it is not possible to apply the same methods. Only numerical results were found and we hope, in next studies, to go one step further in finding pure analytical solutions.

Acknowledgements

We acknowledge the financial support provided by Fundação Cearense de Apoio ao Desenvolvimento Científico e Tecnológico (FUNCAP), the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), and FUNCAP/CNPq/PRONEX.

References

[1] Lisa Randall, Raman Sundrum, Phys. Rev. Lett. 83 (1999) 4690–4693, arXiv:hep-th/9906064.
[2] D. Bazeia, L. Losano, Phys. Rev. D 73 (2006) 025016, arXiv:hep-th/0511193.
[3] Yu-Xiao Liu, Jie Yang, Zhen-Hua Zhao, Chun-E. Fu, Yi-Shi Duan, Phys. Rev. D 80 (2009) 065019, arXiv:0904.1785 [hep-th].
[4] Zhen-Hua Zhao, Yu-Xiao Liu, Hai-Tao Li, Class. Quantum Gravity 27 (2010) 185001, arXiv:0911.2572 [hep-th].

[5] Jun Liang, Yi-Shi Duan, Phys. Lett. B 681 (2009) 172–178.
[6] Zhen-Hua Zhao, Yu-Xiao Liu, Hai-Tao Li, Yong-Qiang Wang, Phys. Rev. D 82 (2010) 084030, arXiv:1004.2181 [hep-th].
[7] Zhen-Hua Zhao, Yu-Xiao Liu, Yong-Qiang Wang, Hai-Tao Li, J. High Energy Phys. 1106 (2011) 045, arXiv:1102.4894 [hep-th].
[8] Mirjam Cvetic, Marko Robnik, Phys. Rev. D 77 (2008) 124003, arXiv:0801.0801 [hep-th].
[9] R.R. Landim, G. Alencar, M.O. Tahim, R.N. Costa Filho, J. High Energy Phys. 1108 (2011) 071, arXiv:1105.5573 [hep-th].
[10] G. Alencar, R.R. Landim, M.O. Tahim, R.N. Costa Filho, J. High Energy Phys. 1301 (2013) 050, arXiv:1207.3054 [hep-th].
[11] Milton Abramowitz, Irene A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Dover Publications, New York, 1964.
[12] H.-T. Li, Y.-X. Liu, Z.-H. Zhao, H. Guo, Phys. Rev. D 83 (2011) 045006, arXiv:1006.4240 [hep-th].