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Correction to "A Principle of Subsequences in Probability Theory: The Central Limit Theorem"

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Condition (VIII) of the paper can only be satisfied in general if we replace $t_{\sqrt{k}}(\cdot)$ there by $t_{\sqrt{n(k)}}(\cdot)$. This vitiates the rest of the proof which has to be completed as follows.

We shall choose a subsequence F_0 such that any subsequence $\{f_n\}$ of F_0 besides satisfying (I)–(VII) fulfills the following also:

(VIII) $$|E(f_n \mid f_1, \ldots, f_{n-1})| \leqslant 2^{-n};$$

and

(IX) $$\sum_{n=1}^{\infty} n^{-1/2} \cdot \int_{\{|f_n| > n^{1/2}\}} |f_n| \, dP < \infty.$$

That (VIII) now can be fulfilled is contained in Lemma 2. The possibility of (IX) will be proved at the end. We now put $g_n = t_{\sqrt{n}}(f_n)$, $h_n = f_n - g_n$, $\alpha_n = E_{n-1}(g_n)$, and $\xi_n = (g_n - \alpha_n)$, where $E_{n-1}(\cdot) = E(\cdot \mid f_1 \cdots f_{n-1})$. We now prove that

(1) $$E_{n-1}(\xi_n^2) \leqslant 4C, \quad (\text{cf.(VII)})$$

(2) $$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} \xi_k^2 = \theta \quad \text{a.e.} \quad (\text{cf.(V)})$$

and

(3) $$\lim_{n\to\infty} \left\{ n^{-1/2} \sum_{k=1}^{n} f_k - n^{-1/2} \sum_{k=1}^{n} \xi_k \right\} = 0 \quad \text{a.e.}$$

Then by Lemma 6 and Theorem 2, the proof of Theorem 1 will be complete.

As regards (1), we use

$$\xi_n^2 \leqslant 2(g_n^2 + \alpha_n^2),$$

whence

$$E_{n-1}(\xi_n^2) \leqslant 2E_{n-1}(g_n^2) + 2E_{n-1}(\alpha_n^2)$$
$$\leqslant 4C$$

by VII and the fact that $\alpha_n^2 = (E_{n-1}g_n)^2 \leqslant E_{n-1}g_n^2 \leqslant C$. For (2) and (3), we note first that by (IX),

$$\int \sum n^{-1/2} \mid E_{n-1}(h_n) \mid dP \leqslant \sum n^{-1/2} \int \mid h_n \mid dP$$
$$\leqslant \sum n^{-1/2} \int_{\{|f_n|>n^{1/2}\}} \mid f_n \mid dP < \infty,$$

whence $\sum n^{-1/2} \mid E_{n-1}(h_n) \mid < \infty$ a.e. This gives from the definition of α_n and condition (VIII) that

$$\sum n^{-1/2} \mid \alpha_n \mid < \infty$$

which implies via Kronecker's lemma that

$$\lim_{n\to\infty} n^{-1/2} \sum_{k=1}^{n} \mid \alpha_k \mid = 0 \quad \text{a.e.}$$

So

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} \alpha_k^2 = 0 \quad \text{a.e.}$$

since

$$\sum_{k=1}^{n} \alpha_k^2 \leqslant C^{1/2} \sum_{k=1}^{n} \mid \alpha_n \mid \quad \text{a.e.}$$

Also

$$\left| n^{-1} \sum_{k=1}^{n} g_k \alpha_k \right| \leqslant \left(n^{-1} \sum_{k=1}^{n} g_k^2 \right)^{1/2} \cdot \left(n^{-1} \sum_{k=1}^{n} \alpha_k^2 \right)^{1/2}$$
$$\to 0 \quad \text{a.e.}$$

since

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} g_k^2 = \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} f_k^2 = \theta \quad \text{a.e.}$$

by (IV) and (V). Finally, $\xi_k^2 = g_k^2 + \alpha_k^2 - 2g_k\alpha_k$ gives (2). For (3), we note that

$$\left| n^{-1/2} \sum_{k=1}^{n} (f_k - \xi_k) \right| = \left| n^{-1/2} \sum_{k=1}^{n} (f_k - g_k + \alpha_k) \right|$$

$$\leqslant \left| n^{-1/2} \sum_{k=1}^{n} h_k \right| + \left| n^{-1/2} \sum_{k=1}^{n} \alpha_k \right| \to 0 \quad \text{a.e.}$$

because

$$\sum_{n=1}^{\infty} n^{-1/2} \int | h_n | \, dP < \infty \quad \text{by} \quad \text{(IX)}.$$

We now come to the justification of (IX). For this the following lemma is sufficient:

LEMMA. *Let* $\sup\{\| f \|_2 : f \in F\} = M < \infty$. *Then we can choose a subsequence* F_0 *such that for any subsequence* $\{f_n\}$ *of* F_0, (IX) *holds.*

The proof of this is similar to that of Lemma 1 in my paper "A general strong law," *Invent. Math.* **9**, 235–245 (1970) and so I shall be brief. Choose $F_0 = \{f_n\}$ to be such that

$$\lim_{n\to\infty} P(j < | f_n | \leqslant (j+1)) = a_j$$

exists for $j = 0, 1, 2, \ldots$, and so that

$$P(j < | f_n | \leqslant (j+1)) \leqslant a_j + 2^{-j}$$

for $0 \leqslant j \leqslant n$. Then $\sum j^2 a_j \leqslant M$, $\sum a_j \leqslant 1$. Also

$$\int_{\{|f_n| > n^{1/2}\}} | f_n | \, dP \leqslant \sum_{j=[n^{1/2}]}^{n} (a_j + 2^{-j})(j+1) + \int_{\{|f_n|>n\}} | f_n | \, dP$$

$$= c_n + b_n .$$

But

$$b_n \leqslant \| f_n \|_2 \{P(| f_n | > n)\}^{1/2}$$

$$\leqslant \| f_n \|_2^2 / n = M/n.$$

So

$$\sum n^{-1/2} b_n < \infty.$$

Also,

$$\begin{aligned}\sum_{n=1}^{\infty} n^{-1/2} c_n &\leqslant \sum_{j=1}^{\infty} (j+1)(a_j + 2^{-j}) \sum_{n=1}^{j^2} n^{-1/2} \\ &\leqslant D \cdot \sum (j+1)^2 (a_j + 2^{-j}) \\ &< \infty\end{aligned}$$

(where D is a positive constant).

This completes the proof of the Lemma since any subsequence of $\{f_n\}$ will also satisfy the conditions imposed on $\{f_n\}$ itself.

The proof of Theorem 1 is herewith complete.

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