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On the Existence of Solutions to the Equation $u_{tt} = u_{xxt} + \sigma(u_x)_x$

GRAHAM ANDREWS

*Department of Mathematics, Heriot–Watt University, Edinburgh EH14 4AS, Scotland**

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1. INTRODUCTION

In this paper we consider the mixed boundary-initial value problem for the partial differential equation

$$u_{tt} = u_{xxt} + \sigma(u_x)_x, \qquad x \in (0, 1)\ t \in [0, T], \tag{1.1}$$

with initial conditions

$$u(x, 0) = u_0(x), \qquad x \in (0, 1), \tag{1.2}$$

$$u_t(x, 0) = u_1(x), \qquad x \in (0, 1), \tag{1.3}$$

and boundary conditions either

$$u(0, t) = u(1, t) = 0, \qquad t \in [0, T], \tag{1.4}$$

or

$$\sigma(u_x(0, t)) = u(1, t) = 0, \qquad t \in [0, \mathrm{T}]. \tag{1.5}$$

This problem arises when one considers the purely longitudinal motion of a homogeneous bar which, in its original stress-free state, is of uniform cross-section and unit length. The displacement of a cross-section of the bar at time t is given by $u(x, t)$. Thus condition (1.4) corresponds to the case when both ends of the bar are fixed, while condition (1.5) corresponds to the case when one end is stress-free.

If $T(x, t)$ denotes the stress on a cross-section of the bar at time t, then the equation of motion takes the form

$$\rho_0 u_{tt} = T_x, \qquad x \in (0, 1)\ t \in [0, T] \tag{1.6}$$

where ρ_0 is the density of the bar in its original configuration. We obtain equation (1.1) from (1.6) by making the constitutive assumption

$$T(x, t) = \lambda u_{xt}(x, t) + \sigma(u_x(x, t)), \qquad \lambda > 0 \tag{1.7}$$

* Present address: Energy Technology Support Unit, AERE Harwell, Oxfordshire, OX11 ORA, U.K.

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and then setting $\rho_0 = \lambda = 1$. That is, we are assuming that the bar is composed of a visco-elastic material of the rate type. This is the simplest model of a material whose stress depends on the history of the motion.

The addition of the visco-elastic term u_{xt} in the constitutive assumption makes the problem more tractable than the equation of one-dimensional non-linear elasticity

$$u_{tt} = \sigma(u_x)_x . \tag{1.8}$$

It is well-known that, even for smooth initial data, global smooth solutions to (1.8) do not, in general, exist, as some second derivative of the solution may become infinite in a finite time (see MacCamy and Mizel [12]).

Equation (1.1), together with (1.2), (1.3) and (1.4), was first treated by Greenberg, MacCamy and Mizel ([10]). They assumed that the function σ was monotonic, i.e.

$$\sigma'(\xi) > 0 \qquad \text{for all} \qquad \xi \in \mathscr{R} \tag{1.9}$$

and that the initial data was smooth, specifically

$$u_0 \in C^4([0\ \ 1]), \qquad u_1 \in C^2([0, 1]).$$

Under these assumptions they were able to show the existence of a unique smooth solution which decays to the zero solution as $t \to \infty$. (See also Greenberg [8] and Greenberg and MacCamy [9].)

Dafermos [6] treated the somewhat more general problem in which $T = \sigma(u_x, u_{xt})$. He made the hypotheses

$$\sigma_p(p, q) \geqslant K > 0 \qquad \text{for all} \qquad p, q \in \mathscr{R} \tag{1.10}$$

$$| \sigma_p(p, q) | \leqslant N(\sigma_q(p, q))^{1/2} \qquad \text{for some } N > 0. \tag{1.11}$$

Then, taking initial data u_0 and u_1 in $C^{2,\alpha}([0, 1])$, he was able to prove the existence of a unique global smooth solution. In the case when the viscoelastic term is linear, that is when the stress is given by (1.7), condition (1.11) can be replaced by the weaker condition $\sigma_p(p, q) \geqslant -N$ for some $N \geqslant 0$. Note that Dafermos made no monotonicity assumption, analogous to (1.9), concerning the elastic part of the stress. As he points out, this makes the problem of asymptotic behaviour rather more interesting.

A different approach to equation (1.1) was initiated by Tsutsumi [14]. He used the Galerkin method to obtain a global "weak" solution in the Sobolev space $W^{1,p}(0, 1)$. As well as imposing a growth condition on σ, it is essential for his method of proof to impose the monotonicity assumption (1.9). Unfortunately Tsutsumi's proof runs into technical problems; he asserts that if a Banach space V can be compactly imbedded into a Banach space W, then $L^2((0, T); V)$ can be compactly imbedded into $L^2((0, T); W)$; this is not true, as the example $V = W = \mathscr{R}$ shows. These problems can, however, be overcome; see, for example, the methods used by Clements [5] when considering periodic solutions of (1.1).

In this paper we separate the problems of local and global existence of solutions to (1.1). First we prove the local existence of weak solutions, in a sense to be made more precise in section 2, under a mild hypothesis on σ and, in particular, without imposing a monotonicity condition. Then we see what additional hypotheses are sufficient to prove global existence. This approach makes clear the purpose of each restriction on the function σ. The existence theorem gives a weak solution in the Sobolev space $W^{1,\infty}(0, 1)$; it is hoped that this will be a good space in which to tackle the problem of the asymptotic behaviour of solutions to equation (1.1) when σ is not montone.

A full statement of the results, along with some definitions and remarks on notation, is given in section 2. After some lemmas on the Green's function of the heat equation in section 3, we prove the main local existence theorem in section 4. This theorem is proved with the help of a fixed point theorem due to Krasnosel'skii (see Proposition 2.2). It does not seem to be possible to use instead the contraction mapping principle or Schauder's fixed point theorem. The proofs of the results in sections 3 and 4 will be given for the case when the boundary conditions are given by (1.4), but we will remark on the changes which are needed when the boundary conditions are given by (1.5).

In section 5 we discuss the problem of global existence of solutions. Under quite mild conditions on σ, which do not imply monotonicity, we are able to prove the existence of a global weak solution in the space $W^{1,\infty}(0, 1)$. When the boundary conditions are given by (1.5) we assume that $\sigma(z)z > 0$ whenever $|z| \geqslant h$, for some $h > 0$. When the boundary conditions are given by (1.4) we have to impose the stronger hypothesis that $(\sigma(z_1) - \sigma(z_2))(z_1 - z_2) > 0$ for all $|z_1 - z_2| \geqslant h$. These global existence results are proved by using a priori estimates obtained from a maximum principle for the function $q(x, t)$ given by:

$$q(x, t) = \int_0^x u_t(z, t)\, dz - u_x(x, t), \qquad t \in [0, T],\ x \in [0, 1],$$

for the boundary condition (1.5), or by

$$q(x, t) = \int_{x_0}^x u_t(z, t)\, dz - u_x(x, t) + u_x(x_0, t), \qquad t \in [0, T],\ x, x_0 \in [0, 1],$$

for the boundary condition (1.4). Note that q is not a locally defined function; it was demonstrated by Chueh, Conley and Smoller (see [4]) that there is no locally defined function which will give the required bounds on the solution.

2. Statement of Results

We denote the norm of the space $L^p(0, 1)$, $1 \leqslant p \leqslant \infty$, by $\| \cdot \|_p$. The Sobolev space $W^{m,p}(0, 1)$, $m = 1, 2, 3, \ldots$, $1 \leqslant p \leqslant \infty$, consists of those functions u belonging to $L^p(0, 1)$ with weak derivatives $d^j u/dx^j$, $j \leqslant m$, also belonging to $L^p(0, 1)$. $W^{m,p}(0, 1)$ is a Banach space under the norm

$$\| u \|_{m,p} = \| u \|_p + \left\| \frac{du}{dx} \right\|_p + \cdots + \left\| \frac{d^m u}{dx^m} \right\|_p . \tag{2.1}$$

By Sobolev's imbedding theorem all elements of $W^{1,p}(0, 1)$ are continuous functions on $[0, 1]$. Also all elements of $W^{1,p}(0, 1)$ for which $u(0) = 0$ and/or $u(1) = 0$ satisfy

$$\| u \|_p \leqslant \sup_{x \in [0,1]} | u(x) | \leqslant \left\| \frac{du}{dx} \right\|_p \tag{2.2}$$

Define

$$W_0^{1,p}(0, 1) = \{u \in W^{1,p}(0, 1) : u(0) = u(1) = 0\}$$

Inequality (2.2) shows that $\| du/dx \|_p$ forms an equivalent norm for the Banach subspace $W_0^{1,p}(0, 1)$ of $W^{1,p}(0, 1)$. For a sequence $\{u_n\}$ in $L^\infty(0, 1)$, u_n converges weak$*$ in $L^\infty(0, 1)$ to u, which we write as $u_n \rightharpoonup^* u$ in $L^\infty(0, 1)$, if and only if

$$\int_0^1 u_n(x)\, \phi(x)\, dx \to \int_0^1 u(x)\, \phi(x)\, dx, \qquad \text{for every} \quad \phi \in L^1(0, 1).$$

For a sequence $\{v_n\}$ in $W^{1,\infty}(0, 1)$ we define weak$*$ convergence as follows: $v_n \rightharpoonup^* v$ in $W^{1,\infty}(0, 1)$ if and only if $v_n \rightharpoonup^* v$ in $L^\infty(0, 1)$ and $dv_n/dx \rightharpoonup^* dv/dx$ in $L^\infty(0, 1)$.

Later on we will make use of the following result.

Proposition 2.1. Let $\{u_n\}$ be a bounded sequence in $W^{1,\infty}(0, 1)$. Then there exists a subsequence $\{u_\mu\}$ and a function $w \in W^{1,\infty}(0, 1)$ such that $u_\mu \rightharpoonup^* w$ in $W^{1,\infty}(0, 1)$.

Let $G(x, y, t)$ be the Green's function for the heat equation on $(0, 1) \times (0, \infty)$ with zero Dirichlet boundary data. Suppose, for the moment, that u_0 and u_1 are smooth and that $u(x, t)$ is a smooth solution to the problem (1.1) – (1.4). Then

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) u_t = \sigma(u_x)_x .$$

Applying the inverse of the heat operator to both sides of this equation we obtain

$$u_t(x, t) = \int_0^1 G(x, y, t)\, u_1(y)\, dy + \int_0^t \int_0^1 G(x, y, t - \tau)\, \sigma(u_y(y, \tau))_y\, dy\, d\tau$$

so that

$$u(x, t) = u_0(x) + \int_0^t \int_0^1 G(x, y, s)\, u_1(y)\, dy\, ds$$
$$- \int_0^t \int_0^s \int_0^1 G_y(x, y, s - \tau)\, \sigma(u_y(y, \tau))\, dy\, d\tau\, ds. \qquad (2.3)$$

Thus any smooth solution of the mixed boundary-initial value problem (1.1) — (1.4) is also a solution to the integral equation (2.3). Our main local existence theorem states that, under certain hypotheses, there exists a unique solution $u(x, t)$ to equation (2.3) for a sufficiently small time interval $[0, T]$. We will look for a solution to equation (2.3) in the space $X(T)$ defined by

$$X(T) = C([0, T]; W_0^{1,\infty}(0, 1)).$$

This is a Banach space under the norm

$$\| u \|_{X(T)} = \sup_{t \in [0,T]} \{\| u(\cdot, t)\|_{1,\infty}\}$$

or equivalently, using (2.2)

$$\| u \|_{X(T)} = \sup_{t \in [0,T]} \{\| u_x(\cdot, t)\|_{\infty}\}. \qquad (2.4)$$

Our hypotheses on u_0, u_1 and σ are as follows:

(H1) The initial data u_0 and u_1 satisfy $u_0 \in W_0^{1,\infty}(0, 1)$, $u_1 \in W_0^{1,2}(0, 1)$.

(H2) The function $\sigma : \mathscr{R} \to \mathscr{R}$ is locally Lipschitz continuous, that is, for each bounded subset A of $\mathscr{R}$ there exists a constant $\alpha(A)$ with

$$| \sigma(z_1) - \sigma(z_2) | \leqslant \alpha(A) \, | z_1 - z_2 | , \qquad \text{for all} \qquad z_1, z_2 \in A.$$

THEOREM I. *Under hypotheses* (H1) *and* (H2) *there exists a unique solution* $u \in X(T)$ *to the integral equation* (2.3) *provided* $T > 0$ *is sufficiently small.*

The proof of Theorem I depends on the following fixed point theorem due to Krasnosel'skii (see [11] page 143).

PROPOSITION 2.2. *Let A be a closed, bounded, convex subset of a Banach space X. Let T and S be operators defined on A with values in X and satisfying the conditions*

(a) $Tx + Sy \in A$ whenever $x, y \in A$,

(b) T is a contraction on A, that is, there exists a constant $k < 1$ such that

$$\| Tx - Ty \| \leqslant k \| x - y \| , \qquad \text{for all} \qquad x, y \in A,$$

(c) S is continuous and compact.

Then there is at least one point x^* in A such that

$$Tx^* + Sx^* = x^*.$$

This proposition was later shown to be a special case of the existence theorem for k-set contractions.

In Lemma 4.5 we show that the solution $u \in X(T)$ to the integral equation (2.3) is also a weak solution to the mixed boundary-initial value problem (1.1) — (1.4) in the sense that u satisfies $u(x, 0) = u_0(x)$ and

$$-\int_0^t \int_0^1 u_s(x, s)\, \phi_s(x, s)\, dx\, ds + \int_0^t \int_0^1 u_{xs}(x, s)\, \phi_x(x, s)\, dx\, ds$$
$$+ \int_0^t \int_0^1 \sigma(u_x(x, s))\, \phi_x(x, s)\, dx\, ds$$
$$= \int_0^1 u_1(x)\, \phi(x, 0)\, dx - \int_0^1 u_t(x, t)\, \phi(x, t)\, dx \tag{2.6}$$

for every ϕ in the set

$$\{\phi : \phi \in C([0, T]; W_0^{1,1}(0, 1)), \phi_t \in C([0, T]; W_0^{1,2}(0, 1))\}.$$

Similar results on local existence hold when the boundary conditions are given by (1.5) rather than (1.4).

In order to prove that the solution to (2.3) exists for all time we have to place further restrictions on the function σ. When the boundary conditions are given by (1.5) that is

$$\sigma(u_x(0, t)) = u(1, t) = 0, \qquad t \in [0, T],$$

we can obtain global existence by assuming that;

(H3) there exists a constant $h > 0$ such that

$$\sigma(z)z > 0, \qquad \text{for all} \qquad |z| \geqslant h.$$

THEOREM II. *Under hypotheses* (H1), (H2) *and* (H3) *there exists a unique* $u \in X(T)$ *which satisfies the integral equation* (2.3) *and the boundary conditions* (1.5) *for any* $T > 0$. *Moreover*

$$\| u(\cdot, t) \|_{1,\infty} \leqslant C(\| u_0 \|_{1,\infty}, \| u_1 \|_{1,2}), \qquad \text{for all} \qquad t \in [0, T].$$

When the boundary conditions are given by (1.4), that is

$$u(0, t) = u(1, t) = 0, \qquad t \in [0, T],$$

a somewhat stronger condition on σ is needed.

(H4) There exists a constant $h > 0$ such that

$$(\sigma(z_1) - \sigma(z_2))(z_1 - z_2) > 0 \qquad \text{whenever} \qquad |z_1 - z_2| \geqslant h.$$

Note that (H4) implies (H3).

THEOREM III. *Under hypotheses* (H1), (H2) *and* (H4) *there exists a unique* $u \in X(T)$ *which satisfies the integral equation* (2.3) *and the boundary conditions* (1.4), *for any* $T > 0$. *Moreover*

$$\| u(\cdot, t)\|_{1,\infty} \leqslant C(\| u_0 \|_{1,\infty}, \| u_1 \|_{1,2}), \qquad \text{for all} \qquad t \in [0, T].$$

3. THE GREEN'S FUNCTION FOR THE LINEAR HEAT EQUATION

In this section we prove several results which deal with the Green's function for the linear heat equation on $(0, 1) \times (0, \infty)$, that is,

$$u_t = u_{xx}, \qquad x \in (0, 1), t > 0.$$

When the boundary conditions are

$$u(0, t) = u(1, t) = 0, \qquad \text{for all} \qquad t > 0$$

the Green's function is given explicitly by

$$G(x, y, t) = \frac{1}{(4\pi t)^{1/2}} \times \sum_{n=-\infty}^{\infty} \left\{ \exp\left(\frac{-(x - y + 2n)^2}{4t}\right) - \exp\left(\frac{-(x + y + 2n)^2}{4t}\right) \right\} \tag{3.1}$$

(see Friedman [7] p. 84). Throughout C will denote a positive constant.

LEMMA 3.1. *For every* $f \in L^\infty(0, 1)$,

$$\left\| \int_0^1 G_{yx}(x, y, t) f(y)\, dy \right\|_\infty \leqslant Ct^{-1} \| f \|_\infty, \qquad t > 0 \tag{3.2}$$

$$\left\| \int_0^1 G_{yxx}(x, y, t) f(y)\, dy \right\|_\infty \leqslant Ct^{-3/2} \| f \|_\infty, \qquad t > 0. \tag{3.3}$$

Proof. Using Hölder's inequality

$$\left| \int_0^1 G_{yx}(x, y, t) f(y)\, dy \right| \leqslant \int_0^1 | G_{yx}(x, y, t) f(y)|\, dy$$
$$\leqslant \left(\int_0^1 | G_{yx}(x, y, t)|\, dy \right) \|f\|_\infty . \tag{3.4}$$

From (3.1) we have that

$$G_{yx}(x, y, t) = \frac{1}{(4\pi t)^{1/2}} \sum_{n=-\infty}^{\infty} \left\{ \frac{1}{2t} \exp\left(\frac{-(x-y+2n)^2}{4t} \right) \right.$$
$$+ \frac{1}{2t} \exp\left(\frac{-(x+y+2n)^2}{4t} \right)$$
$$- \frac{(x-y+2n)^2}{4t^2} \exp\left(\frac{-(x-y+2n)^2}{4t} \right)$$
$$\left. - \frac{(x+y+2n)^2}{4t^2} \exp\left(\frac{-(x+y+2n)^2}{4t} \right) \right\}.$$

For $t > 0$ this series is uniformly convergent in x and y, so we may integrate term by term. Now,

$$\int_0^1 \frac{1}{2t^{1/2}} \exp\left(\frac{-(x-y+2n)^2}{4t} \right) dy \leqslant \int_{(x+2n-1)/2t^{1/2}}^{(x+2n)/2t^{1/2}} e^{-y^2}\, dy.$$

Hence,

$$\sum_{n=-\infty}^{\infty} \frac{1}{(4\pi t)^{1/2}} \int_0^1 \frac{1}{2t} \exp\left(\frac{-(x-y+2n)^2}{4t} \right) dy$$
$$\leqslant \frac{1}{2\pi^{1/2} t} \int_{-\infty}^{\infty} e^{-y^2}\, dy \leqslant Ct^{-1}.$$

Similarly

$$\sum_{n=-\infty}^{\infty} \frac{1}{(4\pi t)^{1/2}} \int_0^1 \frac{1}{2t} \exp\left(\frac{-(x+y+2n)^2}{4t} \right) dy \leqslant Ct^{-1}.$$

Also,

$$\int_0^1 \left(\frac{1}{2t^{1/2}} \right) \left(\frac{(x-y+2n)^2}{4t} \right) \exp\left(\frac{-(x-y+2n)^2}{4t} \right) dy$$
$$\leqslant \int_{(x+2n-1)/2t^{1/2}}^{(x+2n)/2t^{1/2}} y^2 e^{-y^2}\, dy$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{(4\pi t)^{1/2}} \int_0^1 \frac{(x-y+2n)^2}{4t^2} \exp\left(\frac{-(x-y+2n)^2}{4t} \right) dy$$
$$\leqslant \frac{1}{\pi^{1/2} t} \int_{-\infty}^{\infty} y^2 e^{-y^2}\, dy \leqslant Ct^{-1}$$

Similarly,

$$\sum_{n=-\infty}^{\infty} \frac{1}{(4\pi t)^{1/2}} \int_0^1 \frac{(x+y+2n)^2}{4t^2} \exp\left(\frac{-(x+y+2n)^2}{4t}\right) dy \leqslant Ct^{-1}.$$

Hence,

$$\int_0^1 | G_{yx}(x, y, t)| \, dy \leqslant Ct^{-1}. \tag{3.5}$$

Thus (3.4) and (3.5) give the required result.

In order to prove inequality (3.3) we note that

$$\begin{aligned} G_{yxx}(x, y, t) = \frac{1}{(4\pi t)^{1/2}} \sum_{n=-\infty}^{\infty} \Bigg\{ & \frac{-3(x-y+2n)}{4t^2} \exp\left(\frac{-(x-y+2n)^2}{4t}\right) \\ & - \frac{3(x+y+2n)}{4t^2} \exp\left(-\frac{(x+y+2n)^2}{4t}\right) \\ & + \frac{(x-y+2n)^3}{8t^3} \exp\left(-\frac{(x-y+2n)^2}{4t}\right) \\ & + \frac{(x+y+2n)^3}{8t^3} \exp\left(-\frac{(x+y+2n)^2}{4t}\right)\Bigg\}. \end{aligned}$$

For $t > 0$ this series is uniformly convergent in x and y, so that we may integrate term by term to obtain

$$\begin{aligned} \int_0^1 | G_{yxx}(x, y, t)| \, dy &\leqslant Ct^{-1/2} \left\{\int_{-\infty}^{\infty} | y | \, e^{-y^2} \, dy + \int_{-\infty}^{\infty} | y |^3 \, e^{-y^2} \, dy\right\} \\ &\leqslant Ct^{-3/2} \end{aligned}$$

and by Hölder's inequality we arrive at inequality (3.3).

LEMMA 3.2. *For every* $f \in L^\infty(0, 1)$

$$\left\| \frac{\partial}{\partial x} \left(\int_0^t \int_0^1 G_y(x, y, t-s) f(y) \, dy \, ds\right)\right\|_\infty \leqslant C \, \| f \|_\infty , \qquad t \geqslant 0. \tag{3.6}$$

Proof. Consider the heat equation on $(0, 1) \times (0, \infty)$ with zero Neumann boundary data, i.e.

$$v_t = v_{xx} , \qquad x \in (0, 1), \qquad t > 0 \tag{3.7}$$

$$v_x(0, t) = v_x(1, t) = 0, \qquad t > 0 \tag{3.8}$$

$$v(x, 0) = f(x), \qquad x \in (0, 1). \tag{3.9}$$

Let $H(x, y, t)$ be the Green's function for the above boundary-initial value problem, so that the solution $v(x, t)$ is given by

$$v(x, t) = \int_0^1 H(x, y, t) f(y)\, dy. \tag{3.10}$$

For any $t > 0$, $v(x, t)$ is infinitely differentiable with respect to x in $(0, 1)$ and

$$\frac{\partial^m v}{\partial x^m}(x, t) = \int_0^1 \frac{\partial^m H}{\partial x^m}(x, y, t) f(y)\, dy \tag{3.11}$$

It can easily be shown that $G_y(x, y, t) = -H_x(x, y, t)$, so that

$$\begin{aligned}\int_0^1 G_y(x, y, t-s) f(y)\, dy &= -\int_0^1 H_x(x, y, t-s) f(y)\, dy \\ &= -v_x(x, t-s).\end{aligned}$$

In a similar manner to Lemma 3.1 we can show that

$\| v_x(\cdot, t)\|_\infty \leqslant Ct^{-\frac{1}{2}} \| f \|_\infty$, hence for almost all $x \in (0, 1)$

$$\begin{aligned}\lim_{\epsilon \to 0+} \int_0^{t-\epsilon} \int_0^1 G_y(x, y, t-s) f(y)\, dy &= \int_0^t \int_0^1 G_y(x, y, t-s) f(y)\, dy \\ &= \int_0^t v_x(x, t-s)\, ds. \end{aligned} \tag{3.12}$$

Using Lemma 3.1 we have that for $\epsilon > 0$

$$\begin{aligned}\frac{\partial}{\partial x}\left(\int_0^{t-\epsilon} \int_0^1 G_y(x, y, t-s) f(y)\, dy\, ds\right) &= \int_0^{t-\epsilon} \int_0^1 G_{yx}(x, y, t-s) f(y)\, dy\, ds \\ &= -\int_0^{t-\epsilon} v_{xx}(x, t-s)\, ds \\ &= \int_0^{t-\epsilon} v_s(x, t-s)\, ds \\ &= v(x, \epsilon) - v(x, t)\end{aligned}$$

Thus for almost all $x \in (0, 1)$

$$\lim_{\epsilon \to 0+} \frac{\partial}{\partial x}\left(\int_0^{t-\epsilon} \int_0^1 G_y(x, y, t-s) f(y)\, dy\, ds\right) = f(x) - v(x, t) \tag{3.13}$$

Hence, by (3.12) and (3.13),

$$\frac{\partial}{\partial x}\left(\int_0^t \int^1 G_y(s, y, t-s) f(y)\, dy\, ds\right) = f(x) - v(x, t)$$

which immediately implies the result.

LEMMA 3.3. *For every* $f \in L^\infty(0, 1)$

$$\int_0^1 G_y(0, y, t) f(y)\, dy = \int_0^1 G_y(1, y, t) f(y)\, dy = 0, \qquad \text{for all} \quad t > 0. \quad (3.14)$$

Proof. As in the proof of lemma 3.2, let $v(x, t)$ denote the solution to the heat equation on $(0, 1) \times (0, \infty)$ with zero Neumann boundary data and with initial data $f(x)$ and let $H(x, y, t)$ denote the corresponding Green's function. Then

$$v(x, t) = \int_0^1 H(x, y, t) f(y)\, dy$$

and since $G_y(x, y, t) = -H_x(x, y, t)$ we have that

$$v_x(x, t) = \int_0^1 H_x(x, y, t) f(y)\, dy = -\int_0^1 G_y(x, y, t) f(y)\, dy.$$

But $v_x(0, t) = v_x(1, t) = 0$ for $t > 0$, and the result follows.

4. THE PROOF OF THEOREM I

For positive constants R and K we define two subsets $A(R, K)$ and $B(R, K)$ of the Banach space $X(T)$ as follows;

$$A(R, K) = \{u \in X(T) : \| u \|_{X(T)} \leqslant R, \quad \| u(t_1) - u(t_2)\|_{1,\infty} \leqslant K \mid t_1 - t_2 \mid^\gamma \text{ for all } t_1, t_2 \in [0, T]\} \quad (4.1)$$

where $\frac{1}{2} < \gamma < \frac{3}{4}$,

$$B(R, K) = \{u \in A(R, K) : \sup_{t \in [0, T]} \{\| u(\cdot, t)\|_{2,\infty}\} \leqslant R\} \quad (4.2)$$

Clearly $A(R, K)$ is a closed, bounded and convex subset of $X(T)$. Define mappings $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{C}$ on the set $A(R, K)$ by

$$(\mathscr{F} w)(x, t) = -\int_0^t \int_0^s \int_0^1 G_y(x, y, s-\tau)\, \sigma(w_y(y, \tau))\, dy\, d\tau\, ds + \Phi(x, t) \quad (4.3)$$

$$(\mathscr{G}w)(x, t) = -\int_0^t \int_0^s \int_0^1 G_y(x, y, s-\tau)\{\sigma(w_y(y, \tau)) - \sigma(w_y(y, s))\}\, dy\, d\tau\, ds \tag{4.4}$$

$$(\mathscr{C}w)(x, t) = -\int_0^t \int_0^s \int_0^1 G_y(x, y, s-\tau)\, \sigma(w_y(y, s))\, dy\, d\tau\, ds + \Phi(x, t) \tag{4.5}$$

$$\Phi(x, t) = u_0(x) + \int_0^t \int_0^1 G(x, y, s)\, u_1(y)\, dy\, ds. \tag{4.6}$$

so that $\mathscr{F}w = \mathscr{G}w + \mathscr{C}w$. For clarity we divide the proof of Theorem I into a number of lemmas whose aim is to show that $\mathscr{G}$ and $\mathscr{C}$ satisfy the hypotheses of Proposition 2.2, which will imply that $\mathscr{F}$ has a fixed point.

LEMMA 4.1. *Under hypotheses* (H1) *and* (H2) *we can choose* $R > 0$ *and* $K > 0$ *such that, for a sufficiently small* $T > 0$, $\mathscr{G}w_1 + \mathscr{C}w_2 \in A(R, K)$ *for all* w_1, $w_2 \in A(R, K)$.

Proof. Let $w_1, w_2 \in A(R, K)$ and let $\tau, s \in [0, T]$. Then, using (H2) and the definition of $A(R, K)$

$$|\sigma(w_{1x}(x, \tau)) - \sigma(w_{1x}(x, s))| \leqslant \alpha(R)\, |w_{1x}(x, \tau) - w_{1x}(x, s)|$$

for almost all $x \in [0, 1]$. So, by the Holder continuity of functions in $A(R, K)$,

$$|\sigma(w_{1x}(x, \tau)) - \sigma(w_{1x}(x, s))| \leqslant K\alpha(R)\, |\tau - s|^\gamma \tag{4.7}$$

for almost all $x \in [0, 1]$ and for all $\tau, s \in [0, T]$.
Now, the expression

$$\int_0^1 G_y(x, y, s-\tau)\{\sigma(w_{1y}(y, \tau)) - \sigma(w_{1y}(y, s))\}\, dy$$

is differentiable with respect to x and its derivative is

$$\int_0^1 G_{yx}(x, y, s-\tau)\{\sigma(w_{1y}(y, \tau)) - \sigma(w_{1y}(y, s))\}\, dy$$

Also, for almost all $x \in [0, 1]$, using Lemma 3.1,

$$\begin{aligned} \left| \int_0^1 G_{yx}(x, y, s-\tau)\{\sigma(w_{1y}(y, \tau)) - \sigma(w_{1y}(y, s))\}\, dy \right| &\leqslant C\, |s-\tau|^{-1}\, \|\sigma(w_{1y}(y, \tau)) - \sigma(w_{1y}(y, s))\|_\infty \\ &\leqslant C\, |s-\tau|^{-1}(K\alpha(R)\, |\tau - s|^\gamma) \end{aligned} \tag{4.8}$$

Hence by the theorem on differentiation of an integral

$$\frac{\partial \mathscr{G}}{\partial x} w_1(x, t) = -\int_0^t \int_0^s \int_0^1 G_{yx}(x, y, s-\tau)\{\sigma(w_{1y}(y, \tau)) - \sigma(w_{1y}(y, s))\}\, dy\, d\tau\, ds$$

and

$$\left\| \frac{\partial \mathscr{G}}{\partial x} w_1(\cdot, t) \right\|_\infty \leqslant \int_0^t \int_0^s CK\alpha(R) \mid s-\tau \mid^{\nu-1} d\tau\, ds \leqslant CK\alpha(R)\, t^{1+\nu}.$$

$$\| \mathscr{G} w_1 \|_{X(T)} \leqslant CK\alpha(R) T^{1+\nu}. \tag{4.9}$$

For the mapping $\mathscr{C}$ we use Lemma 3.2 to obtain

$$\begin{aligned}\left\| \frac{\partial \mathscr{C}}{\partial x} w_2(\cdot, t) \right\|_\infty &\leqslant \int_0^t \left\| \frac{\partial}{\partial x} \left(\int_0^s \int_0^1 G_y(x, y, s-\tau)\, \sigma(w_{2y}(y, s))\, dy\, d\tau \right) \right\|_\infty ds \\ &\quad + \| \Phi(\cdot, t) \|_{1,\infty} \\ &\leqslant C \int_0^t \| \sigma(w_{2y}(y, s)) \|_\infty\, ds + \| \Phi(\cdot, t) \|_{1,\infty}.\end{aligned} \tag{4.10}$$

Also,

$$\| \Phi(\cdot, t) \|_{1,\infty} \leqslant \| u_0 \|_{1,\infty} + \int_0^t \left\| \int_0^1 G_x(x, y, s)\, u_1(y)\, dy \right\|_\infty ds.$$

Now $G_x(x, y, s) = -H_y(x, y, s)$, where H is given by (3.10). Therefore on integrating by parts, using the fact that $u_1(0) = u_1(1) = 0$,

$$\begin{aligned}\| \Phi(\cdot, t) \|_{1,\infty} &\leqslant \| u_0 \|_{1,\infty} + \int_0^t \left\| \int_0^1 H(x, y, s) \frac{du_1}{dy}(y)\, dy \right\|_\infty ds \\ &\leqslant \| u_0 \|_{1,\infty} + \int_0^t \left(\int_0^1 \mid H(x, y, s) \mid^2 dy \right)^{1/2} \| u_1 \|_{1,2}\, ds \\ &\leqslant \| u_0 \|_{1,\infty} + CT^{3/4} \| u_1 \|_{1,2}.\end{aligned} \tag{4.11}$$

Hence, from (4.10) and (4.11)

$$\| \mathscr{C} w_2 \|_{X(T)} \leqslant C\alpha(R) T + \| u_0 \|_{1,\infty} + CT^{3/4} \| u_1 \|_{1,2} \tag{4.12}$$

and from (4.9) and (4.12) we deduce that for all $w_1, w_2 \in A(R, K)$

$$\begin{aligned}\| \mathscr{G} w_1 + \mathscr{C} w_2 \|_{X(T)} &\leqslant \| \mathscr{G} w_1 \|_{X(T)} + \| \mathscr{C} w_2 \|_{X(T)} \\ &\leqslant CK\alpha(R) T^2 + C\alpha(R) T + \| u_0 \|_{1,\infty} + CT^{3/4} \| u_1 \|_{1,2}\end{aligned} \tag{4.13}$$

Now choose R so that $\| u_0 \|_{1,\infty} \leqslant R/2$. Then by choosing T sufficiently small we have that

$$\| \mathscr{G}w_1 + \mathscr{C}w_2 \|_{X(T)} \leqslant R \qquad \text{for all } w_1, w_2 \in A(R, K). \tag{4.14}$$

Next we show that $\mathscr{G}w_1 + \mathscr{C}w_2$ satisfies the Hölder condition with respect to t whenever $w_1, w_2 \in A(R, K)$. Let $t_1, t_2 \in [0, T]$ with $t_1 > t_2$.

$$\left\| \frac{\partial \mathscr{G}}{\partial x} w_1(\cdot, t_1) - \frac{\partial \mathscr{G}}{\partial x} w_1(\cdot, t_2) \right\|_\infty$$
$$\leqslant \int_{t_2}^{t_1} \int_0^s \left\| \int_0^1 G_{yx}(x, y, s-\tau)\{\sigma(w_{1y}(y, \tau)) - \sigma(w_{1y}(y, s))\}\, dy \right\|_\infty d\tau\, ds$$

and using Lemma 3.1 and inequality (4.7) we obtain

$$\left\| \frac{\partial \mathscr{G}}{\partial x} w_1(\cdot, t_1) - \frac{\partial \mathscr{G}}{\partial x} w_1(\cdot, t_2) \right\|_\infty \leqslant CK\alpha(R)T \mid t_1 - t_2 \mid^\gamma. \tag{4.15}$$

Also, by the same method as was used to obtain inequality (4.12) it follows that

$$\left\| \frac{\partial \mathscr{C}}{\partial x} w_2(\cdot, t_1) - \frac{\partial \mathscr{C}}{\partial x} w_2(\cdot, t_2) \right\|_\infty \leqslant CT^{1-\gamma}\alpha(R) \mid t_1 - t_2 \mid^\gamma$$
$$+ CT^{3/4-\gamma} \mid t_1 - t_2 \mid^\gamma \| u_1 \|_{1,2}. \tag{4.16}$$

Now choose K so that $C(\alpha(R)T^{1/4} + \| u_1 \|_{1,2})T^{3/4-\gamma} \leqslant K/2$, and then choose T sufficiently small that $C\alpha(R)T \leqslant \frac{1}{2}$. Inequalities (4.15) and (4.16) now imply that $\mathscr{G}w_1 + \mathscr{C}w_2$ satisfies the required continuity condition in t, that is

$$\| \mathscr{G}w_1(\cdot, t_1) + \mathscr{C}w_2(\cdot, t_1) - \mathscr{G}w_1(\cdot, t_2) - Cw_2(\cdot, t_2) \|_{1,\infty} \leqslant K \mid t_1 - t_2 \mid^\gamma$$

for all $w_1, w_2 \in A(R, K)$.

Finally we show that $\mathscr{G}w(x, t)$ and $\mathscr{C}w(x, t)$ satisfy the boundary conditions for any $w \in A(R, K)$. For any $t \in [0, T]$, $\sigma(w_y(\cdot, t)) \in L^\infty(0, 1)$, so we can use Lemma 3.3 which immediately implies that $\mathscr{G}w(0, t) = \mathscr{G}w(1, t) = 0$ for all $t \in [0, T]$. Since $u_0(0) = u_0(1) = 0$ we have that $\Phi(0, t) = \Phi(1, t) = 0$ for all $t \in [0, T]$ and Lemma 3.3 implies that $\mathscr{C}w(0, t) = \mathscr{C}w(1, t) = 0$.

LEMMA 4.2. *The set $B(R, K)$, given by* (4.2), *is a pre-compact subset of $X(T)$ for any choice of positive constants R, K and T.*

Proof. By Sobolev's imbedding theorem $W^{2,\infty}(0, 1)$ can be compactly imbedded into $W^{1,\infty}(0, 1)$. For any $w \in B(R, K)$ and $t \in [0, T]$ $w(\cdot, t)$ lies in a bounded subset of $W^{2,\infty}(0, 1)$ and hence in the compact subset $\{w \in W^{1,\infty}(0, 1)$: $\| w \|_{2,\infty} \leqslant R\}$ of $W^{1,\infty}(0, 1)$. Also any $w \in B(R, K)$ satisfies

$$\| w(\cdot, t_1) - w(\cdot, t_2) \|_{1,\infty} \leqslant K \mid t_1 - t_2 \mid^{\gamma}, \tfrac{1}{2} < \gamma < \tfrac{3}{4}$$

and hence $B(R, K)$ is an equi-continuous family of functions in $X(T)$. By the infinite-dimensional version of the Ascoli-Arzela theorem (see Yosida [15] p. 85), $B(R, K)$ is a pre-compact subset of $X(T)$.

LEMMA 4.3. *$\mathscr{G}$ as a map from $A(R, K)$ into $X(T)$ is compact and continuous, provided that T is sufficiently small.*

Proof. To prove that $\mathscr{G}$ is a compact map it is sufficient to show that $\mathscr{G}w \in B(R, K)$ whenever $w \in A(R, K)$.

$$\left\| \frac{\partial^2 \mathscr{G}}{\partial x^2} w(\cdot, t) \right\|_{\infty}$$
$$\leqslant \int_0^t \int_0^s \left\| \int_0^1 G_{yxx}(x, y, s - \tau)\{\sigma(w_y(y, \tau)) - \sigma(w_y(y, s))\} \, dy \right\|_{\infty} d\tau \, ds.$$

Now use inequality (3.3) of Lemma 3.1

$$\begin{aligned}\left\| \frac{\partial^2 \mathscr{G}}{\partial x^2} w(\cdot, t) \right\|_{\infty} &\leqslant C \int_0^t \int_0^s \mid s - \tau \mid^{-3/2} \| \sigma(w_y(y, \tau)) - \sigma(w_y(y, s))\|_{\infty} \, d\tau \, ds \\ &\leqslant C \int_0^t \int_0^s \mid s - \tau \mid^{-3/2} K\alpha(R) \mid s - \tau \mid^{\gamma} d\tau \, ds \\ &\leqslant CK\alpha(R) \, T^{\gamma+1/2}.\end{aligned}$$

Combining this with inequality (4.9) we see that

$$\| \mathscr{G}w(\cdot, t) \|_{2,\infty} \leqslant CK\alpha(R)(T^{\gamma+1} + T^{\gamma+1/2}).$$

Hence, provided T is sufficiently small

$$\sup_{t \in [0, T]} \{\| \mathscr{G}w(\cdot, t)\|_{2,\infty}\} \leqslant R$$

so that $\mathscr{G}w \in B(R, K)$ whenever $w \in A(R, K)$.
To prove that $\mathscr{G}$ is continuous, let $\{w_n\}$ be a sequence in $A(R, K)$ with $w_n \to w$ in $X(T)$, so that $w \in A(R, K)$. Then

$$
\begin{aligned}
&\| \mathcal{G}w(\cdot, t) - \mathcal{G}w_n(\cdot, t)\|_\infty \\
&\quad \leqslant \int_0^t \int_0^s \Big\| \int_0^1 G_y(x, y, s-\tau)\{\sigma(w_y(y, \tau)) \\
&\qquad - \sigma(w_y(y, s)) - \sigma(w_{n_y}(y, \tau)) + \sigma(w_{n_y}(y, s))\}\, dy \Big\|_\infty \, d\tau\, ds \\
&\quad \leqslant \int_0^t \int_0^s | s - \tau |^{-1/2} \| \sigma(w_y(y, \tau)) - \sigma(w_y(y, s)) \\
&\qquad - \sigma(w_{n_y}(y, \tau)) + \sigma(w_{n_y}(y, s))\|_\infty \, d\tau\, ds \\
&\quad \leqslant CT^{3/2}\alpha(R) \| w - w_n \|_{X(T)} .
\end{aligned}
$$

Hence,

$$
\sup_{t\in[0, T]} \{\| \mathcal{G}w(\cdot, t) - \mathcal{G}w_n(\cdot, t)\|_\infty\} \leqslant CT^{3/2}\alpha(R) \| w - w_n \|_{X(T)}
$$

so that $\mathcal{G}w_n \to \mathcal{G}w$ in $C([0, T]; L^\infty(0, 1))$.
Now, for sufficiently small T, $\mathcal{G}$ is a compact map on $A(R, K)$, so there exists a subsequence $\{\mathcal{G}w_\mu\}$ of $\{\mathcal{G}w_n\}$ which converges in $X(T)$. Thus there exists $\xi \in C([0, T]; L^\infty(0, 1))$ such that

$$
\frac{\partial \mathcal{G}}{\partial x} w_\mu \to \xi \qquad \text{in} \quad C([0, T]; L^\infty(0, 1)).
$$

But since $w \in A(R, K)$, $\mathcal{G}w \in A(R, K)$ for T sufficiently small. Therefore $\xi = \partial \mathcal{G}w/\partial x$ and hence $\mathcal{G}$ is continuous on $A(R, K)$. This completes the proof of the lemma.

LEMMA 4.4. *The map $\mathcal{C}$ is a contraction on $A(R, K)$, provided that T is sufficiently small.*

Proof. *Let* $w_1, w_2 \in A(R, K)$ then,

$$
\begin{aligned}
&\left\| \frac{\partial \mathcal{C}}{\partial x} w_1(\cdot, t) - \frac{\partial \mathcal{C}}{\partial x} w_2(\cdot, t) \right\|_\infty \\
&\qquad \leqslant \int_0^t \left\| \frac{\partial}{\partial x} \left(\int_0^s \int_0^1 G_y(x, y, s-\tau)\{\sigma(w_{1y}(y, s)) - \sigma(w_{2y}(y, s))\}\, dy\, d\tau \right) \right\|_\infty ds
\end{aligned}
$$

and by Lemma 3.2 we obtain

$$
\begin{aligned}
\left\| \frac{\partial \mathcal{C}}{\partial x} w_1(\cdot, t) - \frac{\partial \mathcal{C}}{\partial x} w_2(\cdot, t) \right\|_\infty &\leqslant C \int_0^t \| \sigma(w_{1y}(y, s)) - \sigma(w_{2y}(y, s))\|_\infty \, ds \\
&\leqslant C\alpha(R) \int_0^t \| w_1(\cdot, s) - w_2(\cdot, s)\|_{1,\infty}\, ds.
\end{aligned}
$$

Hence,

$$\| \mathscr{C} w_1 - \mathscr{C} w_2 \|_{X(T)} \leqslant C\alpha(R) T \| w_1 - w_2 \|_{X(T)}$$

and choosing $T > 0$ so that $C\alpha(R)T \leqslant \frac{1}{2}$ we see that $\mathscr{C}$ is a contraction map on $A(R, K)$.

By Lemmas 4.1 to 4.4 we have verified all the hypotheses of the fixed point theorem given in Proposition 2.2 for the mappings $\mathscr{G}$ and $\mathscr{C}$ on the subset $A(R, K)$ of the Banach space $X(T)$. Thus the map $\mathscr{F} = \mathscr{C} + \mathscr{G}$ has at least one fixed point $u \in A(R, K)$ which satisfies

$$u(x, t) = u_0(x) + \int_0^t \int_0^1 G(x, y, s)\, u_1(y)\, dy\, ds - \int_0^t \int_0^s \int_0^1 G_y(x, y, s - \tau)\, \sigma(u_y(y, \tau))\, dy\, d\tau\, ds,$$

whenever $u_0 \in W_0^{1,\infty}(0, 1)$, $u_1 \in W_0^{1,2}(0, 1)$ and σ satisfies (H2).

Remark. Although initially $u_t(\cdot, 0) = u_1(\cdot) \in W_0^{1,2}(0, 1)$, for any $t > 0$ $u_t(\cdot, t) \in W_0^{1,\infty}(0, 1)$ since

$$u_t(x, t) = \int_0^1 G(x, y, t)\, u_1(y)\, dy - \int_0^t \int_0^1 G_y(x, y, t - \tau)\, \sigma(u_y(y, \tau))\, dy\, d\tau$$

and therefore

$$\begin{aligned} \| u_t(\cdot, t) \|_{1,\infty} &\leqslant \left\| \int_0^1 G_x(x, y, t)\, u_1(y)\, dy \right\|_\infty \\ &\quad + \left\| \int_0^t \int_0^1 G_y(x, y, t - \tau)\, \sigma(u_y(y, \tau))\, dy\, d\tau \right\|_{1,\infty} \\ &\leqslant C t^{-1/4} \| u_1 \|_{1,2} + CK\alpha(R)\, t^\gamma + C\alpha(R). \end{aligned}$$

Remark. It does not seem to be possible to work in the space $W^{1,p}(0, 1)$ with $1 \leqslant p < \infty$ in place of $W^{1,\infty}(0, 1)$ and still use the method of Theorem I, except in a very special case which we remark on below. For example, we might take $u_0 \in W_0^{1,p}(0, 1)$, assume that $| \sigma(z) | \leqslant C | z |^{p-1}$ and try to prove the existence of a solution $u \in C([0, T]; W_0^{1,p}(0, 1))$. However when $u(\cdot, t) \in W_0^{1,p}(0, 1)$, $\sigma(u_x(\cdot, t)) \in L^{p/(p-1)}(0, 1)$, and for every $s > 0$

$$\int_0^s \int_0^1 G_y(x, y, s - \tau)\, \sigma(w_y(y, \tau))\, dy\, d\tau \in W^{1,p/(p-1)}(0, 1),$$

but is not in the space $W^{1,p}(0, 1)$, so our method breaks down. This is because for the equation

$$v_t(x, t) = v_{xx}(x, t) + f(x, t)$$

$f(\cdot, t) \in L^p(0, 1)$ implies that $v(\cdot, t) \in W^{2,p}(0, 1)$. So for

$$u_{tt}(x, t) = u_{xxt}(x, t) + \sigma(u_x\, x, t))_x$$

$\sigma(u_x(\cdot, t)) \in L^{p/(p-1)}(0, 1)$ will give $u(\cdot, t) \in W^{1,p/(p-1)}(0, 1)$ instead of $W^{1,p}(0, 1)$.

Remark. We can obtain slightly different results by making the stronger assumption that σ is uniformly Lipschitz continuous, that is, there exists a constant α such that

$$|\sigma(z_1) - \sigma(z_2)| \leqslant \alpha\, |z_1 - z_2|, \qquad \text{for all } z_1, z_2 \in \mathscr{R}. \tag{4.17}$$

If σ is everywhere differentiable (4.17) implies that

$$|\sigma'(z)| \leqslant \alpha, \qquad \text{for all } z \in \mathscr{R}.$$

Now, by taking initial data $u_0, u_1 \in W_0^{1,2}(0, 1)$, we can prove that there exists a solution $u \in C([0, T]; W_0^{1,2}(0, 1))$ to the integral equation (2.3) for any $T > 0$. The method used is similar to that of Theorem I except that we can now work in the space $W^{1,2}(0, 1)$ in place of $W^{1,\infty}(0, 1)$. The reason that this case is an exception to the previous remark is that $u(\cdot, t) \in W^{1,2}(0, 1)$ implies that $\sigma(u_x(\cdot, t)) \in L^2(0, 1)$ and for every $s \geqslant 0$

$$\int_0^s \int_0^1 G_y(x, y, s - \tau)\, \sigma(u_y(y, \tau))\, dy\, d\tau \in W^{1,2}(0, 1)$$

as required.

Note also that we automatically obtain global existence when σ satisfies (4.17). This hypothesis is, however, too restrictive to be considered as a model for a viscoelastic material.

All that remains in Theorem I is to prove that the solution is unique. To do this we first show that any solution to the integral equation is also a "weak" solution to the partial differential equation.

LEMMA 4.5. *Let $u \in C([0, T]; W_0^{1,\infty}(0, 1))$ be a solution to the integral equation* (2.3). *Then u also satisfies $u(\cdot, 0) = u_0$ and*

$$\begin{aligned} -\int_0^t \int_0^1 u_s(x, s)\, \phi_s(x, s)\, dx\, ds &+ \int_0^t \int_0^1 u_{xs}(x, s)\, \phi_x(x, s)\, dx\, ds \\ &+ \int_0^t \int_0^1 \sigma(u_x(x, s))\, \phi_x(x, s)\, dx\, ds \\ &= \int_0^1 u_1(x)\, \phi(x, 0)\, dx - \int_0^1 u_t(x, t)\, \phi(x, t)\, dx \end{aligned} \tag{4.18}$$

for every ϕ in the set

$$\{\phi : \phi \in C([0, T]; W_0^{1,1}(0, 1)), \phi_t \in C([0, T]; W_0^{1,2}(0, 1))\}.$$

Proof. First assume that $\phi(x, t)$ is smooth. Now

$$u_t(x, t) = \int_0^1 G(x, y, t)\, u_1(y)\, dy - \int_0^t \int_0^1 G_y(x, y, t-\tau)\, \sigma(u_y(y, \tau))\, dy\, d\tau.$$

Multiplying by ϕ_t and integrating over $(0, 1) \times (0, t)$ we obtain

$$\begin{aligned}\int_0^t \int_0^1 & u_s(x, s)\, \phi_s(x, s)\, dx\, ds \\ &= \int_0^t \int_0^1 \phi_s(x, s) \left(\int_0^1 G(x, y, s)\, u_1(y)\, dy\right) dx\, ds \\ &\quad - \int_0^t \int_0^1 \phi_s(x, s) \left(\int_0^s \int_0^1 G_y(x, y, s-\tau)\, \sigma(u_y(y, \tau))\, dy\, d\tau\right) dx\, ds. \qquad (4.19)\end{aligned}$$

Also

$$\begin{aligned}u_{xt}(x, t) &= \int_0^1 G_x(x, y, t)\, u_1(y)\, dy \\ &\quad - \frac{\partial}{\partial x}\left(\int_0^t \int_0^1 G_y(x, y, t-\tau)\, \sigma(u_y(y, \tau))\, dy\, d\tau\right).\end{aligned}$$

Multiplying by ϕ_x and integrating over $(0, 1) \times (0, t)$ we obtain

$$\begin{aligned}\int_0^t \int_0^1 & u_{xs}(x, s)\, \phi_x(x, s)\, dx\, ds \\ &= \int_0^t \int_0^1 \phi_x(x, s) \left(\int_0^1 G_x(x, y, s)\, u_1(y)\, dy\right) dx\, ds \\ &\quad + \int_0^t \int_0^1 \phi_{xx}(x, t) \left(\int_0^s \int_0^1 G_y(x, y, s-\tau)\, \sigma(u_y(y, \tau))\, dy\, d\tau\right) dx\, ds. \qquad (4.20)\end{aligned}$$

Now subtract (4.19) from (4.20), interchange the order of integration in the integrals on the r.h.s. of the resulting equation, integrate by parts and use the properties of the Green's function $G(x, y, t)$. In this way we obtain equation (4.18) for all smooth functions ϕ. We can then pass to the limit to obtain the full result.

We now prove that solutions to the integral equation are *unique*. In fact we prove the following stronger result which gives the extent to which the solution depends continuously on the initial data.

PROPOSITION 4.6. *Let* $\{u_0^{(n)}(x)\}$ *be a sequence in* $W_0^{1,\infty}(0, 1)$ *and let* $\{u_1^{(n)}(x)\}$ *be a sequence in* $W_0^{1,2}(0, 1)$ *such that*

$$u_0^{(n)} \to u_0 \quad \text{in} \quad W^{1,2}(0, 1) \quad \text{and} \quad u_0^{(n)} \xrightarrow{*} u_0 \text{ in } W^{1,\infty}(0, 1)$$
$$u_1^{(n)} \to u_1 \quad \text{in} \quad W^{1,2}(0, 1).$$

Let $u^{(n)}(x, t)$ be the solution to the integral equation (2.3) with initial data $u_0^{(n)}$ and $u_1^{(n)}$ and let $u(x, t)$ be the solution with initial data u_0 and u_1. Then there exists $T > 0$ such that every solution $u^{(n)}$ exists on $[0, T]$ and there exists $R > 0$ such that

$$\sup_{t\in[0,T]} \{\| u^{(n)}(\cdot, t)\|_{1,\infty}\} \leqslant R \qquad \text{for every} \quad n. \tag{4.21}$$

Moreover for any $t \in [0, T]$, as $n \to \infty$

$$u^{(n)}(\cdot, t) \to u(\cdot, t) \qquad \text{in} \quad W^{1,2}(0, 1) \tag{4.22}$$
$$u^{(n)}(\cdot, t) \xrightarrow{*} u(\cdot, t) \qquad \text{in} \quad W^{1,\infty}(0, 1) \tag{4.23}$$
$$u_t^{(n)}(\cdot, t) \to u_t(\cdot, t) \qquad \text{in} \quad W^{1,2}(0, 1) \tag{4.24}$$

and for $t > 0$

$$u_t^{(n)}(\cdot, t) \xrightarrow{*} u_t(\cdot, t) \qquad \text{in} \quad W^{1,\infty}(0, 1) \tag{4.25}$$

Proof. Since the sequence $\{u_0^{(n)}\}$ converges weak $*$ in $W_0^{1,\infty}(0, 1)$ and $\{u_1^{(n)}\}$ converges in $W_0^{1,2}(0, 1)$ we can find a constant $R > 0$, independent of n such that

$$\| u_0^{(n)}(\cdot)\|_{1,\infty} \leqslant \frac{R}{2} \qquad \text{for all} \quad n$$
$$\| u_1^{(n)}(\cdot)\|_{1,2} \leqslant R \qquad \text{for all} \quad n.$$

In the earlier part of the proof of Theorem I, the interval of existence $[0, T]$ for a solution was determined only by $\| u_0 \|_{1,\infty}$ and $\| u_1 \|_{1,2}$. Thus we can choose an interval $[0, T]$ depending only on R and therefore independent of n as required. Moreover as Lemma 4.1 shows, on $[0, T]$ we have that

$$\sup_{t\in[0,T]} \| u^{(n)}(\cdot, t)\|_{1,\infty} \leqslant R \qquad \text{for every} \quad n$$

as required.

It remains to prove relations (4.22)-(4.25). Let $w^{(n)}(x, t) = u^{(n)}(x, t) - u(x, t)$. From Lemma 4.5 we have that

$$-\int_0^t \int_0^1 w_s^{(n)}(x, s)\, \phi_s(x, s)\, dx\, ds + \int_0^t \int_0^1 w_{xs}^{(n)}(x, s)\, \phi_x(x, s)\, dx\, ds$$

$$+ \int_0^t \int_0^1 \{\sigma(u_x^{(n)}(x, s)) - \sigma(u_x(x, s))\}\, \phi_x(x, s)\, dx\, ds$$

$$= \int_0^1 w_1^{(n)}(x)\, \phi(x, 0)\, dx - \int_0^1 w_t^{(n)}(x, t)\, \phi(x, t)\, dx \tag{4.26}$$

holds for every $\phi \in C([0, T];\ W_0^{1,1}(0, 1))$ for which $\phi_t \in C([0, T];\ W_0^{1,2}(0, 1))$. In particular (4.26) must hold when $\phi = w^{(n)}$, so that

$$-\int_0^t \| w_s^{(n)}(\cdot, s)\|_2^2\, ds + \int_0^t \int_0^1 w_{xs}^{(n)}(x, s)\, w_x^{(n)}(x, s)\, dx\, ds$$

$$+ \int_0^t \int_0^1 \{\sigma(u^{(n)}(x, s)) - \sigma(u_x(x, s))\}\, w_x^{(n)}(x, s)\, dx\, ds$$

$$= \int_0^1 w_1^{(n)}(x)\, w_0^{(n)}(x)\, dx - \int_0^1 w_t^{(n)}(x, t)\, w^{(n)}(x, t)\, dx \tag{4.27}$$

where $w_0^{(n)} = u_0^{(n)} - u_0$ and $w_1^{(n)} = u_1^{(n)} - u_1$. Now

$$\int_0^t \int_0^1 w_{xs}^{(n)}(x, s)\, w_x^{(n)}(x, s)\, dx\, ds = \frac{1}{2} \int_0^t \frac{d}{ds} \left(\int_0^1 w_x^{(n)}(x, s)^2\, dx \right) ds$$

$$= \frac{1}{2} \| w_x^{(n)}(\cdot, t)\|_2^2 - \frac{1}{2} \| w_0^{(n)} \|_{1,2}^2 \tag{4.28}$$

Substituting (4.28) into (4.27), estimating the remaining terms and using Gronwall's inequality gives us that, as $n \to \infty$

$u^{(n)}(\cdot, t) \to u(\cdot, t)$ is $W^{1,2}(0, 1)$ for any $t \in [0, T]$.

Since, for $t \in [0, T]$, $\| u^{(n)}(\cdot, t) \|_{1,\infty} \leq R$, we can use Proposition 2.1 to show that there exists a subsequence $\{u^{(n_k)}(\cdot, t)\}$ which converges weak$*$ in $W^{1,\infty}(0, 1)$ to a limit which must be $u(\cdot, t)$ as $u^{(n)}(\cdot, t)$ converges strongly to $u(\cdot, t)$ in $W^{1,2}(0, 1)$. This proves (4.23). The results for $u_t(x, t)$ follow from the above results.

COROLLARY 4.7. *There exists a unique solution u to the integral equation* (2.3) within the subset $A(R, K)$ of $X(T)$. This completes the proof of Theorem I.

Remark. Theorem I remains true if the boundary conditions (1.4) are replaced by (1.5). In this case we work in the Banach space;
$X(T) = \{u \in C([0, T];\ W^{1,\infty}(0, 1))$ such that $u(1, t) = 0$ for all $t \in [0, T]\}$. We also assume that the initial conditions satisfy $u_0 \in W^{1,\infty}(0, 1)$ with $u_0(1) = 0$ and $u_1 \in W_0^{1,2}(0, 1)$. In the integral equation we replace $G(x, y, t)$ by the Green's

function for the heat equation on $(0, 1) \times (0, \infty)$ together with the boundary condition

$$u_x(0, t) = u(1, t) = 0 \qquad \text{for all } t > 0.$$

Since $u_x(x, t)$ is not necessarily continuous in x we cannot say that the boundary condition at $x = 0$ is satisfied in the usual sense. However it can be shown that for any given $\epsilon > 0$ there exists $\delta > 0$ such that for any $t \in [0, T]$

$$\operatorname*{ess\,sup}_{x\in[0,\delta]}\{| u_x(x, t) - \alpha |\} \leqslant \epsilon$$

where α is a root of $\sigma(z) = 0$, provided that u_0 satisfies the boundary condition in the same way.

Remark. The method used in the proof of Theorem I can be adapted to prove regularity results. For example, assume that u_0, $u_1 \in W^{2,p}(0, 1)$ for some $p \geqslant 1$ and that $u_0(0) = u_0(1) = u_1(0) = u_1(1) = 0$. Then, by Theorem I there exists a unique solution $u \in C([0, T]; W_0^{1,\infty}(0, 1))$ to the integral equation (2.3), where the size of T depends only on $\| u_0 \|_{1,\infty}$ and $\| u_1 \|_{1,2}$. To obtain smoother solutions we also have to assume that $\sigma \in C^1(\mathscr{R})$ and that σ' is locally Lipschitz, we then have that $u(\cdot, t) \in W^{2,p}(0, 1)$ and $u_t(\cdot, t) \in W^{2,p}(0, 1)$ for every $t \in [0, T]$. To prove this result we work in the space

$$Z(T) = \{u \in C([0, T]; W^{2,p}(0, 1)) : u(0, t) = u(1, t) = 0 \quad \text{for all } t \in [0, T]\}$$

together with the norm

$$\| u \|_{Z(T)} = \sup_{t\in[0,T]} \{e^{-\delta t} \| u(\cdot, t)\|_{2,p}\}$$

where $\delta > 0$. The introduction of the factor $e^{-\delta t}$ allows us to use the same interval $[0, T]$ as occurs in the proof of Theorem I. (See Chu and Diaz [3]).

Similar results hold for higher ordar Sobolev spaces $W^{m,p}(0, 1)$ with $m \geqslant 3$.

5. Global Existence

Since we would not expect to obtain global existence for all possible choices of σ, we would like to find conditions on σ which are sufficient to give a priori bounds on $\| u(\cdot, t) \|_{1,\infty}$ and $\| u_t(\cdot, t) \|_{1,2}$, but which, hopefully, do not imply that σ is monotone nor that σ is uniformly Lipschitz continuous. Neither the energy equation, (5.5.), nor an application of Gronwall's Lemma to the integral equation provide the required estimates. Instead we use, in Theorems 5.2 and 5.3 a maximum principle method which is similar to certain techniques in the theory of nonlinear parabolic equations (see Chueh, Conley and Smoller [4]).

Throughout this section we will assume, for convenience, that $\sigma \in C^2(\mathscr{R})$.

We deal with the two types of boundary condition separately. Firstly we consider the case

$$\sigma(u_x(0, t)) = u(1, t) = 0, \qquad \text{for all } t \in [0, T]. \tag{5.1}$$

In Theorem 5.2 we will show that the hypothesis

(H3) there exists $h > 0$ such that

$$\sigma(z)z > 0, \qquad \text{for all } |z| \geqslant h$$

is sufficient to prove the existence of a bound on $\| u(\cdot, t) \|_{1,\infty}$.
Secondly we condier the case

$$u(0, t) = u(1, t) = 0, \qquad \text{for all } t \in [0, T]. \tag{5.2}$$

In Theorem 5.3 we will show that the hypothesis

(H4) there exists $h > 0$ such that

$$(\sigma(z_1) - \sigma(z_2))(z_1 - z_2) > 0 \quad \text{whenever} \quad |z_1 - z_2| \geqslant h$$

is sufficient to prove the existence of a bound on $\| u(\cdot, t) \|_{1,\infty}$ in this case. Note that (H4) implies (H3).

In the following lemma, which applies to both types of boundary condition, we use the energy equation associated with the initial-boundary value problem to obtain a priori estimates in $L^2(0, 1)$.

LEMMA 5.1. *Let $u(x, t)$ be a C^3 solution on $[0, T]$ of*

$$u_{tt} = u_{xxt} + \sigma(u_x)_x, \quad t \in [0, T], \quad x \in (0, 1), \tag{5.3}$$

together with the initial conditions

$$u(x, t) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in (0, 1), \tag{5.4}$$

and either of the boundary conditions (5.1) or (5.2). Then $u(x, t)$ also satisfies the energy equation

$$\frac{1}{2} \| u_t(\cdot, t)\|_2^2 + \int_0^1 W(u_x(x, t))\, dx + \int_0^t \| u_{xs}(\cdot, s)\|_2^2\, ds$$
$$= \frac{1}{2} \| u_1(\cdot)\|_2^2 + \int_0^1 W\left(\frac{du_0}{dx}(x)\right) dx \tag{5.5}$$

where

$$W(z) = \int_0^z \sigma(\xi)\, d\xi. \tag{5.6}$$

If σ satisfies hypothesis (H3), then

$$\| u_t(\cdot, t) \|_2 \leqslant L, \qquad \text{for all } t \in [0, T], \tag{5.7}$$

where L depends only on $\| u_0 \|_{1,\infty}$, $\| u_1 \|_2$ and σ.

Proof. First we note that smooth solutions to equation (5.3) exist on $[0, T]$ by the remark on regularity made in section 4. Multiply equation (5.3) by $u_t(x, t)$ and integrate over (0, 1).

$$\int_0^1 u_{tt} u_t \, dx = \int_0^1 u_{xxt} u_t \, dx + \int_0^1 \sigma(u_x)_x \, u_t \, dx$$

with either type of boundary condition we can integrate by parts to obtain

$$\frac{1}{2}\left(\frac{d}{dt} \| u_t \|_2^2\right) = -\| u_{xt} \|_2^2 - \int_0^1 \sigma(u_x)\, u_{xt}\, dx$$

$$\frac{d}{dt}\left(\frac{1}{2} \| u_t \|_2^2 + \int_0^1 W(u_x)\, dx\right) = -\| u_{xt} \|_2^2$$

and integrating over $(0, t)$ immediately gives the energy equation (5.5). Now we note that hypothesis (H3) implies that for some $J \in \mathscr{R}$

$$W(z) \geqslant J, \qquad \text{for all } z \in \mathscr{R}$$

so, from the energy equation

$$\frac{1}{2} \| u_t(\cdot, t) \|_2^2 \leqslant \frac{1}{2} \| u_1(\cdot) \|_2^2 + \int_0^1 W\left(\frac{du_0}{dx}(x)\right) dx + J$$

which immediately gives inequality (5.7).

Remark. A solution $u(x, t)$ to the integral equation (2.3) with initial data $u_0 \in W_0^{1,\infty}(0, 1)$ and $u_1 \in W_0^{1,2}(0, 1)$ also satisfies the enrgy equation (5.5). This can be proved by approximating u_0 and u_1 by a series of smooth functions and using Proposition 4.6.

THEOREM 5.2. *Let $u(x, t)$ be a C^3 solution to equation* (5.3) *on* $[0, T]$, *with initial condition* (5.4) *and boundary conditions*

$$\sigma(u_x(0, t)) = u(1, t) = 0, \qquad \text{for all} \quad t \in [0, T].$$

If σ satisfies hypothesis (H3), then

$$\| u(\cdot, t) \|_{1,\infty} \leqslant M, \qquad \text{for all} \quad t \in [0, T] \tag{5.8}$$

where M depends only on $\| u_0 \|_{1,\infty}$, $\| u_1 \|_2$ and σ.

Proof. Define a mapping q from the space $C^3([0, T] \times [0, 1])$ into the space $C^2([0, T] \times [0, 1])$ by

$$q(w(x, t)) = \int_0^x w_t(z, t)\, dz - w_x(x, t), \qquad \text{for} \quad t \in [0, T], x \in [0, 1]. \tag{5.9}$$

Thus for a fixed function w and a fixed $x \in [0, 1]$, the function $q(w(x, \cdot))$ maps $[0, T]$ into $\mathscr{R}$. In particular when $u(x, t)$ is a smooth solution to equation (5.3) then,

$$\begin{aligned}
\frac{\partial}{\partial t}(q(u(x, t))) &= \int_0^x u_{tt}(z, t)\, dz - u_{xt}(x, t) \\
&= \int_0^x \{u_{zzt}(z, t) + \sigma(u_z(z, t))_z\}\, dz - u_{xt}(x, t) \\
&= [u_{zt}(z, t) + \sigma(u_z(z, t))]_{z=0}^{z=x} - u_{xt}(x, t) \\
&= \sigma(u_x(x, t)) - \{u_{xt}(0, t) + \sigma(u_x(0, t))\}.
\end{aligned}$$

Hence, by the boundary condition at $x = 0$

$$\frac{\partial}{\partial t}(q(u(x, t))) = \sigma(u_x(x, t)).^1 \tag{5.10}$$

By Lemma 5.1 there exists an M such that

$$\sup_{x \in [0,1]} \left(\left| \int_0^x u_t(z, t)\, dz \right| \right) \leqslant \frac{M}{3} \qquad \text{for all} \quad t \in [0, T] \tag{5.11}$$

and

$$\sup_{x \in [0,1]} | u_x(x, 0)| \leqslant \frac{M}{3} \tag{5.12}$$

so that

$$\sup_{x \in [0,1]} | q(u(x, 0))| \leqslant \frac{2M}{3}.$$

[1] Assuming that the zeros of σ are isolated.

We also assume that $M/3 > h$, where h is the constant appearing in (H3). Now fix $k > 1$. Suppose that for some $x \in [0, 1]$ there exists $t \in [0, T]$ such that

$$q(u(x, t)) > \frac{2kM}{3}.$$

Since the map $t \to q(u(x, t))$, for this fixed $x \in [0, 1]$, is continuously differentiable, the intermediate value theorem implies that there exists $t^* \in (0, t)$ such that

$$q(u(x, t^*)) = \frac{2kM}{3} \tag{5.13}$$

$$q(u(x, t)) \leqslant \frac{2kM}{3} \qquad \text{for all} \quad t \in [0, t^*]$$

$$\frac{\partial q}{\partial t}(u(x, t^*)) \geqslant 0. \tag{5.14}$$

Now (5.11) and (5.13) imply that $u_x(x, t^*) \leqslant -M/3$, so, since we assumed that $M/3 > h$,

$$u_x(x, t^*) < -h.$$

But by hypothesis (H3)

$$\sigma(u_x(x, t^*))\, u_x(x, t^*) > 0.$$

Hence $\sigma(u_x(x, t^*)) < 0$, which, by (5.10), implies that

$$\frac{\partial q}{\partial t}(u(x, t^*)) < 0$$

which contradicts inequality (5.14).

If $q(u(x, t)) < 2kM/3$ for some $t \in [0, T]$, we obtain a similar contradiction. Hence,

$$|q(u(x, t))| \leqslant \frac{2M}{3} \qquad \text{for all} \quad t \in [0, T]. \tag{5.15}$$

Therefore, from (5.11) and (5.15)

$$|u_x(x, t)| \leqslant M \qquad \text{for all} \quad t \in [0, T].$$

That is, provided M is large enough to satisfy (5.11) and $M/3 > h$, then for any $x \in [0, 1]$, $|u_x(x, 0)| \leqslant M/3$ implies that $|u_x(x, t)| \leqslant M$ for all $t \in [0, T]$. Hence, if $\| u_0 \|_{1,\infty} \leqslant M/3$ then $\| u(\cdot, t) \|_{1,\infty} \leqslant M$, as required.

Next we deal with the Dirichlet boundary conditions (5.2). The method used is similar to the proof of Theorem 5.2, but it needs certain modifications and instead of (H3) we have to assume that σ satisfies the stronger condition (H4).

THEOREM 5.3. *Let* $u(x, t)$ *be a* C^3 *solution to equation* (5.3) *on* $[0, T]$ *with initial data* (5.4) *and boundary conditions*

$$u(0, t) = u(1, t) = 0 \qquad \text{for all} \quad t \in [0, T].$$

If σ satisfies hypothesis (H4), then

$$\| u(\cdot, t) \|_{1,\infty} \leqslant M \qquad \text{for all} \quad t \in [0, T] \tag{5.16}$$

where M depends only on $\| u_0 \|_{1,\infty}$, $\| u_1 \|_2$ and σ.

Proof. Using Lemma 5.1, let M be a constant such that

$$\sup_{x\in[0,1]} \left(\left| \int_{x_0}^{x} u_t(z, t)\, dz \right| \right) \leqslant \frac{M}{3} \qquad \text{for all} \quad t \in [0, T] \tag{5.17}$$

and

$$\sup_{x\in[0,1]} (| u_x(x, 0)|) \leqslant \frac{M}{6}. \tag{5.18}$$

We also assume that $M/3 > h$ where h is the constant appearing in hypothesis (H4). Define the mapping q by

$$q(w(x, t)) = \int_{x_0}^{x} w_t(z, t)\, dz - w_x(x, t) + w_x(x_0, t)$$

where x_0 is any point in $[0, 1]$. Then, when $u(x, t)$ is a smooth solution to (5.3),

$$\frac{\partial}{\partial t} (q(u(x, t))) = \sigma(u_x(x, t)) - \sigma(u_x(x_0, t)). \tag{5.19}$$

Also, by (5.17) and (5.18)

$$\sup_{x\in[0,1]} | q(u(x, 0))| \leqslant \frac{2M}{3}. \tag{5.20}$$

Fix $k > 1$. Suppose that for some $x \in [0, 1]$ there exists $t \in [0, T]$ such that

$$q(u(x, t)) > \frac{2kM}{3}.$$

Then, by the intermediate value theorem, there exists $t^* \in [0, T]$ such that

$$q(u(x, t^*)) = \frac{2kM}{3} \tag{5.21}$$

$$q(u(x, t)) \leqslant \frac{2kM}{3} \qquad \text{for all} \quad t \in [0, t^*]$$

$$\frac{\partial q}{\partial t}(u(x, t^*)) \geqslant 0. \tag{5.22}$$

Now (5.17) and (5.21) imply that

$$-u_x(x, t^*) + u_x(x_0, t^*) \geqslant \frac{M}{3} > h > 0 \tag{5.23}$$

so that, by hypothesis (H4)

$$[\sigma(u_x(x, t^*)) - \sigma(u_x(x_0, t^*))]\,[u_x(x, t^*) - u_x(x_0, t^*)] > 0$$

and hence by (5.19) and (5.23)

$$\frac{\partial}{\partial t}(q(u(x, t^*))) = \sigma(u_x(x, t^*)) - \sigma(u_x(x_0, t^*)) < 0$$

which contradicts (5.22).

If $q(u(x, t)) < -2kM/3$ for some $t \in [0, T]$ we obtain a similar contradiction. Hence

$$|q(u(x, t))| \leqslant \frac{2M}{3} \qquad \text{for all} \quad t \in [0, T] \tag{5.24}$$

which implies that

$$|u_x(x, t) - u_x(x_0, t)| \leqslant M \qquad \text{for all} \quad t \in [0, T]. \tag{5.25}$$

Now inequality (5.25) holds for any $x_0 \in [0, 1]$. In particular, using the boundary conditions, Rolle's theorem and the smoothness of $u(\cdot, t)$, we can choose $x_0 \in [0, 1]$ such that

$$u_x(x_0, t) = 0.$$

Hence, provided M satisfies (5.17) and $M/3 > h$,

$$|u_x(x, 0)| \leqslant \frac{M}{6} \quad \text{implies that} \quad |u_x(x, t)| \leqslant M \quad \text{for all} \quad t \in [0, T].$$

Thus $\| u_0 \|_{1,\infty} \leqslant M/6$ implies that $\| u(\cdot, t) \|_{1,\infty} \leqslant M$ for any $t \in [0, T]$ which completes the proof of Theorem 5.3.

In order to prove global existence we also require an a priori estimate on $\| u_t(\cdot, t) \|_{1,2}$. This follows from the estimate on $\| u(\cdot, t) \|_{1,\infty}$ since

$$u_t(x, t) = \int_0^1 G(x, y, t)\, u_1(y)\, dy - \int_0^t \int_0^1 G_y(x, y, t - \tau)\, \sigma(u_y(y, \tau))\, dy\, d\tau$$

and therefore

$$\begin{aligned}
\| u_t(\cdot, t)\|_{1,2} &\leqslant \left\| \int_0^1 G_x(x, y, t)\, u_1(y)\, dy \right\|_2 \\
&\quad + \int_0^t \left\| \int_0^1 G_{yx}(x, y, t - \tau)\{\sigma(u_y(y, \tau)) - \sigma(u_y(y, t))\}\, dy \right\|_2 d\tau \\
&\quad + \left\| \int_0^t \int_0^1 G_{yx}(x, y, t - \tau)\, \sigma(u_y(y, t))\, dy\, d\tau \right\|_2 \\
&\leqslant C \| u_1 \|_{1,2} \\
&\quad + C \int_0^t | t - \tau |^{-1}\, \alpha(M) K\, | t - \tau |^{\gamma}\, d\tau + C\alpha(M) \| u(\cdot, t)\|_{1,\infty} \\
&\leqslant C(\| u_1 \|_{1,2} + t^{\gamma}\alpha(M)K + \alpha(M) \| u(\cdot, t)\|_{1,\infty}).
\end{aligned}$$

Thus $\| u_t(\cdot, t) \|_{1,2}$ is bounded whenever $\| u(\cdot, t) \|_{1,\infty}$ is bounded. So far in this section we have assumed that $u(x, t)$ is smooth. Using Proposition 4.6 we now extend the results of Theorems 5.2 and 5.3 to any solution of the integral equation for which $u_0 \in W^{1,\infty}(0, 1)$ and $u_1 \in W_0^{1,2}(0, 1)$ and so prove global existence.

THEOREM II. *Under hypotheses* (H1), (H2) *and* (H3) *there exists a unique* $u \in X(T)$ *which satisfies the integral equation* (2.3) *and the boundary conditions* (1.5) *for any* $T > 0$. *Moreover*

$$\| u(\cdot, t) \|_{1,\infty} \leqslant C(\| u_0 \|_{1,\infty}, \| u_1 \|_{1,2}), \qquad \text{for all} \quad t \in [0, T].$$

THEOREM III. *Under hypotheses* (H1), (H2) *and* (H4) *there exists a unique* $u \in X(T)$ *which satisfies* (2.3) *and* (1.4) *for any* $T > 0$. *Moreover*

$$\| u(\cdot, t) \|_{1,\infty} \leqslant C(\| u_0 \|_{1,\infty}, \| u_1 \|_{1,2}), \qquad \text{for all} \quad t \in [0, T].$$

Proof. We will only give the proof of Theorem III as the proof of Theorem II is nearly identical.

The first step is to approximate $u_0(x)$ and $u_1(x)$ by smooth functions $u_0^{(\epsilon)}(x)$ and $u_1^{(\epsilon)}(x)$ which vanish for $x = 0$ and $x = 1$. Let $\epsilon > 0$ and define

$$\begin{aligned} v_0(x) &= u_0((1 - 4\epsilon)x + 2\epsilon), && \text{for} \quad x \in (2\epsilon, 1 - 2\epsilon) \\ &\quad 0, && \text{for} \quad x \in [0, 2\epsilon] \cup [1 - 2\epsilon, 1]. \end{aligned}$$

Since $u_0 \in W_0^{1,\infty}(0, 1)$, $v_0 \in W_0^{1,\infty}(0, 1)$ and v_0 has compact support in $(0, 1)$. Moreover

$$\| v_0 \|_{1,\infty} \leqslant \| u_0 \|_{1,\infty} \, .$$

Let $\rho \in C_0^\infty(\mathscr{R})$ be non-negative and such that $\rho(x) = 0$ if $| x | \geqslant 1$, and $\int_{-\infty}^{\infty} \rho(x)dx = 1$. For any $\epsilon > 0$ the function $\rho_\epsilon(x) = \epsilon^{-1}\rho(x/\epsilon)$ is non-negative, belongs to $C_0^\infty(\mathscr{R})$, satisfies $\rho_\epsilon(x) = 0$ if $| x | \geqslant \epsilon$ and $\int_{-\infty}^{\infty} \rho_\epsilon(x)dx = 1$. The function ρ_ϵ is called a mollifier. If we take the convolution of ρ_ϵ with v_0, that is

$$(\rho_\epsilon * v_0)(x) = \int_0^1 \rho_\epsilon(x - y)\, v_0(y)\, dy$$

then $\rho_\epsilon * v_0 \in C_0^\infty(0, 1)$. It is now easy to show that (see Adams [1])

$$\lim_{\epsilon \to 0} \| \rho_\epsilon * v_0 - u_0 \|_{1\,2} = 0 \tag{5.26}$$

and

$$\| \rho_\epsilon * v_0 \|_{1,\infty} \leqslant \| v_0 \|_{1,\infty} \leqslant \| u_0 \|_{1,\infty} \, . \tag{5.27}$$

Thus $\{\rho_\epsilon * v_0\}$ forms a bounded set in $W_0^{1,\infty}(0, 1)$, so we can find a sequence $\epsilon^n \to 0$ such that $\rho_{\epsilon_n} * v_0 \overset{*}{\rightharpoonup} u_0$ in $W^{1,\infty}(0, 1)$. Let $u_0^{(n)}(x) = \rho_{\epsilon_n} * v_0(x)$, then as $n \to \infty$

$$\begin{aligned} u_0^{(n)} &\to u_0 && \text{in} \quad W^{1,2}(0, 1) \\ u_0^{(n)} &\overset{*}{\rightharpoonup} u_0 && \text{in} \quad W^{1,\infty}(0, 1). \end{aligned}$$

In a similar fashion we can construct a sequence $\{u_1^{(n)}\}$ of functions in $W_0^{1,2}(0, 1)$ such that as $n \to \infty$

$$u_1^{(n)} \to u_1 \qquad \text{in} \quad W^{1,2}(0, 1).$$

Let M be the constant such that

$$\| u_0^{(n)} \|_{1,\infty} \leqslant M \qquad \text{and} \qquad \| u_1^{(n)} \|_{1,2} \leqslant M. \tag{5.28}$$

Let $u^{(n)}(x, t)$ be the solution to the integral equation with initial data $u_1^{(n)}$ and $u_0^{(n)}$, and let $u(x, t) \in X(T)$ be the solution with initial data u_0 and u_1. By

Theorem I we can choose $T > 0$ sufficiently small so that every solution $u^{(n)}$ and u exists on $[0, T]$, and by Proposition 4.6 we have that for every $t \in [0, T]$

$$u^{(n)}(\cdot, t) \overset{*}{\rightharpoonup} u(\cdot, t) \quad \text{in} \quad W^{1,\infty}(0, 1) \tag{5.29}$$

$$u_t^{(n)}(\cdot, t) \rightharpoonup u_t(\cdot, t) \quad \text{in} \quad W^{1,2}(0, 1). \tag{5.30}$$

But using Theorem 5.3 (or Theorem 5.2 in the proof of Theorem II) inequalities (5.28) imply that there exists a constant $N > 0$, independent of n such that for all $t \in [0, T]$

$$\| u^{(n)}(\cdot, t)\|_{1,\infty} \leqslant N \quad \text{and} \quad \| u_t^{(n)}(\cdot, t)\|_{1,2} \leqslant N \quad \text{for all} \quad n.$$

Hence from (5.29) and (5.30)

$$\| u(\cdot, t) \|_{1,\infty} \leqslant N \quad \text{for all} \quad t \in [0, T] \tag{5.31}$$

$$\| u_t(\cdot, t) \|_{1,2} \leqslant N \quad \text{for all} \quad t \in [0, T]. \tag{5.32}$$

The existence of a unique solution on an arbitrary interval $[0, T]$ now follows from (5.31) and (5.32) by a standard method (Reed [13] page 9).

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