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Asymptotic behaviour of two-point functions in multi-species models

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Abstract

We extract the long-distance asymptotic behaviour of two-point correlation functions in massless quantum integrable models containing multi-species excitations. For such a purpose, we extend to these models the method of a large-distance regime re-summation of the form factor expansion of correlation functions. The key feature of our analysis is a technical hypothesis on the large-volume behaviour of the form factors of local operators in such models. We check the validity of this hypothesis on the example of the $SU(3)$-invariant XXX magnet by means of the determinant representations for the form factors of local operators in this model. Our approach confirms the structure of the critical exponents obtained previously for numerous models solvable by the nested Bethe Ansatz.

0. Introduction

Form factor expansions, and hence form factors, play an important role in the characterisation of correlation functions. Over the last few decades, there has been a significant progress in describing the form factors and the associated expansions for so-called quantum integrable systems. First progress in characterising the form factors has been achieved for massive models

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directly in the infinite volume. Archetypes of such models are given by the massive integrable quantum field theories in $1 + 1$ dimensions. In such a setting, the form factors were characterised as solutions to a set of bootstrap equations [31,33,72]. The resolution of the bootstrap program allowed for an explicit description of the form factors of local operators in numerous models and of some of their intrinsic properties [1,12,26,31,42,51,72]. One should also mention the significant progress in conforming the representation theory of quantum affine algebras to the description of the spectrum of the XXZ spin chain Hamiltonian [13]. This progress allowed to access to multiple integral representations for the form factors in various massive spin chain models [4,30,43], again directly in the infinite volume.

More recently, the calculation of form factors of finite volume quantum integrable models associated to rank one Lie algebras was undertaken within the algebraic Bethe Ansatz [22]. The approach builds on two major ingredients: on the one hand the solution of the quantum inverse scattering problem [39,50] and, on the other hand, determinant representations for the norms [40] and scalar products [67] of Bethe vectors. Within such a setting one obtains finite-size determinant representations for the form factors, see *e.g.* [39,41]. Typically, when the model's volume goes to infinity – the so-called thermodynamic limit –, so does the size of the matrix whose determinant is evaluated. The very structure of the limit depends strongly on whether the underlying model exhibits a massless or a massive spectrum. The massive case is easier to deal with in that, individual form factors decay as integer powers of the volume L. This integer power in L decay allows one to replace discrete sums appearing in a finite volume form factor expansion by a series of multiple integrals, once the thermodynamic limit is taken. The investigation of the large-volume behaviour of a specific form factor in the massive regime of the XXZ chain was carried out in [28]. Later, the analysis was extended to all form factors of the longitudinal spin operator in [19]. The main complication associated with a massless spectrum is that the form factors are expected to vanish as some, generically, non-integer power of the volume [11]. The presence of such vanishing strongly complicates the analysis. First results relative to extracting the leading in L behaviour out of the determinant representations were obtained in [68]. They concerned the form factors of the current operator in the non-linear Schrödinger model. The technique of analysis developed there was improved and extended in [34,36] where the large-volume behaviour of so-called particle–hole form factors in the massless regime of the XXZ chain was obtained. See also [17,18,45] where the analysis of the low-temperature limit of so-called thermal form factors in a massless model at finite temperature has been carried out.

The main issue with the non-integer decay in the volume of individual form factors is that it does not allow one to replace the finite-volume form factor expansions by series of multiple integrals. In fact, for a finite spatial and/or temporal separation between the operators it has been impossible, so far, to write any meaningful form factor series expansion in the thermodynamic limit. Even though intractable in general, form factor series expansion for massless models have recently been discovered to be manageable in the limit of large spatial separations between the operators building up the correlator from which the expansion originates. Indeed, when the volume is large but finite, the large-distance/time asymptotic behaviour of such series can be extracted by means of a variant of the saddle-point method. The evaluation of the leading contribution to the correlator is achieved through the evaluation of certain multidimensional sums over the massless excitations of the model. After re-summing, one can already take the infinite volume limit, hence accessing to the large-distance asymptotic behaviour of the correlator. This approach has been developed in [35,37,38] and culminated in the construction of a direct mapping [46] between the zero energy excitation sector of a massless model and the free boson field theory.

The results which have been reminded above mostly concern integrable models built over rank one algebras, typically $GL(2)$ or its q-deformation. The correlation functions and form factors of models built over higher rank algebras were much less studied, this despite their relevance to the physics of super-Yang–Mills theories [23,65] or condensed matter models of mixed particle species [5,6,49]. Multiple integral representations for the form factors of massive quantum field theories associated with various higher rank algebras were obtained, *e.g.*, in the works [2,3,71]. However, so far, only few results were obtained relatively to quantum integrable models in finite volume. Indeed, although many models underlying to higher rank Lie algebras can still be solved by a variant of the Bethe Ansatz – the so-called nested Bethe Ansatz [47,48,80] – the structure of these model's Bethe vectors becomes highly intricate hence inducing numerous new technical complications to the calculation of correlation functions. In fact, until very recently, only a determinant representation for the norm [62] of Bethe vectors in certain rank 2 models was known. Recently, some progress has been achieved relatively to simplifying the scheme of constructing the Bethe vectors as well as to the calculation of their scalar product [9,20,32,52,54], this in the context of spin chain models based on either $GL(N)$ or its q-deformation. For models enjoying of an $SU(3)$ symmetry, several expressions – either in terms of determinants or multiple-integrals – were obtained for scalar products between Bethe vectors [7,8,75,76]. The determinant based representations for the scalar products appeared effective enough so as to lead to determinant based representations for the form factors [10,55]. Very recently, some progress has been made relatively to characterising the scalar products in models related to a q-deformation of a rank 2 algebra [53,56], what ultimately led to a determinant representation for some specific instances of scalar products [70]. The aforementioned results could also be generalised to the case of a two-component Bose gas. A set of recurrence relations satisfied by the form factors in this model have been obtained in [61] and some special cases of solutions were given. Later, determinant representations for all form factors in the model were obtained in [57–59], this on the basis of preparatory results obtained in [66].

Despite the lack of explicit representation for multi-point correlation functions in higher rank models, one can still expect to build on the universality principle so as to describe some of the properties of the correlators, their large-distance asymptotics in particular. Building on Cardy's [11] relations between the $1/L$ corrections to the ground and excited state's energies on the one hand and critical exponents on the other one, Izergin, Korepin and Reshetikhin [29] predicted the critical exponents driving the long-distance asymptotic behaviour of two-point functions in a large class of higher rank massless quantum integrable models. The predictions for the critical exponents obtained in [29] were in fact extending Cardy's approach in that the large-distance regime was not described by an effective $c = 1$ conformal field theory but, rather, by a direct sum thereof. This was needed due to the different values of Fermi velocities associated with the different branches of massless excitations in such models. The $1/L$ corrections to the excitation energies of the Hubbard model were obtained in [77] for the model at half-filling, which led to a purely conformal spectrum of low-lying excitations. These results were extended to the case of the Hubbard model away from half-filling in [79], and it was apparent that, again, one needs to consider a direct sum of $c = 1$ conformal field theories so as to describe the long-distance asymptotics of correlation functions in this model. This point of view was developed in [24] and some specific limits (strong coupling, magnetic field close to zero or to its critical value) of the critical exponents were studied in [25]. Asymptotics of two-point functions in this model were also predicted on the basis of a generalisation of the Luttinger liquid concept to multi-species excitations in [63,64].

The aim of the paper is to test the aforementioned predictions through an *ab initio* calculation. On top of providing a firm ground to the universality predictions, this paper demonstrates that upon certain reasonable hypothesis on the structure of the spectrum and form factors of local operators – which we explicitly check to be true in the case of the $SU(3)$-invariant XXX magnet – it is possible to adapt the large-distance form factor summation technique developed in [35,37,38,46] to the setting of multi-species excitations. Our microscopic approach confirms the predictions of [24,29,63,64,79]. We shall describe the structure of the large-distance asymptotics we obtain on the example of a specific correlator in the $SU(3)$-invariant XXX magnet, the two-point function $\langle \mathtt{E}^{21}_{1+m}\mathtt{E}^{12}_{1}\rangle$ involving the local operators given by the elementary matrices E^{ij} localised on sites 1 and $1+m$ of the chain. The results obtained in the core of this paper demonstrate that the following large-m expansion holds

$$\langle \mathtt{E}^{21}_{1+m}\mathtt{E}^{12}_{1}\rangle \simeq \sum_{\boldsymbol{\ell}\in\mathbb{Z}^2} (-1)^{m\ell^{(1)}} \cdot \mathrm{e}^{\mathrm{i}m(2\boldsymbol{\ell}+\mathfrak{n}_{12})\cdot\boldsymbol{\sigma}_F} \cdot \frac{\left|\mathcal{F}_{\boldsymbol{\ell},\boldsymbol{\kappa}_{12}}\big(\mathtt{E}^{12}_1\big)\right|^2}{\big(2\pi m\big)^{\Delta_{\boldsymbol{\ell},\boldsymbol{\kappa}_{12}}}} \tag{0.1}$$

with

$$2\Delta_{\boldsymbol{\ell},\boldsymbol{\kappa}_{12}} = \Big(2\boldsymbol{\ell}+\mathfrak{n}_{12}, \mathcal{Z}\mathcal{Z}^{\mathrm{t}}\cdot(2\boldsymbol{\ell}+\mathfrak{n}_{12})\Big) + \big(\boldsymbol{\kappa}_{12}, \big[\mathcal{Z}\mathcal{Z}^{\mathrm{t}}\big]^{-1}\boldsymbol{\kappa}_{12}\big)\,. \tag{0.2}$$

In the above expansion

- the sum runs over two-dimensional integer valued vectors $\boldsymbol{\ell} = \big(\ell^{(1)}, \ell^{(2)}\big)$ which label the possible Umklapp excitations over the model's ground state.
- The vector $\mathfrak{n}_{12}$ originates from the fine structure of the class of excited states connected to the ground state by the operator $\mathtt{E}^{12}_1$.
- The contribution of each Umklapp excitation produces an oscillatory factor with phase $2\mathrm{i}m\,\boldsymbol{\ell}\cdot\boldsymbol{\sigma}_F$ in which $\boldsymbol{\sigma}_F = \big(\sigma_F^{(1)}, \sigma_F^{(2)}\big)$ is built up from the two Fermi momenta associated with the two Fermi zones of the model.
- There is an additional oscillatory factor with phase $\mathrm{i}m\,\mathfrak{n}_{12}\cdot\boldsymbol{\sigma}_F$ associated with the shift in the momentum of the class of excited states connected by the operator $\mathtt{E}^{12}_1$ to the ground state.
- The quantity $|\mathcal{F}_{\boldsymbol{\ell},\boldsymbol{\kappa}_{12}}\big(\mathtt{E}^{12}_1\big)|^2$ represents the thermodynamic limit of the $\mathtt{E}^{12}_1$ form factor, taken between the ground state and the lowest energy excited state corresponding to the above mentioned $\boldsymbol{\ell}$-Umklapp excitations and properly normalised in the model's volume.
- Finally, the critical exponent is given by a scalar product which involves a 2×2 matrix $\mathcal{Z}$ that can be expressed in terms of the dressed charge matrix of the model.

The paper is organised as follows. In Section 1 we present the general framework that a model needs to fulfil so that the approach of the paper becomes applicable. In Section 2 we show how the general structure described in Section 1 does allow one to access to the large-distance asymptotic behaviour of the two-point functions in the model. We then conform the obtained result to the case of the $SU(3)$-invariant XXX magnet and to the Hubbard model away from half-filling. In Section 3 we build on the determinant representation for the form factors of local operators in the $SU(3)$-invariant XXX magnet so as to prove that, indeed, this rank 2 quantum integrable model does fit into the framework described in Section 1. The paper contains several appendices were we postpone some of the technical handlings. Appendix A provides a reminder of the description of the thermodynamic limit of higher rank models. In Section A.1 we briefly recall the derivation of the large-L expansion of the vector counting function associated with nested Bethe Ansatz solvable models. In Section A.2, we establish several identities satisfied by the solutions to linear integral equations that are of interest to the problem. In Section A.3 we build on these results so as to obtain the leading large-L expansion of the excitation energies and momenta relatively to the ground state in such models. Appendix B contains the proof of the smooth-discrete part factorisation of form factors in the $SU(3)$-invariant XXX magnet. We have also gathered our basic notations in Appendix C so as to ease the reading of this article.

1. The general setting

In this section we discuss several general features present in quantum integrable models solvable by the nested Bethe Ansatz. The number of nestings necessary to construct the Bethe vectors will be denoted by $r-1$, where r corresponds to the rank of the Lie algebra over which the integrable model is constructed. We will focus on lattice integrable models, namely those whose Hilbert space $\mathfrak{h}=\mathfrak{h}_1\otimes\cdots\otimes\mathfrak{h}_L$ admits a tensor product decomposition into a product of so-called local spaces $\mathfrak{h}_k$ called the sites of the model. We will assume that these spaces are all isomorphic to some base space $\mathfrak{h}_k\simeq\mathfrak{h}_{\mathrm{base}}$. The Hamiltonian of the model is assumed to take the form

$$\mathbf{H}=\mathbf{H}_0+\sum_{k=1}^{r}h^{(k)}\mathbf{Q}^{(k)}\,. \tag{1.1}$$

$\mathbf{H}_0$ represents the base Hamiltonian and $\mathbf{Q}^{(k)}$, $k=1,\dots,r$, a set of r independent conserved charges $\big[\mathbf{H}_0,\mathbf{Q}^{(k)}\big]=0$ that take into account the possibility of coupling the base $\mathbf{H}_0$ Hamiltonian with external fields $h^{(1)},\dots,h^{(r)}$. Explicit examples of base Hamiltonians $\mathbf{H}_0$ and of its associated charges $\mathbf{Q}^{(k)}$ will be discussed in Sub-section 2.2, this for the case of the $SU(3)$-invariant XXX magnet and the Hubbard model. Here, we only mention that, typically, the auxiliary conserved charges $\mathbf{Q}^{(k)}$ have integer eigenvalues which are expressible, in a simple way, in terms of the number of roots arising in the nested Bethe Ansatz equations.

The base Hilbert space $\mathfrak{h}_{\mathrm{base}}$ is assumed to be endowed with an algebraic basis $\mathbf{O}^{(\alpha)}$ of operators where the superscript $\alpha=1,2,\dots$ runs through some finite or infinite set depending on the dimensionality of $\mathfrak{h}_{\mathrm{base}}$. These operators can then be raised into operators $\mathbf{O}_n^{(\alpha)}$ on $\mathfrak{h}$ by tensoring them with the identity

$$\mathbf{O}_n^{(\alpha)}=\underbrace{\mathrm{id}\otimes\cdots\otimes\mathrm{id}}_{n-1\ \text{times}}\otimes\,\mathbf{O}^{(\alpha)}\otimes\underbrace{\mathrm{id}\otimes\cdots\otimes\mathrm{id}}_{L-n\ \text{times}}\,. \tag{1.2}$$

We will call such operators local since they only act non-trivially on *one* of the spaces appearing in the tensor product decomposition of $\mathfrak{h}$. We will assume in the following that the algebraic basis $\mathbf{O}^{(\alpha)}$ used to build the operators $\mathbf{O}_n^{(\alpha)}$ is chosen in such a way that the operators $\mathbf{O}_n^{(\alpha)}$ only connect eigenstates of the operators $\mathbf{Q}^{(k)}$ having definite eigenvalues.

1.1. Bethe equations and basic observables

The eigenvectors of the Hamiltonian (1.1) that are constructed by the nested Bethe Ansatz are called Bethe vectors and we denote them by $|\,\Lambda\,\rangle$. The Bethe vectors $|\,\Lambda\,\rangle$ are parametrised by a collection

$$\Lambda=\Big\{\{\lambda_a^{(k)}\}_{a=1}^{N_\Lambda^{(k)}}\,,\,k=1,\dots,r\Big\} \tag{1.3}$$

of Bethe roots, split into r families of "roots" $\{\lambda_a^{(k)}\}_{a=1}^{N_\Lambda^{(k)}}$ that we call species. The notation introduced in (1.3) indicates that the set Λ contains $N_\Lambda^{(k)}$ roots of species k. Note that, when passing from one Bethe vector to another, the number of roots of a given species may change. The roots Λ not only parametrise the Bethe vector $|\,\Lambda\,\rangle$ but also all the observables of the model associated with it: the energy, momentum and, more generally, the form factor of local operators involving the state $|\,\Lambda\,\rangle$. For $|\,\Lambda\,\rangle$ to be an eigenvector of the model's Hamiltonian, the roots Λ have to satisfy a system of Bethe equations which, in its logarithmic variant, takes the general form:

$$L\sigma_0^{(k)}(\lambda_a^{(k)}) + \sum_{\ell=1}^{r}\sum_{b=1}^{N_\Lambda^{(k)}}\theta_{k\ell}(\lambda_a^{(k)},\lambda_b^{(\ell)}) = m_{a;\Lambda}^{(k)} - \frac{1+N_\Lambda^{(k)}-\mathfrak{n}_\Lambda^{(k)}}{2}, \tag{1.4}$$

where the shift integer $\mathfrak{n}_\Lambda^{(k)}$ can be expressed in terms of the condition below on the sums of integers

$$w_\Lambda^{(k)} = \sum_{\substack{\ell=1,\neq k\\ \ell:\theta_{k\ell}\neq 0}}^{r} N_\Lambda^{(\ell)} + \big(N_\Lambda^{(k)}+1\big)\mathbf{1}_{\theta_{kk}=0} \quad \text{as} \quad \mathfrak{n}_\Lambda^{(k)} = \begin{cases} 0 & \text{if} \quad w_\Lambda^{(k)} \in 2\mathbb{N} \\ 1 & \text{if} \quad w_\Lambda^{(k)} \in 2\mathbb{N}+1 \end{cases}. \tag{1.5}$$

Here $\mathbf{1}_{\theta_{kk}=0}=1$ if the function θ_{kk} vanishes identically and $\mathbf{1}_{\theta_{kk}=0}=0$ otherwise. The functions appearing in the *lhs* of (1.4) are the bare momenta $\sigma_0^{(k)}$ and the bare phases $\theta_{k\ell}$ of the excitations. Throughout this paper, we shall assume that the bare phase is a function of the sole difference of arguments and that it is symmetric in its indices

$$\theta_{k\ell}(\lambda,\mu) = \theta_{\ell k}(\lambda,\mu) = \vartheta_{k\ell}(\lambda-\mu)\,. \tag{1.6}$$

We shall as well take for granted that all bare momenta and bare phases are odd functions, namely that

$$\sigma_0^{(k)}(-\lambda) = -\sigma_0^{(k)}(\lambda) \qquad \text{and} \qquad \theta_{k\ell}(\lambda,\mu) = \theta_{\ell k}(-\mu,-\lambda) = -\theta_{\ell k}(\mu,\lambda)\,, \\ \text{for} \quad k,\ell \in [\![\,1\,;\,r\,]\!]\,. \tag{1.7}$$

We do stress that although such an assumption allows for a few technical simplifications, it is not essential at any stage of the handlings that will follow. Finally, the logarithmic Bethe equations involve integers $m_{a;\Lambda}^{(k)} \in \mathcal{I}^{(k)}$ which take value in a model-dependent set $\mathcal{I}^{(k)} \supset [\![\,1\,;\,N_\Lambda^{(k)}\,]\!]$. For models with an unbounded kth bare momentum – such as the multi-component Bose gas [80] – one has, typically, $\mathcal{I}^{(k)}=\mathbb{Z}$ while, for models with a bounded kth bare momentum – such as the $GL(N)$-invariant XXX magnet [47,48] –, one has $\mathcal{I}^{(k)} = [\![\,-M_-^{(k)}\,;\,M_+^{(k)}\,]\!]$ for some integers $M_\pm^{(k)}$.

Clearly, a choice of roots Λ does fix the value of the integers $m_{a;\Lambda}^{(k)}$ arising in the *rhs* of the logarithmic Bethe equations (1.4). However, it could happen that the correspondence is not injective, meaning that two distinct collections of Bethe roots could give rise to exactly the same collection of integers. In the following, we shall however assume that, if one restricts one's attention to a subset of solutions to the Bethe equations called particle–hole excited states, then there is a one-to-one correspondence between collections of integers and solutions to the Bethe Ansatz equations. Recently, this property was show to hold, for L large-enough, in the case of the XXZ spin-1/2 chain [44] and we believe that the same mechanism will be at play in the more general setting we consider in the present paper.

• **The ground state**

We shall assume that the model has a non-degenerate ground state.[1] We shall always denote by

$$\Omega = \big\{ \{\omega_a^{(k)}\}_{a=1}^{N_\Omega^{(k)}}, k = 1, \dots, r \big\} \tag{1.8}$$

the collection of Bethe roots giving rise to the model's ground state. Furthermore, we shall assume that the set of integers associated with the ground state takes the form

$$m_{a;\Omega}^{(k)} = a \,, \quad a = 1, \dots, N_\Omega^{(k)} \quad \text{and} \quad k = 1, \dots, r \tag{1.9}$$

and that, for the ground state, one has $\mathfrak{n}_\Omega^{(k)} = 0$ for $k = 1, \dots, r$. The value of the integers $N_\Omega^{(k)}$ of kth-species roots for the ground state is fixed by the external fields $h^{(k)}$, $k = 1, \dots, r$ which couple $\mathbf{H}_0$ to the auxiliary conserved charges $\mathbf{Q}^{(k)}$.

When taking the thermodynamic limit $L \to +\infty$ of the model, we will assume that all the integers $N_\Omega^{(k)}$, $k = 1, \dots, r$ grow with L in such a way that $\lim_{L\to+\infty} \big(N_\Omega^{(k)}/L\big) = D^{(k)}$, with $D^{(k)} > 0$, is finite and fixed once for all. Furthermore, these integers will be such that the two endpoints $M_{-/+}^{(k)}$ of the set $\mathcal{I}^{(k)}$ obey

$$M_-^{(k)} \to +\infty \quad \text{and} \quad M_+^{(k)} - N_\Omega^{(k)} \to +\infty \quad \text{as} \quad L \to +\infty \,. \tag{1.10}$$

In this paper we shall build on the assumption that the ground state roots have the densification property, meaning that

$$\frac{1}{L} \sum_{a=1}^{N_\Omega^{(k)}} f\big(\omega_a^{(k)}\big) \underset{L\to+\infty}{\longrightarrow} \int_{-q^{(k)}}^{q^{(k)}} f(s) \cdot \rho^{(k)}(s) \mathrm{d}s \,. \tag{1.11}$$

In other words, the kth species ground state Bethe roots will form, when $L \to +\infty$, a dense distribution on a finite interval $[-q^{(k)}; q^{(k)}]$ with density $\rho^{(k)}$. The interval $[-q^{(k)}; q^{(k)}]$ will be called the Fermi zone associated with the kth species roots, or kth Fermi zone for short. The specific value of the endpoints $q^{(k)}$ of this interval is fixed by the values taken by the densities $D^{(a)}$, $a = 1, \dots, r$.

We do stress that for the XXZ spin-1/2 chain which is a rank one ($r = 1$) model one can prove [81] that, indeed, the choice of integers (1.9) does give rise to the roots parameterising the ground state in the sector with $N_\Omega^{(1)}$ particles, and that, in this sector, the ground state is non-degenerate. The key feature is to first, investigate the model at specific values of its coupling constant where it reduces to free fermions and, thus, where the logarithmic Bethe equations become explicitly solvable giving rise to an explicit formula for the energies of the eigenstates. The second step consists then in using a continuity argument adjoined with a non-degeneracy of the ground state. Finally, the densification property of the Bethe roots for the ground state can be established by studying the large-L behaviour of solutions to a non-linear integral equation [44] of Destri–deVega type.

Unfortunately, such reasoning cannot, in general, be reproduced for higher rank models as the Bethe equations do not seem to enjoy the presence of a free fermion point. Some arguments

[1] Should the mode have a finitely degenerate ground state, there would still be no problem to apply the present setting, although some of the handlings would become bulkier.

were given in the literature regarding to the fact that the $c \to 0^+$ limit of a multi-species boson gas interacting through two-body Dirac delta functions of strength c has its ground state given by (1.9). It has been conjectured that such a description holds, in fact, for all c [73,80]. It is as well important to keep in mind that there are several examples of higher rank models which are believed *not* to follow our assumptions on the ground state roots (1.9). For instance, it appears [74] that the ground state of a spin chain built out of a mixture of $r-1$ species of fermions and one species of bosons is obtained by solving logarithmic Bethe equations with $m^{(1)}_{a;\Omega} = a$, $a = 1, \dots, N^{(k)}_{\Omega}$ and $N^{(k)}_{\Omega}$ growing to infinity as described above, this for $k = 1, \dots r-1$ but with $N^{(r)}_{\Omega} = 0$. *A priori*, our setting *does not* directly include these models, although we do trust that it can be appropriately modified so as to encompass these cases as well.

• The excited states

Throughout this paper, we shall only focus on the particle–hole excited states, hence waving-off the bound states. Since, *in fine* our aim is to study the asymptotic behaviour of two-point functions, this does not constitute an important limitation. Indeed, the bound states are expected to contribute solely to corrections that are exponentially small in the distance of separation between the operators.

Recall that $N^{(k)}_{\Omega}$ denotes the number of kth species roots building up the ground state. We consider excited states having a finite excitation energy relatively to the ground state, in the thermodynamic limit $L \to +\infty$. They are realised in terms of the Bethe vectors $|\,\Lambda\,\rangle$ built out of $N^{(k)}_{\Lambda}$ roots of species k, $k = 1, \dots, r$, such that

$$N^{(k)}_{\Lambda} \;=\; N^{(k)}_{\Omega} + \kappa^{(k)} \tag{1.12}$$

with $\kappa^{(k)}$ being some integers (depending on the excitation of interest) which are kept *bounded* in L.

In the language of logarithmic Bethe equations, a particle–hole excitation corresponds to a solution to (1.4) associated with an "almost" contiguous distribution of integers, namely:

$$\begin{aligned} m^{(k)}_{a;\Lambda} &= a \qquad \text{for} \qquad a \in [\![\, 1\,;\, N^{(k)}_{\Lambda} \,]\!] \setminus \{h^{(k)}_1, \dots, h^{(k)}_{n^{(k)}}\} \qquad \text{and} \\ m^{(k)}_{h^{(k)}_b;\Lambda} &= p^{(k)}_b \qquad \text{for} \quad b = 1, \dots, n^{(k)}\,. \end{aligned} \tag{1.13}$$

Here

$$\begin{aligned} & h^{(k)}_1 < \cdots < h^{(k)}_{n^{(k)}}, \;\; h^{(k)}_a \in [\![\, 1\,;\, N^{(k)}_{\Lambda} \,]\!] \qquad \text{and} \\ & p^{(k)}_1 < \cdots < p^{(k)}_{n^{(k)}}, \;\; p^{(k)}_a \in \mathcal{I}^{(k)} \setminus [\![\, 1\,;\, N^{(k)}_{\Lambda} \,]\!] \end{aligned} \tag{1.14}$$

are integers labelling, on a microscopic level, the particles and holes building up the excitation. The numbers $n^{(k)}$, with $k = 1, \dots, r$, count the number of particle–hole excitations in the kth species sector.

Owing to the presumed property that the ground state's shifts all vanish $\mathfrak{n}^{(k)}_{\Omega} = 0$, one can express the shifts $\mathfrak{n}^{(k)}_{\Lambda}$ for the excited state solely in terms of the κ's. Indeed, defining the vector

$$\upsilon^{(k)}_{\Lambda} = \sum_{\substack{\ell=1, \neq k \\ \ell:\theta_{k\ell} \neq 0}}^{r} \kappa^{(\ell)} + \kappa^{(k)} \mathbf{1}_{\theta_{kk}=0} \qquad \text{one has} \quad \mathfrak{n}^{(k)}_{\Lambda} \;=\; \begin{cases} 1 & \text{if} \quad \upsilon^{(k)}_{\Lambda} \in 2\mathbb{Z}+1 \\ 0 & \text{if} \quad \upsilon^{(k)}_{\Lambda} \in 2\mathbb{Z} \end{cases} . \tag{1.15}$$

The relationship between the $\kappa^{(k)}$'s and the $\mathfrak{n}_{\Lambda}^{(k)}$'s entails that

$$\boldsymbol{\kappa}\cdot\boldsymbol{\mathfrak{n}} \equiv \sum_{k=1}^{r}\kappa^{(k)}\mathfrak{n}_{\Lambda}^{(k)} \in 2\mathbb{Z}\,. \tag{1.16}$$

Indeed, $\boldsymbol{\kappa}\cdot\boldsymbol{\mathfrak{n}} \in 2\mathbb{Z}$ is equivalent to $\boldsymbol{\kappa}\cdot\boldsymbol{v}_{\Lambda} \in 2\mathbb{Z}$. However, owing to $\theta_{k\ell}(\lambda,\mu)=\theta_{\ell k}(-\mu,-\lambda)$, one can recast the last scalar product as

$$\boldsymbol{\kappa}\cdot\boldsymbol{v}_{\Lambda} = \sum_{\substack{k=1\\ k:\theta_{kk}=0}}^{r}\kappa^{(k)}\big(\kappa^{(k)}+1\big) + 2\sum_{\substack{k>\ell\\ \ell:\theta_{k\ell}\neq 0}}\kappa^{(k)}\kappa^{(\ell)} \ \in 2\mathbb{Z}\,, \tag{1.17}$$

so that the relation (1.16) is fulfilled.

• The energy and momentum

When L is finite, the momentum and energy of an excited state $|\,\Lambda\,\rangle$ (measured relatively to the ground state ones) take the form:

$$\mathcal{P}^{(\mathrm{ex})}_{\Lambda;\Omega} = \mathcal{P}_{\Lambda} - \mathcal{P}_{\Omega} = \sum_{k=1}^{r}\Bigg\{\sum_{a=1}^{N_{\Lambda}^{(k)}}\sigma_0^{(k)}\big(\lambda_a^{(k)}\big) - \sum_{a=1}^{N_{\Omega}^{(k)}}\sigma_0^{(k)}\big(\omega_a^{(k)}\big)\Bigg\} \tag{1.18}$$

$$\mathcal{E}^{(\mathrm{ex})}_{\Lambda;\Omega} = \mathcal{E}_{\Lambda} - \mathcal{E}_{\Omega} = \sum_{k=1}^{r}\Bigg\{\sum_{a=1}^{N_{\Lambda}^{(k)}}\varepsilon_0^{(k)}\big(\lambda_a^{(k)}\big) - \sum_{a=1}^{N_{\Omega}^{(k)}}\varepsilon_0^{(k)}\big(\omega_a^{(k)}\big)\Bigg\}\,. \tag{1.19}$$

We remind that $\Omega=\big\{\{\omega_a^{(k)}\}_{a=1}^{N_{\Omega}^{(k)}}\big\}_{k=1}^{r}$ is the set of Bethe roots associated with the ground state. The function $\sigma_0^{(k)}$, resp. $\varepsilon_0^{(k)}$, corresponds to the bare momentum, resp. bare energy, of an excitation in the kth-species sector. They are both model dependent. Furthermore, the functions $\varepsilon_0^{(k)}$ depend explicitly on the external magnetic fields $h^{(1)},\dots,h^{(r)}$.

• Linear integral equation describing the thermodynamic limit

The observables of the model in thermodynamic limit are characterised in terms of a collection of special functions defined as solutions to auxiliary linear integral equations. We shall present the solutions which will be of interest to our study. However, first, we need to introduce a few notation.

By $\boldsymbol{D}$, $\boldsymbol{\mathfrak{n}}$, $\boldsymbol{\kappa}$, resp. $\boldsymbol{f}(\lambda)$, we shall denote the following r-dimensional vectors, resp. vector-valued functions:

$$\boldsymbol{f}(\omega) = \begin{pmatrix} f^{(1)}(\omega)\\ \vdots\\ f^{(r)}(\omega)\end{pmatrix}\,,\quad \boldsymbol{D} = \begin{pmatrix} D^{(1)}\\ \vdots\\ D^{(r)}\end{pmatrix}\,,\quad \boldsymbol{\mathfrak{n}} = \begin{pmatrix} \mathfrak{n}_{\Lambda}^{(1)}\\ \vdots\\ \mathfrak{n}_{\Lambda}^{(r)}\end{pmatrix} \quad\text{and}\quad \boldsymbol{\kappa} = \begin{pmatrix} \kappa^{(1)}\\ \vdots\\ \kappa^{(r)}\end{pmatrix}\,. \tag{1.20}$$

Furthermore, we introduce an integral operator $\mathbf{K}$ acting on vector valued functions $\boldsymbol{f}(\lambda)$ as

$$\Big[\Big(\mathrm{id}+\mathbf{K}\Big)[\boldsymbol{f}](\omega)\Big]^{(k)} = \boldsymbol{f}^{(k)}(\omega) + \sum_{\ell=1}^{r}\int_{-q^{(\ell)}}^{q^{(\ell)}}\partial_{\mu}\theta_{k\ell}(\omega,\mu)\cdot\boldsymbol{f}^{(\ell)}(\mu)\cdot\mathrm{d}\mu\,. \tag{1.21}$$

Note that the integral kernel of $\mathbf{K}$ is built out of the first order derivatives of the bare phases θ_{ks} arising in (1.4). We shall assume that the operator $\mathrm{id} + \mathbf{K}$ is invertible and denote its inverse by $\mathrm{id} - \mathbf{R}$. This inverse acts on vector valued functions as

$$\Big[\Big(\mathrm{id} - \mathbf{R}\Big)[\boldsymbol{f}](\omega)\Big]^{(k)} = \boldsymbol{f}^{(k)}(\omega) - \sum_{\ell=1}^{r} \int_{-q^{(\ell)}}^{q^{(\ell)}} R_{k\ell}(\omega, \mu) \cdot \boldsymbol{f}^{(\ell)}(\mu) \cdot \mathrm{d}\mu \,. \tag{1.22}$$

The domains of integrations $[-q^{(k)}\,;q^{(k)}]$ correspond to the kth species Fermi zone. The endpoint $q^{(k)}$ of this zone should be chosen in such a way that the configuration of Bethe roots condensing on $[-q^{(k)}\,;q^{(k)}]$ does indeed realise the minimum of the energy. This requirement can be rephrased in terms of the dressed energies $\varepsilon^{(k)}$ which are the components of the vector dressed energy defined as the solution to the linear integral equation:

$$\Big(\mathrm{id} + \mathbf{K}\Big)[\boldsymbol{\varepsilon}](\omega) = \boldsymbol{\varepsilon}_0(\omega) \qquad \text{with} \quad \big[\boldsymbol{\varepsilon}_0(\omega)\big]^{(k)} = \varepsilon_0^{(k)}(\omega) \,. \tag{1.23}$$

The endpoints $q^{(k)}$ are chosen in such a way that the associated dressed energies vanish on their respective Fermi boundaries $\varepsilon^{(k)}\big(q^{(k)}\big) = 0$ and, furthermore, satisfy

$$\begin{aligned} &\varepsilon^{(k)}(\omega) < 0 \quad \text{for} \quad \omega \in \,]-q^{(k)}\,;q^{(k)}[\qquad \text{and} \\ &\varepsilon^{(k)}(\omega) > 0 \quad \text{for} \quad \omega \in \mathbb{R} \setminus [-q^{(k)}\,;q^{(k)}] \,. \end{aligned} \tag{1.24}$$

Thus, it is in fact the equation $\varepsilon^{(k)}\big(q^{(k)}\big) = 0$ that ought to be taken as the definition of $q^{(k)}$ associated with the model's ground state in the presence of the external fields $h^{(k)}$.

The other functions of interest are the vector dressed momentum $\boldsymbol{\sigma}$, the vector dressed phase $\boldsymbol{\Phi}_s$ associated with species s and the dressed charge matrix $\boldsymbol{Z}$. They are defined as the solutions to the linear integral equations:

$$\Big(\mathrm{id} + \mathbf{K}\Big)[\boldsymbol{\sigma}](\omega) = \boldsymbol{\sigma}_0(\omega) + \sum_{s=1}^{r} \frac{D^{(s)}}{2} \Big(\boldsymbol{\Xi}_s\big(\omega, q^{(s)}\big) + \boldsymbol{\Xi}_s\big(\omega, -q^{(s)}\big)\Big) \tag{1.25}$$

$$\Big(\mathrm{id} + \mathbf{K}\Big)[\boldsymbol{\Phi}_s(*, z)](\omega) = \boldsymbol{\Xi}_s(\omega, z) \qquad \text{where} \qquad \Big(\boldsymbol{\Xi}_s(\omega, z)\Big)^{(k)} = \theta_{ks}\big(\omega, z\big) \tag{1.26}$$

$$\Big(\mathrm{id} + \mathbf{K}\Big)[\boldsymbol{Z}](\omega) = I_r \tag{1.27}$$

where I_r is the $r \times r$ identity matrix and the $*$ indicates the argument of the vector function on which the matrix integral operator $\mathrm{id} + \mathbf{K}$ acts. The action of $\mathbf{K}$ on matrix valued functions is defined column-wise. By construction, the vector dressed momentum satisfies $2\sigma^{(k)}\big(q^{(k)}\big) = D^{(k)}$. The symmetry properties (1.7) then imply that σ is an odd vector function $\boldsymbol{\sigma}(\lambda) = -\boldsymbol{\sigma}(-\lambda)$ while $\boldsymbol{Z}$ is an even matrix function $\boldsymbol{Z}(\lambda) = \boldsymbol{Z}(-\lambda)$.

- **The counting function and thermodynamic limit of eigenvalues**

It is convenient to characterise the roots Λ of a particle–hole excited state in terms of its collection of counting functions

$$\widehat{\xi}_{\Lambda}^{(k)}(\omega) = \sigma_0^{(k)}(\omega) + \frac{1}{L} \sum_{\ell=1}^{r} \sum_{b=1}^{N_{\Lambda}^{(k)}} \theta_{k\ell}\big(\omega, \lambda_b^{(\ell)}\big) + \frac{N_{\Lambda}^{(k)} + 1 - \mathfrak{n}_{\Lambda}^{(k)}}{2L} \quad , \quad k = 1, \dots, r \,. \tag{1.28}$$

These counting functions are, by construction, such that, when evaluated at the Bethe roots belonging to Λ, it holds

$$\widehat{\xi}^{(k)}_{\Lambda}\big(\lambda^{(k)}_a\big) = \frac{m^{(k)}_{a;\Lambda}}{L} . \tag{1.29}$$

We shall lay our analysis on the hypothesis that, for L large enough, the counting functions associated with particle–hole excited states are all strictly increasing: $\big(\widehat{\xi}^{(k)}_{\Lambda}\big)'(\omega) > 0$ for $\omega \in \mathbb{R}$. This being taken for granted, it is useful to introduce a set of auxiliary background roots $\big\{\hat{\lambda}^{(k)}_a\big\}$ which are defined as

$$\widehat{\xi}^{(k)}_{\Lambda}\big(\hat{\lambda}^{(k)}_a\big) = \frac{a}{L} \qquad \text{for} \quad a \in \Big[\!\!\Big[-\big[-\lim_{\omega\to-\infty}\widehat{\xi}^{(k)}_{\Lambda}(\omega)\big]\, ; \lim_{\omega\to+\infty}\widehat{\xi}^{(k)}_{\Lambda}(\omega)\Big]\!\!\Big] \tag{1.30}$$

where $[x]$ stands for the integer part of x. Then, the Bethe roots can be recast as

$$\Big\{\lambda^{(k)}_a\Big\}_{a=1}^{N^{(k)}} = \bigg\{\Big\{\hat{\lambda}^{(k)}_a\Big\}_{a=1}^{N^{(k)}} \setminus \Big\{\hat{\lambda}^{(k)}_{h^{(k)}_a}\Big\}_{a=1}^{n^{(k)}}\bigg\} \cup \Big\{\hat{\lambda}^{(k)}_{p^{(k)}_a}\Big\}_{a=1}^{n^{(k)}} . \tag{1.31}$$

The pieces of information gathered above allow one to write down a non-linear integral equation satisfied by the counting function which, in its turn, allows one to access to the large-L asymptotic expansion of the latter [14,16]. We refer to Appendix A.1 for a short derivation of the first few terms of this asymptotic expansion which, in fact, contain all the information that is needed for the study of the thermodynamic limit of the model. The conclusion of Appendix A.1 is that the counting function admits the large-L asymptotic expansion:

$$\begin{aligned}\widehat{\boldsymbol{\xi}}_{\Lambda}(\omega) = \boldsymbol{\sigma}(\omega) + \frac{\boldsymbol{D}}{2} + \sum_{s=1}^{r}\frac{\kappa_s}{L}\boldsymbol{\Phi}_s(\omega,q_s) + \frac{1}{L}\sum_{s=1}^{r}\sum_{a=1}^{n^{(s)}}\Big[\boldsymbol{\Phi}_s\Big(\omega,\mu^{(s)}_{p^{(s)}_a}\Big) - \boldsymbol{\Phi}_s\Big(\omega,\mu^{(s)}_{h^{(s)}_a}\Big)\Big] \\ + \boldsymbol{Z}(\omega)\cdot\frac{\boldsymbol{\kappa}-\mathfrak{n}}{2L} + \mathrm{O}\Big(\frac{1}{L^2}\Big)\end{aligned} \tag{1.32}$$

where the remainder is to be understood entry-wise. Some explanations are in order. The parameters $\mu^{(k)}_a$ are defined as the unique solutions to the equations

$$\sigma^{(k)}\big(\mu^{(k)}_a\big) + \frac{D^{(k)}}{2} = \frac{a}{L} . \tag{1.33}$$

The parameters $\mu^{(k)}_a$ correspond to the leading large-L behaviour of the kth-species Bethe roots: $\hat{\lambda}^{(k)}_a - \mu^{(k)}_a = \mathrm{O}\big(L^{-1}\big)$. In fact, $\mu^{(k)}_{h^{(k)}_a}$, resp. $\mu^{(k)}_{p^{(k)}_a}$ should be thought of as the macroscopic counterparts of the integers $p^{(k)}_a$, resp. $h^{(k)}_a$. It will appear convenient, in the following, to associate, with each excited state, the set of macroscopic particle–hole rapidities that arise in the parametrisation of the state:

$$\mathfrak{R}_{\Lambda} = \bigg\{\Big\{\mu^{(k)}_{h^{(k)}_a}\Big\}_{a=1}^{n^{(k)}} ; \Big\{\mu^{(k)}_{p^{(k)}_a}\Big\}_{a=1}^{n^{(k)}} , k = 1,\dots,r\bigg\} . \tag{1.34}$$

We stress that one of the hypothesis used in the derivation of the large-L expansion of the counting function is that $|D^{(k)} - N^{(k)}_{\Omega}/L| = \mathrm{O}\big(L^{-2}\big)$. The latter is, in fact, a constraint on how the thermodynamic limit of the model ought to be taken. The prescription $|D^{(k)} - N^{(k)}_{\Omega}/L| = \mathrm{O}\big(L^{-2}\big)$ ensures that the model's low-energy spectrum has the structure of a direct sum of conformal field theories. Should one rather consider the general case, then one would get order $\mathrm{O}\big(L^{-1}\big)$

corrections to the ground and excited states energies that are oscillatory in L, *cf.* the discussion in [78,79].

It follows from (1.32) that the Bethe roots belonging to the kth species condensate, in the thermodynamic limit, on the interval $[-q^{(k)}; q^{(k)}]$ with a density $\rho^{(k)} = \big(\sigma^{(k)}\big)'$. However, the distribution of roots possesses a small number of microscopic gaps at the rapidities $\hat{\lambda}^{(k)}_{h_a^{(k)}}$ of the holes. There are also additional roots at the rapidities $\hat{\lambda}^{(k)}_{p_a^{(k)}}$ of the particles.

The precise control on the large-L behaviour of the counting function associated with a particle–hole excited state Λ allows one to characterise the excitation energies and momenta. In the large-L limit, these are expressed in terms of the dressed energies $\varepsilon^{(k)}$ (1.23) and momenta $\sigma^{(k)}$ (1.25) of the particles and holes as

$$\mathcal{P}^{(\mathrm{ex})}_{\Lambda;\Omega} = \sum_{k=1}^{r}\bigg\{\mathfrak{n}^{(k)}_{\Lambda}\sigma^{(k)}\big(q^{(k)}\big) + \sum_{a=1}^{n^{(k)}}\Big[p^{(k)}\Big(\mu^{(k)}_{p_a^{(k)}}\Big) - \sigma^{(k)}\Big(\mu^{(k)}_{h_a^{(k)}}\Big)\Big]\bigg\} + \mathrm{O}\Big(\frac{1}{L}\Big) \tag{1.35}$$

$$\mathcal{E}^{(\mathrm{ex})}_{\Lambda;\Omega} = \sum_{k=1}^{r}\sum_{a=1}^{n^{(k)}}\Big[\varepsilon^{(k)}\Big(\mu^{(k)}_{p_a^{(k)}}\Big) - \varepsilon^{(k)}\Big(\mu^{(k)}_{h_a^{(k)}}\Big)\Big] + \mathrm{O}\Big(\frac{1}{L}\Big)\,. \tag{1.36}$$

We refer to Appendix A.3 for more detail on the derivation of these formulae.

- **The ℓ-critical classes**

Among all excited states, one specific class of states singles out: the one corresponding to excited states whose excitation energy $\mathcal{E}^{(\mathrm{ex})}_{\Lambda;\Omega}$ vanishes in the thermodynamic limit. By virtue of the sign properties of the dressed energy (1.24) and of the large-L behaviour of the excitation energy (1.36), this can only happen if the particle–hole rapidities of the kth species collapse, when $L \to +\infty$, on the Fermi boundaries $\pm q^{(k)}$. Such a combination of collapsing rapidities is obtained from any collection Λ of Bethe roots whose associated particle–hole integers can be put in the form

$$\Big\{h_a^{(k)}\Big\}_1^{n^{(k)}} = \Big\{N_\Omega^{(k)} + 1 + \kappa^{(k)} - h_a^{(k;+)}\Big\}_1^{n^{(k)}_{h;+}} \cup \Big\{h_a^{(k;-)}\Big\}_1^{n^{(k)}_{h;-}} \tag{1.37}$$

and

$$\Big\{p_a^{(k)}\Big\}_1^{n^{(k)}} = \Big\{N_\Omega^{(k)} + \kappa^{(k)} + p_a^{(k;+)}\Big\}_1^{n^{(k)}_{p;+}} \cup \Big\{1 - p_a^{(k;-)}\Big\}_1^{n^{(k)}_{p;-}}\,, \tag{1.38}$$

with $k = 1, \dots, r$. The re-centred particle $p_a^{(k;\pm)}$ and hole $h_a^{(k;\pm)}$ integers are assumed to be such that

$$\lim_{L\to+\infty}\bigg\{\frac{1}{L}\cdot\sum_{a=1}^{n^{(k)}_{p;\pm}} p_a^{(k;\pm)}\bigg\} = \lim_{L\to+\infty}\bigg\{\frac{1}{L}\cdot\sum_{a=1}^{n^{(k)}_{h;\pm}} h_a^{(k;\pm)}\bigg\} = 0\,. \tag{1.39}$$

The integers $n^{(k)}_{p/h;\pm}$ correspond to the number of particle $n^{(k)}_{p;\pm}$ and hole $n^{(k)}_{h;\pm}$ excitations in the kth species sector that collapse on the right $(+)$ and left $(-)$ boundaries of the kth species Fermi zone. These numbers are such that

$$n^{(k)}_{h;+} + n^{(k)}_{h;-} = n^{(k)} = n^{(k)}_{p;+} + n^{(k)}_{p;-}\,. \tag{1.40}$$

States whose particle/hole integers are of the form (1.37)–(1.38) will be called critical. One can gather the critical states into sub-classes depending on the value taken by the thermodynamic limit of the excitation momentum attached to these states. Introducing the integer

$$\ell^{(k)} = n^{(k)}_{p;+} - n^{(k)}_{h;+} = n^{(k)}_{h;-} - n^{(k)}_{p;-} \tag{1.41}$$

one finds that the excitation momentum takes the form

$$\mathcal{P}^{(\mathrm{ex})}_{\Lambda;\Omega} = \sum_{k=1}^{r}\Bigg[\big(2\ell^{(k)} + \mathfrak{n}^{(k)}_{\Lambda}\big)\sigma^{(k)}_{F} + \frac{2\pi}{L}\Bigg\{\sum_{a=1}^{n^{(k)}_{p;+}}\big(p^{(k)}_{a;+} - 1\big) + \sum_{a=1}^{n^{(k)}_{h;+}} h^{(k)}_{a;+}\Bigg\}$$
$$- \frac{2\pi}{L}\Bigg\{\sum_{a=1}^{n^{(k)}_{p;-}}\big(p^{(k)}_{a;-} - 1\big) + \sum_{a=1}^{n^{(k)}_{h;-}} h^{(k)}_{a;-}\Bigg\}\Bigg] + \cdots \tag{1.42}$$

In this expansion, $\sigma^{(k)}_{F} = \sigma^{(k)}\big(q^{(k)}\big)$ stands for the Fermi momentum associated with the kth species. The dots ... refer to terms depending on the $p^{(k)}_{a;\pm}, h^{(k)}_{a;\pm}$ but preceded by a L^{-2} prefactor, or to terms of the order $1/L$ but which are *independent* of the re-centred particle–hole integers.

Since excited states belonging to $\boldsymbol{\ell}$-classes are such that all of their associated particle–hole rapidities collapse on the respective left or right Fermi boundaries, the thermodynamic limit of the set (1.34) of macroscopic rapidities $\mathfrak{R}_{\Lambda}$ attached to any such states actually reduces to

$$\mathfrak{R}_{\Lambda} \hookrightarrow \Bigg\{\Big\{q^{(k)}\Big\}_{1}^{n^{(k)}_{p;+}} \bigcup \Big\{-q^{(k)}\Big\}_{1}^{n^{(k)}_{p;-}} \; ; \; \Big\{q^{(k)}\Big\}_{1}^{n^{(k)}_{h;+}} \bigcup \Big\{-q^{(k)}\Big\}_{1}^{n^{(k)}_{h;-}} \; , \; k = 1, \dots, r\Bigg\} . \tag{1.43}$$

Regarding the kth species, there will be $|\ell^{(k)}|$ particles collapsing at $q^{(k)}\mathrm{sgn}\big(\ell^{(k)}\big)$ and $|\ell^{(k)}|$ holes collapsing at $-q^{(k)}\mathrm{sgn}\big(\ell^{(k)}\big)$ plus a certain amount of particle–hole excitations, equal in number, on each of the Fermi boundaries. Most, if not all, pertinent observables associated with the model do not actually “see” the effects of the “swarm” of particle–holes excitations, equal in number, attached to each of the boundaries of the Fermi zones but solely keep track of the number $\ell^{(k)}$ of particle–hole excitations on the right endpoints of the kth-species Fermi zone.

A good example of such a mechanism is the reduction occurring in the thermodynamic limit $\boldsymbol{F}_{\Lambda,\Omega}$ of the finite volume $\widehat{\boldsymbol{F}}_{\Lambda,\Omega}$ vector shift function of the state Λ relatively to the ground state Ω defined as:

$$\widehat{\boldsymbol{F}}_{\Lambda,\Omega}(\omega) = L\cdot\Big(\widehat{\boldsymbol{\xi}}_{\Omega}(\omega) - \widehat{\boldsymbol{\xi}}_{\Lambda}(\omega)\Big) \qquad \text{and} \qquad \boldsymbol{F}_{\Lambda,\Omega}(\omega) = \lim_{L\to+\infty}\Big\{\widehat{\boldsymbol{F}}_{\Lambda,\Omega}(\omega)\Big\} . \tag{1.44}$$

Indeed, when focusing on an excited state belonging to the $\boldsymbol{\ell}$-critical class, one has $\boldsymbol{F}_{\Lambda,\Omega} \hookrightarrow \boldsymbol{F}_{\boldsymbol{\ell};\boldsymbol{\kappa}}$ where

$$\boldsymbol{F}_{\boldsymbol{\ell};\boldsymbol{\kappa}}(\omega) = -\boldsymbol{Z}(\omega)\cdot\frac{\boldsymbol{\kappa} - \boldsymbol{\mathfrak{n}}}{2} + \sum_{s=1}^{r}\Big\{\ell^{(s)}\boldsymbol{\Phi}_{s}\Big(\omega, -q^{(s)}\Big) - \big(\ell^{(s)} + \kappa^{(s)}\big)\boldsymbol{\Phi}_{s}\Big(\omega, q^{(s)}\Big)\Big\} . \tag{1.45}$$

It appears convenient to introduce the dressed phase matrix by $\Phi_{ks}(\lambda, \mu) = \boldsymbol{\Phi}^{(k)}_{s}(\lambda, \mu)$. Straightforward handlings of the matrix linear integral equations allow one to express the dressed charge matrix in terms of the dressed phase matrix as

$$\boldsymbol{Z}_{ks}(\omega) = \delta_{ks} + \Phi_{ks}\big(\omega, -q^{(s)}\big) - \Phi_{ks}\big(\omega, q^{(s)}\big) . \tag{1.46}$$

A slightly less obvious identity is established in Appendix A.2 and relates to a closed expression for the inverse of the matrix $\mathcal{Z}_{sk} = Z_{ks}\big(q^{(k)}\big)$:

$$\big[\mathcal{Z}^{-1}\big]_{ks} = \delta_{ks} - \Phi_{ks}\big(q^{(k)}, -q^{(s)}\big) - \Phi_{ks}\big(q^{(k)}, q^{(s)}\big) . \tag{1.47}$$

The above two identities (1.46) and (1.47) allow one to recast the specific combinations of the shift function

$$\delta^{(k);+}_{\boldsymbol{\ell};\boldsymbol{\kappa}} = F^{(k)}_{\boldsymbol{\ell};\boldsymbol{\kappa}}\big(q^{(k)}\big) + \ell^{(k)} + \kappa^{(k)} \qquad \text{and} \qquad \delta^{(k);-}_{\boldsymbol{\ell};\boldsymbol{\kappa}} = F^{(k)}_{\boldsymbol{\ell};\boldsymbol{\kappa}}\big(-q^{(k)}\big) + \ell^{(k)} , \tag{1.48}$$

or in compact form

$$\delta^{(k);\pm}_{\boldsymbol{\ell};\boldsymbol{\kappa}} = \big[\mathcal{Z}^{\mathrm{t}} \cdot \big(\boldsymbol{\ell} + \frac{\mathfrak{n}}{2}\big)\big]^{(k)} \pm \frac{1}{2}\big[\mathcal{Z}^{-1} \cdot \boldsymbol{\kappa}\big]^{(k)} . \tag{1.49}$$

We remind that $\mathcal{Z}_{sk} = \boldsymbol{Z}_{ks}\big(q^{(k)}\big)$ and that $^{\mathrm{t}}$ stands for the matrix transposition.

It will appear useful in the following to introduce the so-called fundamental representative of an $\boldsymbol{\ell}$-critical class. It is an excited state whose particle–hole integers are packed as tightly as possible. More precisely, given $\boldsymbol{\ell}$, such a state corresponds to the configuration

$$\begin{array}{lcl} \Big\{\{p^{(k)}_{a;+} = a\}_1^{\ell^{(k)}}\,;\, \{\emptyset\}\Big\} \bigcup \Big\{\{\emptyset\}\,;\, \{h^{(k)}_{a;-} = a\}_1^{\ell^{(k)}}\Big\} & \text{if} & \ell^{(k)} \geq 0 \\ \Big\{\{\emptyset\}\,;\, \{h^{(k)}_{a;+} = a\}_1^{-\ell^{(k)}}\Big\} \bigcup \Big\{\{p^{(k)}_{a;-} = a\}_1^{-\ell^{(k)}}\,;\, \{\emptyset\}\Big\} & \text{if} & \ell^{(k)} < 0 \end{array} . \tag{1.50}$$

1.2. The form factors of local operators

We are now finally in position to discuss the structure of the form factors of local operators. We are going to state a conjecture relative to their universal form in nested Bethe Ansatz solvable models. As we shall demonstrate in Section 2, this universal form is responsible for the universality of the critical behaviour of the correlation functions in these models.

The conjecture has been demonstrated to hold for numerous rank 1 models, *cf.* [34,36]. In Section 3, we shall demonstrate that it holds as well for the $SU(3)$-invariant XXX magnet. The proof heavily builds on the results of [8,60] which provide determinant based representations for the form factors of local operators in this model. We do however trust that the conjecture does hold for all models solvable by the nested Bethe Ansatz.

• The main conjecture

Conjecture 1.1. *Let $\mathsf{o}_1^{(\alpha)}$ be a local operator acting on the first site of the model and such that it connects $\boldsymbol{N}$-particle states solely with $\boldsymbol{N} + \boldsymbol{\kappa}_\alpha$ ones, with $\boldsymbol{\kappa}_\alpha^t = \big(\kappa_\alpha^{(1)}, \dots, \kappa_\alpha^{(r)}\big)$. Then the matrix elements of $\mathsf{o}_1^{(\alpha)}$ taken between two Bethe vectors $|\,\Lambda\,\rangle$ and $|\,\Upsilon\,\rangle$ take the form*

$$\left| \frac{\langle\,\Upsilon\,|\mathsf{o}_1^{(\alpha)}|\,\Lambda\,\rangle}{\|\Lambda\| \cdot \|\Upsilon\|} \right|^2 = \widehat{\mathcal{A}}_{\mathsf{o}^{(\alpha)}}(\Upsilon;\Lambda) \cdot \prod_{k=1}^{r} \Big\{ \mathcal{D}\Big(\{\nu_a^{(k)}\}_{a=1}^{N_\Upsilon^{(k)}} \mid \{\lambda_a^{(k)}\}_{a=1}^{N_\Lambda^{(k)}}\Big)\Big[\widehat{\xi}_\Upsilon^{(k)}, \widehat{\xi}_\Lambda^{(k)}\Big]\Big\}$$

$$\textit{where} \qquad \left\{ \begin{array}{l} \Lambda = \Big\{\lambda_a^{(k)}\Big\}_{a=1}^{N_\Lambda^{(k)}} \\ \Upsilon = \Big\{\nu_a^{(k)}\Big\}_{a=1}^{N_\Upsilon^{(k)}} \end{array} \right. \tag{1.51}$$

and the vector integers $\boldsymbol{N}_{\Upsilon}$ *and* $\boldsymbol{N}_{\Lambda}$ *satisfy*

$$N_{\Upsilon}^{(k)} = N_{\Lambda}^{(k)} + \kappa_{\alpha}^{(k)} . \tag{1.52}$$

The decomposition of the form factor is split in two parts: on the one hand the so-called "smooth" part $\widehat{\mathcal{A}}_{\mathsf{O}^{(\alpha)}}$ *and, on the other hand, the product of the so-called discrete parts associated with the kth species of roots.*

- *The factor* $\widehat{\mathcal{A}}_{\mathsf{O}^{(\alpha)}}\big(\Lambda;\Upsilon\big)$ *represents the "smooth" part of the form factor in the sense that its thermodynamic limit is solely described in terms of the thermodynamic limit of the rapidities of the multi-species particles and holes that build up the excited state:*

$$\widehat{\mathcal{A}}_{\mathsf{O}^{(\alpha)}}\big(\Upsilon;\Lambda\big) = \mathcal{A}_{\mathsf{O}^{(\alpha)}}\Big(\mathfrak{R}_{\Upsilon}\,;\,\mathfrak{R}_{\Lambda}\Big)\cdot\Big(1+\mathrm{O}\Big(\frac{1}{L}\Big)\Big) . \tag{1.53}$$

Furthermore, the dependence on the particle–hole rapidities contained in the sets $\mathfrak{R}_{\Upsilon}$ *and* $\mathfrak{R}_{\Lambda}$ *is smooth. The set function* $\mathcal{A}_{\mathsf{O}^{(\alpha)}}$ *enjoys, furthermore, particle–hole reduction properties, namely it holds*

$$\mathcal{A}_{\mathsf{O}^{(\alpha)}}\Big(\mathfrak{R}_{\Upsilon}\,;\,\mathfrak{R}_{\Lambda}\Big)_{|\mu^{(k)}_{h_a^{(k)}}=\mu^{(k)}_{p_b^{(k)}}} = \mathcal{A}_{\mathsf{O}^{(\alpha)}}\Big(\check{\mathfrak{R}}_{\Upsilon}\,;\,\mathfrak{R}_{\Lambda}\Big) , \tag{1.54}$$

where $\check{\mathfrak{R}}_{\Lambda}$ *is the set obtained from* $\mathfrak{R}_{\Lambda}$ *by deleting from the set of kth species particle–hole rapidities the two rapidities:* $\mu^{(k)}_{h_a^{(k)}}$ *and* $\mu^{(k)}_{p_b^{(k)}}$.

- *The factor* $\mathcal{D}$ *represents the universal part of the form factor. It is operator independent in the sense that it only depends on the two-collections of roots* Λ,Υ *parameterising the states connected by the operator. It represents the "discrete" part of the form factor in the sense that its large-L behaviour not only depends on the macroscopic momenta* $\mathfrak{R}_{\Lambda}$ *and* $\mathfrak{R}_{\Upsilon}$ *associated with the two states but also has an explicit dependence on microscopic data of the excited state, namely the particle–holes integers.* $\mathcal{D}$ *contains all the universal part of the structure of a form factor and reads*

$$\mathcal{D}\Big(\{\nu_a\}_{a=1}^{N_\nu}\mid\{\lambda_a\}_{a=1}^{N_\lambda}\Big)\Big[\widehat{\xi}_{\mu},\widehat{\xi}_{\lambda}\Big] = \prod_{a=1}^{N_\lambda}\left\{\frac{\sin^2\big[\pi\widehat{F}_{\nu;\lambda}(\lambda_a)\big]}{\pi L\widehat{\xi}_{\lambda}'(\lambda_a)}\right\}\cdot\prod_{a=1}^{N_\nu}\left\{\frac{1}{\pi L\widehat{\xi}_{\nu}'(\nu_a)}\right\} \cdot\frac{\prod\limits_{a<b}^{N_\nu}(\nu_a-\nu_b)^2\cdot\prod\limits_{a<b}^{N_\lambda}(\lambda_a-\lambda_b)^2}{\prod\limits_{a=1}^{N_\nu}\prod\limits_{b=1}^{N_\lambda}(\nu_a-\lambda_b)^2} \tag{1.55}$$

where we made use of the shorthand notation

$$\widehat{F}_{\nu;\lambda} = L\cdot\big(\widehat{\xi}_{\lambda}-\widehat{\xi}_{\nu}\big) . \tag{1.56}$$

Here, $\widehat{\xi}_{\nu}$ *(resp.* $\widehat{\xi}_{\lambda}$*) are the counting functions associated with the sets of parameters* ν *(resp.* λ*).*

Several remarks are in order. The smooth part is definitely non-generic. It strongly varies from one model to another and also depends non trivially on the operator. This can be explicitly seen on some rank one models where the smooth/discrete part decomposition of the form factors has been obtained. Also the dependence of the smooth part on the operator is manifest on the level of the expressions found in Section 3 for the form factors of the local operators[2] $\mathbf{E}_1^{22}$, $\mathbf{E}_1^{21}$ and $\mathbf{E}_1^{23}$ associated with the $SU(3)$ invariant XXX magnet. However, for the analysis of the large-distance asymptotic behaviour of the correlation functions, only the very broad properties of the smooth part matter, namely the reduction property (1.53).

The full discrete part is realised as a product over the discrete parts associated with each species k. Such a product decomposition means that the species do not interact on the level of the discrete parts. The discrete part attached to the kth species is a function of the kth-component $\widehat{\boldsymbol{F}}^{(k)}_{\Upsilon,\Lambda}$ of the vector shift function of the state Υ relatively to the state Λ, *cf.* (1.44) and (1.55)–(1.56). In fact, to the leading order in L, the large-L behaviour of the discrete part solely depends on the thermodynamic limit of the latter

$$\boldsymbol{F}_{\Upsilon,\Lambda}(\omega) = \lim_{L\to+\infty}\Big\{\widehat{\boldsymbol{F}}_{\Upsilon,\Lambda}(\omega)\Big\}\,. \tag{1.57}$$

Since it will be of no use for the analysis that will follow, we shall *not* discuss here the large-L asymptotic behaviour of the discrete part in the case when the "in" and "out" states involved in the form factor correspond to general particle–hole excitations. There is no problem to obtain the large-L behaviour even in such a general setting and we refer the interested reader to [36] for the corresponding formulae. However, we will now present the large-L behaviour when one of the states is given by the ground state while the other one corresponds to a particle–hole excitation belonging to an $\boldsymbol{\ell}$-class.

• Large-L behaviour for $\boldsymbol{\ell}$-critical states

Let Υ be a particle–hole excited state belonging to the $\boldsymbol{\ell}$-critical class as described previously and let Ω stands for the ground state Bethe roots. Then the discrete part associated with the kth-species admits the large-L behaviour

$$\begin{aligned}
&\mathcal{D}\Big(\{\nu_a^{(k)}\}_{a=1}^{N_\Upsilon^{(k)}}\mid\{\omega_a^{(k)}\}_{a=1}^{N_\Omega^{(k)}}\Big)\Big[\widehat{\xi}_\Upsilon^{(k)},\widehat{\xi}_\Omega^{(k)}\Big]\\
&\quad\sim\frac{\mathcal{D}^{(k)}_{\boldsymbol{\ell};\boldsymbol{\kappa}}}{L^{\Delta^{(k);+}_{\boldsymbol{\ell};\boldsymbol{\kappa}}+\Delta^{(k);-}_{\boldsymbol{\ell};\boldsymbol{\kappa}}}}\cdot\frac{G^2\big(1+\delta^{(k);+}_{\boldsymbol{\ell};\boldsymbol{\kappa}}-\ell^{(k)}\big)\,G^2\big(1-\delta^{(k);-}_{\boldsymbol{\ell};\boldsymbol{\kappa}}+\ell^{(k)}\big)}{G^2\big(1+\delta^{(k);+}_{\boldsymbol{\ell};\boldsymbol{\kappa}}\big)\,G^2\big(1-\delta^{(k);-}_{\boldsymbol{\ell};\boldsymbol{\kappa}}\big)}\\
&\qquad\times\mathcal{R}_{n^{(k)}_{p;+};n^{(k)}_{h;+}}\bigg(\big\{p_a^{(k;+)}\big\}_1^{n^{(k)}_{p;+}}\,;\,\big\{h_a^{(k;+)}\big\}_1^{n^{(k)}_{h;+}}\mid\delta^{(k);+}_{\boldsymbol{\ell};\boldsymbol{\kappa}}-\ell^{(k)}\bigg)\\
&\qquad\cdot\mathcal{R}_{n^{(k)}_{p;-};n^{(k)}_{h;-}}\bigg(\big\{p_a^{(k;-)}\big\}_1^{n^{(k)}_{p;-}}\,;\,\big\{h_a^{(k;-)}\big\}_1^{n^{(k)}_{h;-}}\mid-\delta^{(k);-}_{\boldsymbol{\ell};\boldsymbol{\kappa}}+\ell^{(k)}\bigg)\,.
\end{aligned}\tag{1.58}$$

There are several ingredients in these asymptotics:

- $\boldsymbol{\kappa}$ stands for the vector of kth-species roots number discrepancies between the out and in states

$$\kappa^{(k)} = N_\Upsilon^{(k)} - N_\Omega^{(k)}\,.$$

[2] Here $\mathbf{E}^{ij}$ are the elementary 3×3 matrices, *viz.* $(\mathbf{E}^{ij})_{k\ell}=\delta_{ik}\delta_{j\ell}$.

- $\mathcal{D}^{(k)}_{\boldsymbol{\ell};\boldsymbol{\kappa}}$ is a numerical prefactor that *only* depends on the vector integers $\boldsymbol{\ell}$ labelling the critical class of interest and on the vector integers $\boldsymbol{\kappa}$ characterising the pseudo-particle changing nature of the operator.
- $\mathcal{R}_{n_p,n_h}$ represents the microscopic contribution of the swarm of particles and holes living on the left or right Fermi boundary. It is expressed as

$$\mathcal{R}_{n_p;n_h}\Big(\{p_a\}_1^{n_p}\ ;\ \{h_a\}_1^{n_h}\mid\delta\Big)=\bigg(\frac{\sin[\pi\delta]}{\pi}\bigg)^{2n_h}\cdot\frac{\prod\limits_{a<b}^{n_p}(p_a-p_b)^2\cdot\prod\limits_{a<b}^{n_h}(h_a-h_b)^2}{\prod\limits_{a=1}^{n_p}\prod\limits_{b=1}^{n_h}(p_a+h_b-1)^2}$$
$$\times\prod_{a=1}^{n_p}\bigg\{\frac{\Gamma^2(p_a+\delta)}{\Gamma^2(p_a)}\bigg\}\cdot\prod_{a=1}^{n_h}\bigg\{\frac{\Gamma^2(h_a-\delta)}{\Gamma^2(h_a)}\bigg\}\,.$$

- G stands for the Barnes function. It is a normalisation constant, chosen so that it counterbalances the contribution of the left/right factors $\mathcal{R}_{n_p,n_h}$ in the case when the particle–holes on that boundary are chosen to be the fundamental representatives of the $\boldsymbol{\ell}$-class, *cf.* (1.50).
- The kth species discrete part decays algebraically in the volume with a critical exponent $\Delta^{(k);-}_{\boldsymbol{\ell};\boldsymbol{\kappa}}+\Delta^{(k);+}_{\boldsymbol{\ell};\boldsymbol{\kappa}}$ that is generically non-rational. The collection of the exponents $\Delta^{(k);\pm}_{\boldsymbol{\ell};\boldsymbol{\kappa}}$ provides one with the scaling dimensions associated with the specific operator. These exponents are given as squares

$$\Delta^{(k);\pm}_{\boldsymbol{\ell};\boldsymbol{\kappa}}=\Big(\delta^{(k);\pm}_{\boldsymbol{\ell};\boldsymbol{\kappa}}\Big)^2 \tag{1.59}$$

of the specific combinations (1.48) of the shift function $F^{(k)}_{\boldsymbol{\ell};\boldsymbol{\kappa}}$ (1.45), taken at the right (+) or left (−) Fermi boundaries.

Furthermore, when restricted to the $\boldsymbol{\ell}$-critical class, due to the reduction properties (1.53), the smooth part goes to a constant solely depending on the integers $\boldsymbol{\ell}$ and $\boldsymbol{\kappa}_\alpha$ and on the operator $\mathbf{O}^{(\alpha)}$:

$$\widehat{\mathcal{A}}_{\mathbf{O}^{(\alpha)}}\big(\Upsilon;\Omega\big)\simeq\mathcal{A}_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha}\big(\mathbf{O}^{(\alpha)}\big)\,. \tag{1.60}$$

For further convenience, it is useful to absorb all the constants in a unique term

$$\big|\mathcal{F}_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha}\big(\mathbf{O}^{(\alpha)}\big)\big|^2=\mathcal{A}_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha}\big(\mathbf{O}^{(\alpha)}\big)\cdot\prod_{k=1}^{s}\Big\{\mathcal{D}^{(k)}_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha}\Big\}\,. \tag{1.61}$$

2. Large-distance asymptotic behaviour of two-point functions and applications

2.1. The large-distance asymptotics in the general setting

In this section we argue that provided the setting of the previous section holds, the zero-temperature two-point functions exhibit the large-distance asymptotics

$$\frac{\langle\,\Omega\,|\big[\mathbf{O}^{(\alpha)}_{m+1}\big]^\dagger\cdot\mathbf{O}^{(\alpha)}_1|\,\Omega\,\rangle}{\big\langle\,\Omega\,|\,\Omega\big\rangle}\simeq\sum_{\boldsymbol{\ell}\in\mathbb{Z}^r}\mathrm{e}^{\mathrm{i}m(2\boldsymbol{\ell}+\mathfrak{n}_\alpha)\cdot\boldsymbol{\sigma}_F}\cdot\frac{(-1)^{\frac{m}{2}(\boldsymbol{\kappa},2\boldsymbol{\ell}+\mathfrak{n}_\alpha)}}{\big(2\pi m\big)^{\Delta_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha}}}\cdot\big|\mathcal{F}_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha}\big(\mathbf{O}^{(\alpha)}\big)\big|^2\cdot\big(1+\mathrm{o}(1)\big)\,. \tag{2.1}$$

In the above expansion, the vector $\mathfrak{n}_\alpha$ is defined in terms of $\boldsymbol{\kappa}_\alpha$ as is (1.15) while

$$2\Delta_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha} = \Big(2\boldsymbol{\ell}+\mathfrak{n}_\alpha, \mathcal{Z}\mathcal{Z}^{\mathrm{t}}\cdot(2\boldsymbol{\ell}+\mathfrak{n}_\alpha)\Big) + \Big(\boldsymbol{\kappa}_\alpha, \big[\mathcal{Z}\mathcal{Z}^{\mathrm{t}}\big]^{-1}\boldsymbol{\kappa}_\alpha\Big). \tag{2.2}$$

Note that the asymptotics are indeed real valued since (1.16) ensures that $(\boldsymbol{\kappa},\mathfrak{n}_\alpha)\in 2\mathbb{Z}$ while the form factors $\big|\mathcal{F}_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha}\big(\mathsf{o}^{(\alpha)}\big)\big|^2$ and $\big|\mathcal{F}_{-\boldsymbol{\ell}-\mathfrak{n}_\alpha;\boldsymbol{\kappa}_\alpha}\big(\mathsf{o}^{(\alpha)}\big)\big|^2$ coincide owing to the symmetries of the Bethe equations. The asymptotic expansion (2.1) provides us with the leading large-m asymptotic behaviour of each of the oscillating with the distance harmonics present in the large-m expansion of the two-point function.

In order to derive the result, we follow the strategy of [35]. We start by writing down the form factor expansion of a two-point function:

$$\frac{\langle\,\Omega\,|\big[\mathsf{o}^{(\alpha)}_{m+1}\big]^{\dagger}\cdot\mathsf{o}^{(\alpha)}_{1}|\,\Omega\,\rangle}{\langle\,\Omega\,|\,\Omega\rangle} = \sum_{\{\Upsilon\}} \mathrm{e}^{\mathrm{i}m\mathcal{P}^{(\mathrm{ex})}_{\Upsilon;\Omega}}\cdot\left|\frac{\langle\,\Upsilon\,|\mathsf{o}^{(\alpha)}_{1}|\,\Omega\,\rangle}{\|\Upsilon\|\cdot\|\Omega\|}\right|^2 . \tag{2.3}$$

The sum runs over all the eigenstates of the model characterised by $N^{(k)}_{\Upsilon}=N^{(k)}_{\Omega}+\kappa^{(\alpha)}$. Owing to (1.15), the integer shift $\mathfrak{n}_\Upsilon=\mathfrak{n}_\alpha$ will be constant for all such excited states. In the following, we shall restrict the summation to the part of the spectrum realised in terms of particle–hole excitations. It is expected that all other types of excited states (*viz.* the bound states) only generate exponentially small contributions to the large-distance asymptotics of the correlator. This fact is rather well supported by calculations carried out on the two-point functions in integrable models subordinated to Lie algebras of rank 1 (for instance the XXZ spin $1/2$ chain). In the long-time and large-distance regime, this property quite probably breaks down. In that case, in addition to the terms that will follows from the present analysis, the contributions of the bound states should be added. These, however, go beyond the scope of the present analysis.

By analogy with the case of rank 1 models, we assume that in the large-m regime the sums over particle–hole excitations will localise around the edges of their respective Fermi zones. In other words, the leading contribution to the large-distance asymptotics will issue from excitations belonging to the $\boldsymbol{\ell}$-classes. We also assume that it is enough to take into account solely the leading large-L behaviour of the involved form factor. This leads to

$$\begin{aligned}
&\frac{\langle\,\Omega\,|\big[\mathsf{o}^{(\alpha)}_{m+1}\big]^{\dagger}\cdot\mathsf{o}^{(\alpha)}_{1}|\,\Omega\,\rangle}{\langle\,\Omega\,|\,\Omega\rangle} \\
&\simeq \prod_{k=1}^{r}\sum_{\ell^{(k)}\in\mathbb{Z}}\prod_{\epsilon_k=\pm}\sum_{\substack{n^{(k)}_{p;\epsilon_k},n^{(k)}_{h;\epsilon_k}\\ n^{(k)}_{p;\epsilon_k}-n^{(k)}_{h;\epsilon_k}=\epsilon_k\ell_k}}\;\sum_{p_1^{(k;\epsilon_k)}<\dots<p^{(k;\epsilon_k)}_{n^{(k)}_{p;\epsilon_k}}}\;\sum_{h_1^{(k;\epsilon_k)}<\dots<h^{(k;\epsilon_k)}_{n^{(k)}_{h;\epsilon_k}}} \mathrm{e}^{\mathrm{i}(2\boldsymbol{\ell}+\mathfrak{n}_\alpha)\cdot\boldsymbol{\sigma}_F m} \\
&\times\frac{\big|\mathcal{F}_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha}\big(\mathsf{o}^{(\alpha)}\big)\big|^2}{\prod_{k=1}^{r}\Big\{L^{\Delta^{(k);+}_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha}+\Delta^{(k);-}_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha}}\Big\}}\cdot\prod_{k=1}^{r}\prod_{\epsilon_k=\pm}\Bigg\{\prod_{a=1}^{n_p^{(k;\epsilon_k)}}\mathrm{e}^{\epsilon_k 2\mathrm{i}\pi\frac{m}{L}(p_a^{(k;\epsilon_k)}-1)}\prod_{a=1}^{n_h^{(k;\epsilon_k)}}\mathrm{e}^{\epsilon_k 2\mathrm{i}\pi\frac{m}{L}h_a^{(k;\epsilon_k)}}\Bigg\} \\
&\times\prod_{k=1}^{r}\prod_{\epsilon=\pm}\Bigg\{\frac{G^2\Big(1+\epsilon\delta^{(k);\epsilon}_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha}-\epsilon\ell^{(k)}\Big)}{G^2\Big(1+\epsilon\delta^{(k);\epsilon}_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha}\Big)} \\
&\cdot\mathcal{R}_{n^{(k)}_{p;\epsilon};n^{(k)}_{h;\epsilon}}\Big(\big\{p_a^{(k;\epsilon)}\big\}_1^{n^{(k)}_{p;\epsilon}}\,;\,\big\{h_a^{(k;\epsilon)}\big\}_1^{n^{(k)}_{h;\epsilon}}\mid\epsilon\delta^{(k);\epsilon}_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha}-\epsilon\ell^{(k)}\Big)\Bigg\}.
\end{aligned} \tag{2.4}$$

After reorganising the sums, the large-m behaviour of the two-point function takes the form

$$\frac{\langle \Omega | [\mathbf{o}_{m+1}^{(\alpha)}]^{\dagger} \cdot \mathbf{o}_{1}^{(\alpha)} | \Omega \rangle}{\langle \Omega | \Omega \rangle} \simeq \sum_{\boldsymbol{\ell} \in \mathbb{Z}^r} \mathrm{e}^{\mathrm{i}\boldsymbol{\sigma}_F \cdot (2\boldsymbol{\ell} + \mathfrak{n}_\alpha) m} \cdot \big| \mathcal{F}_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha}(\mathbf{o}^{(\alpha)}) \big|^2 \cdot \prod_{k=1}^{s} \left\{ \frac{\mathcal{L}_{\ell^{(k)}}\Big(\delta^{(k);+}_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha} - \ell^{(k)}; m\Big) \cdot \mathcal{L}_{-\ell^{(k)}}\Big(\delta^{(k);-}_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha} + \ell^{(k)}; -m\Big)}{L^{\Delta^{(k);+}_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha} + \Delta^{(k);-}_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha}}} \right\} . \tag{2.5}$$

The function $\mathcal{L}_\ell(\nu; x)$ corresponds to the so-called restricted sums and can be computed in a closed form [35]:

$$\mathcal{L}_\ell(\nu; x) = \frac{G^2(1+\nu)}{G^2(1+\ell+\nu)} \times \sum_{\substack{n_p, n_h \\ n_p - n_h = \ell}} \sum_{p_1 < \dots < p_{n_p}} \sum_{h_1 < \dots < h_{n_h}} \prod_{a=1}^{n_p} \Big\{ \mathrm{e}^{2\mathrm{i}\pi \frac{x}{L}(p_a - 1)} \Big\} \prod_{a=1}^{n_h} \Big\{ \mathrm{e}^{2\mathrm{i}\pi \frac{x}{L} h_a} \Big\}$$
$$\times \mathcal{R}_{n_p;n_h}\Big(\{p_a\}_1^{n_p} \, ; \, \{h_a\}_1^{n_h} \mid \nu \Big) = \frac{\mathrm{e}^{\mathrm{i}\frac{\pi x}{L}\ell(\ell-1)}}{\Big(1 - \mathrm{e}^{\frac{2\mathrm{i}\pi x}{L}}\Big)^{(\nu+\ell)^2}} . \tag{2.6}$$

After inserting the expression for $\mathcal{L}_\ell(\nu; x)$, one can already take the $L \to +\infty$ limit what yields

$$\frac{\langle \Omega | [\mathbf{o}_{m+1}^{(\alpha)}]^{\dagger} \cdot \mathbf{o}_{1}^{(\alpha)} | \Omega \rangle}{\langle \Omega | \Omega \rangle} \simeq \sum_{\boldsymbol{\ell} \in \mathbb{Z}^r} \mathrm{e}^{\mathrm{i}(2\boldsymbol{\ell} + \mathfrak{n}_\alpha) \cdot \boldsymbol{\sigma}_F m} \cdot \frac{\big| \mathcal{F}_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha}(\mathbf{o}^{(\alpha)}) \big|^2}{\big(-2\mathrm{i}\pi m\big)^{\Delta^{+}_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha}} \big(2\mathrm{i}\pi m\big)^{\Delta^{-}_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha}}} \big(1 + \mathrm{o}(1)\big) . \tag{2.7}$$

The scaling dimensions driving the large-distance asymptotics are obtained by summing-up over the scaling dimensions (1.59) associated with each species

$$\Delta^{\pm}_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha} = \sum_{k=1}^{r} \Delta^{(k);\pm}_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha} . \tag{2.8}$$

It then solely remains to observe that

$$\big(-2\mathrm{i}\pi m\big)^{-\Delta^{+}_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha}} \big(2\mathrm{i}\pi m\big)^{-\Delta^{-}_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha}} = \frac{\mathrm{e}^{\mathrm{i}\frac{\pi}{2}(\Delta^{+}_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha} - \Delta^{-}_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha})}}{\big(2\pi m\big)^{\Delta^{+}_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha} + \Delta^{-}_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha}}} \tag{2.9}$$

and that, due to (1.49)

$$\Delta^{+}_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha} + \Delta^{-}_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha} = \Delta_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha} \quad \text{and} \quad \Delta^{+}_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha} - \Delta^{-}_{\boldsymbol{\ell};\boldsymbol{\kappa}_\alpha} = (\boldsymbol{\kappa}, 2\boldsymbol{\ell} + \mathfrak{n}_\alpha) . \tag{2.10}$$

This entails the claim. □

2.2. *Application to specific models*

2.2.1. *Application to the SU(3) invariant XXX magnet*

The $SU(3)$ invariant *XXX* magnet refers to the bare Hamiltonian

$$\mathbf{H}_{0;XXX} = \sum_{a=1}^{L} \mathcal{P}_{aa+1} \tag{2.11}$$

acting on the Hilbert space $\mathfrak{h}_{XXX} = \bigotimes_{a=1}^{L} \mathfrak{h}_a^{XXX}$ with $\mathfrak{h}_a^{XXX} = \mathbb{C}^3$, $a = 1, \dots, L$. Above, $\mathcal{P}_{ab}$ stands for the permutation operator on $\mathfrak{h}_a^{XXX} \otimes \mathfrak{h}_b^{XXX}$, *viz.* $\mathcal{P}_{ab} = \sum_{k,j} \mathbf{E}_a^{kj} \otimes \mathbf{E}_b^{jk}$. There are two conserved charges

$$\mathbf{Q}_{XXX}^{(1)} = \sum_{a=1}^{L} \mathbf{E}_a^{11} - \mathbf{E}_a^{22} \qquad \text{and} \qquad \mathbf{Q}_{XXX}^{(2)} = \sum_{a=1}^{L} \mathbf{E}_a^{22} - \mathbf{E}_a^{33} . \tag{2.12}$$

The quantum integrability of the model is ensured by the existence of a local Lax matrix $\mathcal{L}_{ab}(\lambda) = R_{ab}\big(\lambda - \mathrm{i}c/2\big)$, where $R_{ab}(\lambda) = I_9 + \mathrm{i}c\mathcal{P}_{ab}/\lambda$, out of which one builds the twisted monodromy matrix

$$\mathbf{T}_{0;1,\dots,L}^{(\beta)}(\lambda) = \mathrm{e}^{\beta E_0^{22}} \cdot \mathcal{L}_{01}(\lambda) \cdots \mathcal{L}_{0L}(\lambda) . \tag{2.13}$$

This monodromy matrix is a 3×3 matrix on the auxiliary space $\mathfrak{h}_0^{XXX}$ whose entries are operators on $\mathfrak{h}_{XXX}$. It is sometimes convenient to represent it as

$$\mathbf{T}_{0;1,\dots,L}^{(\beta)}(\lambda) = \sum_{i,j=1}^{3} \mathbf{T}_{ij}^{(\beta)}(\lambda) \otimes \mathbf{E}_0^{ij} \tag{2.14}$$

and we shall also write $\mathbf{T}_{ij}(\lambda) = \mathbf{T}_{ij}^{(0)}(\lambda)$.

The bare Hamiltonian is then reconstructed out of the logarithmic derivative of the transfer matrix at zero twist

$$\mathbf{H}_{0;XXX} = \mathrm{i}c \frac{\partial}{\partial z} \ln \Big\{ \big(z - \mathrm{i}c/2\big)^L \cdot \mathrm{tr}_0\big[\mathbf{T}_{0;1,\dots,L}^{(0)}(z)\big] \Big\}_{|z=\mathrm{i}c/2} . \tag{2.15}$$

For later convenience, it is useful to discuss the construction of the eigenvectors associated with the β-twisted transfer matrix. This model is built over a rank 2 Lie-algebra: the eigenstates of the β-twisted transfer matrix are parametrised in terms of a collection Λ_β of two species of Bethe roots $\big\{\lambda_a^{(1)}\big\}_1^{N_\Lambda^{(1)}}$ and $\big\{\lambda_a^{(2)}\big\}_1^{N_\Lambda^{(2)}}$. They solve the system of β-twisted Bethe Ansatz equations

$$\left(\frac{\lambda_k^{(1)} + \mathrm{i}c/2}{-\lambda_k^{(1)} + \mathrm{i}c/2} \right)^L = \mathrm{e}^{\beta} (-1)^{N_\Lambda^{(1)}-1} \prod_{a=1}^{N_\Lambda^{(1)}} \left\{ \frac{\lambda_k^{(1)} - \lambda_a^{(1)} + \mathrm{i}c}{\lambda_a^{(1)} - \lambda_k^{(1)} + \mathrm{i}c} \right\} \cdot \prod_{a=1}^{N_\Lambda^{(2)}} \left\{ \frac{\lambda_a^{(2)} - \lambda_k^{(1)} + \mathrm{i}c/2}{\lambda_a^{(2)} - \lambda_k^{(1)} - \mathrm{i}c/2} \right\} \tag{2.16}$$

$$1 = \mathrm{e}^{-\beta} (-1)^{N_\Lambda^{(2)}-1} \prod_{a=1}^{N_\Lambda^{(2)}} \left\{ \frac{\lambda_k^{(2)} - \lambda_a^{(2)} + \mathrm{i}c}{\lambda_a^{(2)} - \lambda_k^{(2)} + \mathrm{i}c} \right\} \cdot \prod_{a=1}^{N_\Lambda^{(1)}} \left\{ \frac{\lambda_k^{(2)} - \lambda_a^{(1)} + \mathrm{i}c/2}{\lambda_k^{(2)} - \lambda_a^{(1)} - \mathrm{i}c/2} \right\} . \tag{2.17}$$

Here and below, whenever we shall write Λ_β, it will be understood that the Bethe roots solve the β-twisted Bethe equations. When the subscript β is dropped, it will mean that the Bethe roots solve the Bethe equations at $\beta = 0$. Finally, given Λ, by Λ_β we mean the solution to the β-twisted Bethe equations which is a smooth deformation in β of the roots Λ.

The eigenvalues of the β-twisted transfer matrix $\mathrm{tr}_0\big[\mathbf{T}_{0;1,\dots,L}^{(\beta)}(z)\big]$ associated with the eigenstate $|\,\Lambda_\beta\,\rangle$ take the form:

$$\tau_\beta\big(z \mid \Lambda_\beta\big) = \left(\frac{z + \mathrm{i}c/2}{z - \mathrm{i}c/2} \right)^L f\Big(\overline{\lambda}^{(1)}, z\Big) + \mathrm{e}^{\beta} f\Big(z, \overline{\lambda}^{(1)}\Big) \cdot k\Big(\overline{\lambda}^{(2)}, z\Big) + k\Big(\overline{\lambda}^{(2)}, z\Big) \tag{2.18}$$

where we have introduced the functions

$$f(x,y) \;=\; \frac{x-y+\mathrm{i}c}{x-y} \quad , \quad k(x,y) \;=\; \frac{x-y+\mathrm{i}c/2}{x-y-\mathrm{i}c/2}\,. \tag{2.19}$$

Above and in the following, whenever parameters belonging to a set appear with a bar, then one should take the product over all representatives of the set, *e.g.*:

$$f\Big(\omega,\overline{\lambda}^{(k)}\Big) \;=\; \prod_{a=1}^{N_\Lambda^{(k)}} f\Big(\omega,\lambda_a^{(k)}\Big)\,. \tag{2.20}$$

The shift functions associated with this model take the form

$$\widehat{\xi}^{(1)}_{\Lambda_\beta}(\omega) = \sigma^{(1)}_{0;XXX}(\omega) - \mathrm{i}\frac{\beta}{2\pi L} + \frac{1}{L}\sum_{a=1}^{N_\Lambda^{(1)}}\vartheta_1\big(\omega-\lambda_a^{(1)}\big) - \frac{1}{L}\sum_{a=1}^{N_\Lambda^{(2)}}\vartheta_2\big(\omega-\lambda_a^{(2)}\big) + \frac{N_\Lambda^{(1)}-\mathfrak{n}^{(2)}_{\Lambda;XXX}+1}{2L}, \tag{2.21}$$

$$\widehat{\xi}^{(2)}_{\Lambda_\beta}(\omega) = \mathrm{i}\frac{\beta}{2\pi L} + \frac{1}{L}\sum_{a=1}^{N_\Lambda^{(2)}}\vartheta_1\big(\omega-\lambda_a^{(2)}\big) - \frac{1}{L}\sum_{a=1}^{N_\Lambda^{(1)}}\vartheta_2\big(\omega-\lambda_a^{(1)}\big) + \frac{N_\Lambda^{(2)}-\mathfrak{n}^{(1)}_{\Lambda;XXX}+1}{2L}\,. \tag{2.22}$$

Above, $\mathfrak{n}^{(a)}_{\Lambda;XXX}=1$ if $N_\Lambda^{(a)}$ is odd and $\mathfrak{n}^{(a)}_{\Lambda}=0$ if $N_\Lambda^{(a)}$ is even. Also we have introduced

$$\vartheta_n(\omega) \;=\; \frac{1}{2\mathrm{i}\pi}\ln\left(\frac{\mathrm{i}c/n+\omega}{\mathrm{i}c/n-\omega}\right) \qquad \text{and} \qquad \sigma^{(1)}_{0;XXX}(\omega) \;=\; \frac{\mathrm{i}}{2\pi}\ln\left(\frac{\mathrm{i}c/2+\omega}{\mathrm{i}c/2-\omega}\right). \tag{2.23}$$

Note that $\sigma^{(1)}_{0;XXX}(\omega)$ is strictly increasing while the $\vartheta_k(\omega)$ are strictly decreasing. Both functions are odd. From the above, one deduces that the bare phase matrix takes the form

$$\begin{pmatrix} \theta_{11}(\lambda,\mu) & \theta_{12}(\lambda,\mu) \\ \theta_{21}(\lambda,\mu) & \theta_{22}(\lambda,\mu) \end{pmatrix} = \begin{pmatrix} \vartheta_1(\lambda-\mu) & -\vartheta_2(\lambda-\mu) \\ -\vartheta_2(\lambda-\mu) & \vartheta_1(\lambda-\mu) \end{pmatrix} \tag{2.24}$$

and thus does indeed satisfy to the general hypothesis stated earlier. The equations defining the ground state of the model take the form

$$\widehat{\xi}^{(k)}_{\Omega}\big(\omega_a^{(k)}\big) = a \quad \text{with} \quad a=1,\dots,N_\Omega^{(k)} \quad \text{and} \quad k=1,2\,. \tag{2.25}$$

Finally, we remind that the shift functions are defined as

$$\widehat{F}^{(k)}_{\Upsilon,\Lambda}(z) \;=\; L\Big(\widehat{\xi}^{(k)}_{\Lambda}\big(z\big) - \widehat{\xi}^{(k)}_{\Upsilon}\big(z\big)\Big)\,. \tag{2.26}$$

The critical exponents arising in the large-distance asymptotics of two-point functions will then be given by the dressed charge matrix for this specific model. It solely remains to list the vector integers $\boldsymbol{\kappa}_\alpha$ associated with the elementary operators of the model and the vector $\boldsymbol{\mathfrak{n}}_\alpha$ attached to the class of excited states arising in the form factor expansion of the two-point functions:

operator $\mathbf{O}^{(\alpha)}$	$\mathbf{E}_1^{aa}$	$\mathbf{E}_1^{12}$	$\mathbf{E}_1^{13}$	$\mathbf{E}_1^{23}$
vector $\boldsymbol{\kappa}_\alpha$	$(0,0)$	$(-1,0)$	$(1,-1)$	$(0,1)$
vector $\mathfrak{n}_\alpha$	$(0,0)$	$(0,1)$	$(1,1)$	$(1,0)$

(2.27)

The vectors associated with the other elementary operators can be obtained by hermitian conjugation. The asymptotic behaviour of the two-point functions $\langle \mathbf{E}_{1+m}^{ba}\mathbf{E}_1^{ab}\rangle$ is then readily deduced from (2.1) by picking the appropriate vectors $\boldsymbol{\kappa}_{ab}$ and $\mathfrak{n}_{ab}$ from the above table.

2.2.2. *The Hubbard model*

The one-dimensional Hubbard model is a rank 2 model of particular interest that has been extensively studied since the seminal calculation of its spectrum by Lieb and Wu through nested Bethe Ansatz method. We refer to [21] for a thorough discussion of the model. Despite the numerous developments relative to the model, not much is known on the exact expression of its correlation functions. In fact, in the present state of the art, solely the norm of the Bethe state was conjectured in [27]. As mentioned in the introduction, the large-distance asymptotic behaviour of two-point functions was obtained on the basis of the $1/L$ corrections to the ground and excited state's energies by means of conformal field theoretic [24] or Luttinger liquid based [64] reasonings.

The bare Hamiltonian of the Hubbard model is defined in terms of fermionic operators obeying

$$\Big\{\mathbf{c}_{j,a}^{\dagger},\mathbf{c}_{k,b}^{\dagger}\Big\} = \Big\{\mathbf{c}_{j,a}^{\dagger},\mathbf{c}_{k,b}^{\dagger}\Big\} = 0 \qquad \Big\{\mathbf{c}_{j,a},\mathbf{c}_{k,b}^{\dagger}\Big\} = \delta_{a,b}\delta_{j,k} \qquad \mathbf{n}_{k,a} = \mathbf{c}_{k,a}^{\dagger},\mathbf{c}_{k,a}. \tag{2.28}$$

It takes the form

$$\mathbf{H}_{0;HB} = -\sum_{k=1}^{L}\sum_{a=\uparrow,\downarrow}\Big\{\mathbf{c}_{j,a}^{\dagger}\mathbf{c}_{j+1,a} + \mathbf{c}_{j+1,a}^{\dagger}\mathbf{c}_{j,a}\Big\} + 2c\sum_{k=1}^{L}\mathbf{n}_{k,\uparrow}\mathbf{n}_{k,\downarrow}\,. \tag{2.29}$$

The model has two conserved charges

$$\mathbf{Q}_{HB}^{(1)} = \sum_{k=1}^{L}\Big(\mathbf{n}_{k,\uparrow}+\mathbf{n}_{k,\downarrow}\Big) \qquad \text{and} \qquad \mathbf{Q}_{HB}^{(2)} = \frac{1}{2}\sum_{k=1}^{L}\Big(\mathbf{n}_{k,\uparrow}-\mathbf{n}_{k,\downarrow}\Big). \tag{2.30}$$

They are interpreted as the total number of particle and the total longitudinal spin operators. The eigenvectors $|\,\Lambda\,\rangle$ of $\mathbf{H}_{0;HB}$ are parametrised by two species of Bethe roots $\{\lambda_a^{(1)}\}_1^{N_\Lambda^{(1)}}$ and $\{\lambda_a^{(2)}\}_1^{N_\Lambda^{(2)}}$. The conserved global charges act on the Bethe vectors as

$$\mathbf{Q}_{HB}^{(1)} = N_\Lambda^{(1)}|\,\Lambda\,\rangle \qquad \text{and} \qquad \mathbf{Q}_{HB}^{(2)} = \frac{1}{2}\Big(N_\Lambda^{(1)} - 2N_\Lambda^{(2)}\Big)|\,\Lambda\,\rangle\,. \tag{2.31}$$

The Bethe equations for the Hubbard model take the form

$$\Big(\mathrm{i}\lambda_k^{(1)} + \sqrt{1-(\lambda_k^{(1)})^2}\Big)^L = \prod_{a=1}^{N_\Lambda^{(2)}}\left\{\frac{\lambda_a^{(2)}-\lambda_k^{(1)}-\mathrm{i}c/2}{\lambda_a^{(2)}-\lambda_k^{(1)}+\mathrm{i}c/2}\right\} \tag{2.32}$$

$$1 = (-1)^{N_\Lambda^{(2)}-1}\prod_{a=1}^{N_\Lambda^{(2)}}\left\{\frac{\lambda_a^{(2)}-\lambda_k^{(2)}+\mathrm{i}c}{\lambda_k^{(2)}-\lambda_a^{(2)}+\mathrm{i}c}\right\}\cdot\prod_{a=1}^{N_\Lambda^{(1)}}\left\{\frac{\lambda_k^{(2)}-\lambda_a^{(1)}+\mathrm{i}c/2}{\lambda_k^{(2)}-\lambda_a^{(1)}-\mathrm{i}c/2}\right\}. \tag{2.33}$$

They give rise to the following shift functions:

$$\widehat{\xi}^{(1)}_{\Lambda}(\omega) = \sigma^{(1)}_{0;HB}(\omega) \; - \; \frac{1}{L}\sum_{a=1}^{N^{(2)}_{\Lambda}} \vartheta_2\big(\omega - \lambda^{(2)}_a\big) \; + \; \frac{N^{(1)}_{\Lambda} - \mathfrak{n}^{(1)}_{\Lambda;HB} + 1}{2L} \tag{2.34}$$

$$\widehat{\xi}^{(2)}_{\Lambda}(\omega) = \frac{1}{L}\sum_{a=1}^{N^{(2)}_{\Lambda}} \vartheta_1\big(\omega - \lambda^{(2)}_a\big) \; - \; \frac{1}{L}\sum_{a=1}^{N^{(1)}_{\Lambda}} \vartheta_2\big(\omega - \lambda^{(1)}_a\big) \; + \; \frac{N^{(2)}_{\Lambda} - \mathfrak{n}^{(2)}_{\Lambda;HB} + 1}{2L} \, . \tag{2.35}$$

Above, the integer shift is defined as

$$\mathfrak{n}^{(1)}_{\Lambda;HB} = \begin{cases} 1 & \text{if} \quad N^{(1)}_{\Lambda} + N^{(2)}_{\Lambda} + 1 \in 2\mathbb{Z}+1 \\ 0 & \text{if} \quad N^{(1)}_{\Lambda} + N^{(2)}_{\Lambda} + 1 \in 2\mathbb{Z} \end{cases} \qquad \text{and}$$

$$\mathfrak{n}^{(1)}_{\Lambda;HB} = \begin{cases} 1 & \text{if} \quad N^{(1)}_{\Lambda} \in 2\mathbb{Z}+1 \\ 0 & \text{if} \quad N^{(1)}_{\Lambda} \in 2\mathbb{Z} \end{cases} \, . \tag{2.36}$$

The phase functions are defined as for the XXX chain while the bare momentum takes the form

$$\sigma^{(1)}_{0;HB}(\omega) \; = \; \frac{-\mathrm{i}}{2\pi} \ln\Big(\mathrm{i}\omega + \sqrt{1-\omega^2}\Big) \, . \tag{2.37}$$

Thus, in the case of the Hubbard model, the bare phase matrix takes the form

$$\begin{pmatrix} \theta_{11}(\lambda,\mu) & \theta_{12}(\lambda,\mu) \\ \theta_{21}(\lambda,\mu) & \theta_{22}(\lambda,\mu) \end{pmatrix} = \begin{pmatrix} 0 & -\vartheta_2(\lambda-\mu) \\ -\vartheta_2(\lambda-\mu) & \vartheta_1(\lambda-\mu) \end{pmatrix} \tag{2.38}$$

and hence does indeed satisfy to the general hypothesis stated earlier. The equations defining the ground state of the model take the form

$$\widehat{\xi}^{(k)}_{\Omega}\big(\omega^{(k)}_a\big) = a \quad \text{with} \quad a = 1, \dots, N^{(k)}_{\Omega} \quad \text{and} \quad k = 1, 2 \, . \tag{2.39}$$

Therefore, in order to characterise the large-distance asymptotics of two-point functions it remains to list the vector integers $\boldsymbol{\kappa}_{\alpha}$ associated with the elementary operators of the model and the vector $\boldsymbol{\mathfrak{n}}_{\alpha}$ attached to the class of excited states arising in the form factor expansion of the two-point functions:

operator $\mathbf{O}^{(\alpha)}$	$\mathbf{S}^z_1$	$\mathbf{S}^+_1$	$\mathbf{n}_{1;\uparrow/\downarrow}$	$\mathbf{c}^{\dagger}_{1;\uparrow}$	$\mathbf{c}^{\dagger}_{1;\downarrow}$
vector $\boldsymbol{\kappa}_{\alpha}$	(0, 0)	(0, −1)	(0, 0)	(1, −1)	(1, 1)
vector $\boldsymbol{\mathfrak{n}}_{\alpha}$	(0, 0)	(1, 0)	(0, 0)	(0, 1)	(0, 1)

(2.40)

The vectors associated with the other elementary operators can be obtained by hermitian conjugation. Note that, on top of the fermionic operators, we have also introduced the local spin operators:

$$\mathbf{S}^z_1 \; = \; \frac{1}{2}\Big(\mathbf{n}_{1,\uparrow} - \mathbf{n}_{1,\downarrow}\Big) \qquad \text{and} \qquad \mathbf{S}^+_1 \; = \; \mathbf{c}^{\dagger}_{1,\uparrow}\mathbf{c}_{1,\downarrow} \, . \tag{2.41}$$

The large-distance expansions obtained for the Hubbard model within our approach do confirm the conformal field theoretic predictions for this model, see *e.g.* [21].

3. The SU(3) invariant XXX magnet as a check of the form factor's structure

In this section, we follow the setting and notation introduced in Section 2.2.1.

3.1. The β-twisted scalar products

The authors of [8] introduced a function

$$\mathcal{S}_\beta\big(\Upsilon_\beta \mid \Lambda\big) = \langle \Upsilon_\beta \mid \Lambda \rangle \tag{3.1}$$

called the β-twisted scalar product. This function depends on two collections of Bethe roots

$$\Upsilon_\beta = \Big\{ \{\mu_a^{(1)}\}_1^{N_\Upsilon^{(1)}}, \{\mu_a^{(2)}\}_1^{N_\Upsilon^{(2)}} \Big\} \qquad \text{and} \qquad \Lambda = \Big\{ \{\lambda_a^{(1)}\}_1^{N_\Lambda^{(1)}}, \{\lambda_a^{(2)}\}_1^{N_\Lambda^{(2)}} \Big\} \tag{3.2}$$

which solve, respectively, the β-twisted Bethe equations of the model (2.16)–(2.17) and those at $\beta = 0$. At $\beta = 0$, it gives rise to the scalar product between the (un-normalised)-state parametrised by Υ and the one parametrised by Λ. Further, the function $\mathcal{S}_\beta\big(\Upsilon_\beta \mid \Lambda\big)$ corresponds [8] to the generating function of the form factors of the $\mathbf{T}_{22}(z)$ entry of the monodromy matrix in that

$$\langle \Upsilon \,|\mathbf{T}_{22}(z)|\, \Lambda \rangle = \frac{\partial}{\partial \beta} \Big\{ \Big[\tau_\beta\big(z \mid \Upsilon_\beta\big) - \tau_0\big(z \mid \Lambda\big) \Big] \cdot \mathcal{S}_\beta\big(\Upsilon_\beta \mid \Lambda\big) \Big\}_{|\beta=0} . \tag{3.3}$$

Proposition 3.1. *The generating function (3.1) admits the representation*

$$\begin{aligned} \mathcal{S}_\beta\big(\Upsilon_\beta \mid \Lambda\big) = (1 - \mathrm{e}^{\beta}) \cdot \mathrm{e}^{N_\Lambda^{(2)} \beta} \cdot & \prod_{\substack{a=1 \\ \mu_a^{(2)} \neq \theta}}^{N_{\Upsilon_\beta}^{(2)}} \Big(1 - \mathrm{e}^{2\mathrm{i}\pi L \widehat{F}^{(2)}_{\Upsilon_\beta;\Lambda}(\mu_a^{(2)})}\Big) \prod_{a=1}^{N_\Lambda^{(1)}} \Big(1 - \mathrm{e}^{2\mathrm{i}\pi L \widehat{F}^{(1)}_{\Upsilon_\beta;\Lambda}(\lambda_a^{(1)})}\Big) \\ & \times \frac{h\big(\overline{\mu}^{(2)}, \theta\big)}{h\big(\overline{\lambda}^{(2)}, \theta\big)} \cdot k\big(\overline{\mu}^{(2)}, \overline{\mu}^{(1)} \cup \overline{\lambda}^{(1)}\big) \cdot k\big(\overline{\lambda}^{(2)}, \overline{\lambda}^{(1)}\big) \\ & \cdot f\big(\overline{\lambda}^{(2)}, \overline{\mu}^{(2)}\big) f\big(\overline{\lambda}^{(1)}, \overline{\mu}^{(1)}\big) \cdot \det\Big[\mathrm{id} + \widehat{\mathbf{U}}_\theta(\Upsilon_\beta; \Lambda)\Big] \end{aligned} \tag{3.4}$$

in which θ can be any of the roots $\mu_1^{(2)}, \ldots, \mu^{(2)}_{N^{(2)}_{\Upsilon_\beta}}$. This representation involves the function

$$h(x, y) = \frac{x - y + \mathrm{i}c}{\mathrm{i}c}. \tag{3.5}$$

The representation also involves the Fredholm determinant of the operator $\mathrm{id} + \widehat{\mathbf{U}}_\theta(\Upsilon_\beta \mid \Lambda)$*, with* $\widehat{\mathbf{U}}_\theta(\Upsilon_\beta \mid \Lambda)$ *an integral operator on* $L^2\big(\mathscr{C}_{\Upsilon_\beta,\Lambda}\big)$*, with*

$$\mathscr{C}_{\Upsilon_\beta,\Lambda} = \Gamma\Big(\{\lambda_a^{(1)}\}_1^{N_\Lambda^{(1)}}\Big) \bigcup \Gamma\Big(\{\mu_a^{(2)}\}_1^{N_{\Upsilon_\beta}^{(2)}}\Big) \tag{3.6}$$

being a small counter-clockwise loop around the indicated above Bethe roots. The operator $\widehat{\mathbf{U}}_\theta$ *admits the following block decomposition relatively to the above partitioning of the contour* $\mathscr{C}_{\Upsilon_\beta,\Lambda}$*:*

$$\widehat{\mathbf{U}}_\theta(\Upsilon_\beta; \Lambda) = \begin{pmatrix} \widehat{\mathbf{U}}_\theta^{(11)}(\Upsilon_\beta; \Lambda) & \widehat{\mathbf{U}}_\theta^{(12)}(\Upsilon_\beta; \Lambda) \\ \widehat{\mathbf{U}}_\theta^{(21)}(\Upsilon_\beta; \Lambda) & \widehat{\mathbf{U}}_\theta^{(22)}(\Upsilon_\beta; \Lambda) \end{pmatrix}. \tag{3.7}$$

The integral kernels of the operators in the block decompositions take the form

$$\widehat{U}_{\theta}^{(11)}(\Upsilon_\beta;\Lambda)(z,z') = \frac{(2\mathrm{i}\pi)^{-1}}{1-\mathrm{e}^{2\mathrm{i}\pi \widehat{F}^{(1)}_{\Upsilon_\beta;\Lambda}(z')}} \frac{f(z',\overline{\lambda}^{(1)})}{f(z',\overline{\mu}^{(1)})} \Bigg\{ \bigg[\frac{1}{\theta - z + \mathrm{i}c/2} - \frac{1}{z'-z+\mathrm{i}c} \bigg] - \bigg[\frac{\mathrm{e}^{-\beta}}{z-z'+\mathrm{i}c} \cdot \frac{k(\overline{\lambda}^{(2)},z')}{k(\overline{\mu}^{(2)},z')} + \frac{1}{\theta - z - \mathrm{i}c/2} \cdot \frac{k(\theta,\overline{\mu}^{(1)})}{k(\theta,\overline{\lambda}^{(1)})} \bigg] \Bigg\} \tag{3.8}$$

$$\widehat{U}_{\theta}^{(21)}(\Upsilon_\beta;\Lambda)(z,z') = \frac{(2\mathrm{i}\pi)^{-1}}{1-\mathrm{e}^{2\mathrm{i}\pi \widehat{F}^{(1)}_{\Upsilon_\beta;\Lambda}(z')}} \frac{f(z',\overline{\lambda}^{(1)})}{f(z',\overline{\mu}^{(1)})} \Bigg\{ k(z,z') \cdot \frac{k(\overline{\lambda}^{(2)},z')}{k(\overline{\mu}^{(2)},z')} - \frac{\mathrm{e}^{\beta}}{f(\theta,z)} \cdot \frac{k(\theta,\overline{\mu}^{(1)})}{k(\theta,\overline{\lambda}^{(1)})} + \frac{\mathrm{i}c}{z'-z+\mathrm{i}c/2} + \frac{\mathrm{i}c}{z-\theta+\mathrm{i}c} \Bigg\} \tag{3.9}$$

in what concerns the first column while the second column reads

$$\widehat{U}_{\theta}^{(12)}(\Upsilon_\beta;\Lambda)(z,z') = \frac{\mathrm{e}^{\beta}\cdot(2\mathrm{i}\pi)^{-1}}{1-\mathrm{e}^{2\mathrm{i}\pi \widehat{F}^{(2)}_{\Upsilon_\beta;\Lambda}(z')}} \frac{f(\overline{\mu}^{(2)},z')}{f(\overline{\lambda}^{(2)},z')} \cdot \Bigg\{ \frac{1}{z'-z-\mathrm{i}c/2} \cdot \frac{k(z',\overline{\mu}^{(1)})}{k(z',\overline{\lambda}^{(1)})} - \frac{1}{\theta-z-\mathrm{i}c/2} \cdot \frac{k(\theta,\overline{\mu}^{(1)})}{k(\theta,\overline{\lambda}^{(1)})} + \frac{1}{\theta-z+\mathrm{i}c/2} - \frac{1}{z'-z+\mathrm{i}c/2} \Bigg\} \tag{3.10}$$

$$\widehat{U}_{\theta}^{(22)}(\Upsilon_\beta;\Lambda)(z,z') = \frac{(2\mathrm{i}\pi)^{-1}}{1-\mathrm{e}^{2\mathrm{i}\pi \widehat{F}^{(2)}_{\Upsilon_\beta;\Lambda}(z')}} \frac{f(\overline{\mu}^{(2)},z')}{f(\overline{\lambda}^{(2)},z')} \cdot \Bigg\{ \frac{\mathrm{e}^{\beta}}{f(z',z)} \cdot \frac{k(z',\overline{\mu}^{(1)})}{k(z',\overline{\lambda}^{(1)})} - \frac{\mathrm{e}^{\beta}}{f(\theta,z)} \cdot \frac{k(\theta,\overline{\mu}^{(1)})}{k(\theta,\overline{\lambda}^{(1)})} + \frac{\mathrm{i}c}{z-\theta+\mathrm{i}c} - \frac{\mathrm{i}c}{z-z'+\mathrm{i}c} \Bigg\} \tag{3.11}$$

Note that in writing the four integral kernels we have dropped out their explicit dependence on the Bethe roots since there is no possibility of confusion. Finally, the vector shift function $\widehat{\boldsymbol{F}}_{\Upsilon_\beta,\Lambda}$ is as defined in (1.44).

Corollary 3.2. *(See [8,62].) The norms of an eigenstate in the $SU(3)$-invariant XXX spin chain admit the determinant representation*

$$\mathcal{S}_\beta(\Upsilon_\beta \mid \Upsilon_\beta) = \big\{k(\overline{\mu}^{(2)},\overline{\mu}^{(1)})\big\}^3 \cdot \mathfrak{X}(\overline{\mu}^{(1)},\overline{\mu}^{(1)}) \cdot \mathfrak{X}(\overline{\mu}^{(2)},\overline{\mu}^{(2)}) \times \big(2\mathrm{i}\pi Lc\big)^{N^{(1)}_{\Upsilon}+N^{(2)}_{\Upsilon_\beta}} \cdot \Big(\widehat{\xi}^{(1)}_{\Upsilon_\beta}\Big)'(\overline{\mu}^{(1)}) \cdot \Big(\widehat{\xi}^{(2)}_{\Upsilon_\beta}\Big)'(\overline{\mu}^{(2)}) \cdot \det\Big[I_{|\Upsilon_\beta|} + \mathcal{K}_{\Upsilon_\beta}\Big]. \tag{3.12}$$

Above, I_n is the n-dimensional identity matrix and $|\Upsilon_\beta| = \#\Upsilon_\beta = N^{(1)}_{\Upsilon_\beta} + N^{(2)}_{\Upsilon_\beta}$. Finally, given any set of Bethe roots Λ, the matrix $\mathcal{K}_\Lambda$ admits the block decomposition

$$\mathcal{K}_\Lambda = \begin{pmatrix} \widehat{K}^{(11)}_\Lambda(\lambda^{(1)}_j,\lambda^{(1)}_k) & \widehat{K}^{(12)}_\Lambda(\lambda^{(1)}_j,\lambda^{(2)}_k) \\ \widehat{K}^{(21)}_\Lambda(\lambda^{(2)}_j,\lambda^{(1)}_k) & \widehat{K}^{(22)}_\Lambda(\lambda^{(2)}_j,\lambda^{(2)}_k) \end{pmatrix} \quad \textit{with} \quad \widehat{K}^{(a1)}_\Lambda(\lambda,\mu) = \frac{K_a(\lambda-\mu)}{L\Big(\widehat{\xi}^{(1)}_\Lambda\Big)'(\mu)} \tag{3.13}$$

$$\widehat{K}^{(12)}_{\Lambda}(\lambda,\mu) \;=\; \frac{K_2(\lambda-\mu)}{L\left(\widehat{\xi}^{(2)}_{\Lambda}\right)'(\mu)}, \qquad \widehat{K}^{(22)}_{\Lambda}(\lambda,\mu) \;=\; \frac{K_1(\lambda-\mu)}{L\left(\widehat{\xi}^{(2)}_{\Lambda}\right)'(\mu)} \tag{3.14}$$

and where the two difference kernels read

$$K_1(\lambda) \;=\; \frac{2c}{2\pi\left(\lambda^2+c^2\right)} \qquad \text{and} \qquad K_2(\lambda) \;=\; \frac{c}{2\pi\left(\lambda^2+c^2/4\right)}\,. \tag{3.15}$$

Note that one has $\vartheta_n' = -K_n$.

Corollary 3.3. *Let Λ, Υ be any two collections of Bethe roots such that $N^{(a)}_{\Lambda} = N^{(a)}_{\Upsilon}$, $a = 1, 2$, and assume that β is purely imaginary. Then, the normalised β-twisted scalar product admits the factorisation*

$$\left|\frac{\mathcal{S}_\beta\left(\Upsilon_\beta \mid \Lambda\right)}{\|\Upsilon_\beta\|\cdot\|\Lambda\|}\right|^2 \;=\; |1-\mathrm{e}^{\beta}|^2\cdot\left(\widehat{\mathcal{W}}\cdot\widehat{\mathcal{R}}^{(22)}\cdot\widehat{\mathcal{D}}\right)\left(\Upsilon_\beta;\Lambda\right). \tag{3.16}$$

The three functions appearing in such a decomposition take the form

$$\widehat{\mathcal{W}}\left(\Upsilon;\Lambda\right) = W\left(\{\mu^{(1)}_a\}_1^{N^{(1)}_{\Upsilon}};\{\lambda^{(1)}_a\}_1^{N^{(1)}_{\Lambda}}\right)\cdot W\left(\{\mu^{(2)}_a\}_1^{N^{(2)}_{\Upsilon}};\{\lambda^{(2)}_a\}_1^{N^{(2)}_{\Lambda}}\right) \cdot\frac{k\left(\overline{\mu}^{(2)},\overline{\lambda}^{(1)}\right)\cdot k\left(\overline{\lambda}^{(1)},\overline{\mu}^{(2)}\right)}{\left|k\left(\overline{\mu}^{(2)},\overline{\mu}^{(1)}\right)\cdot k\left(\overline{\lambda}^{(2)},\overline{\lambda}^{(1)}\right)\right|} \tag{3.17}$$

where we agree upon

$$W(\{y_a\}_1^N;\{z_a\}_1^{N'}) \;=\; \frac{h\left(\overline{y},\overline{z}\right)h\left(\overline{z},\overline{y}\right)}{h\left(\overline{y},\overline{y}\right)h\left(\overline{z},\overline{z}\right)}\,. \tag{3.18}$$

Furthermore, we have set

$$\widehat{\mathcal{R}}^{(22)}\left(\Upsilon_\beta;\Lambda\right) = \prod_{a=1}^{N^{(2)}_{\Upsilon}}\left(\frac{\sin\left[\pi\widehat{F}_2\left(\mu^{(2)}_a\right)\right]}{\sin\left[\pi\widehat{F}_2\left(\lambda^{(2)}_a\right)\right]}\right)^2\cdot\frac{1}{4\sin^2\left[\pi\widehat{F}_2(\theta)\right]}\cdot\left|\frac{h\left(\overline{\mu}^{(2)},\theta\right)}{h\left(\overline{\lambda}^{(2)},\theta\right)}\right|^2 \times\frac{\left|\det\left[\mathrm{id}\,+\,\widehat{\mathbf{U}}_\theta\left(\Upsilon_\beta;\Lambda\right)\right]\right|^2}{\det\left[I_{|\Upsilon_\beta|}+\mathcal{K}_{\Upsilon_\beta}\right]\cdot\det\left[I_{|\Lambda|}+\mathcal{K}_{\Lambda}\right]} \tag{3.19}$$

where $\theta\in\{\mu^{(2)}_a\}_1^{N^{(2)}_{\Upsilon}}$ is arbitrary. Finally,

$$\widehat{\mathcal{D}}\left(\Upsilon;\Lambda\right) \;=\; \mathcal{D}\left(\{\mu^{(1)}_a\}_{a=1}^{N^{(1)}_{\Upsilon}}\mid\{\lambda^{(1)}_a\}_{a=1}^{N^{(1)}_{\Lambda}}\right)\left[\widehat{\xi}^{(1)}_{\Upsilon},\widehat{\xi}^{(1)}_{\Lambda}\right]\cdot\mathcal{D}\left(\{\mu^{(2)}_a\}_{a=1}^{N^{(2)}_{\Upsilon}}\mid\{\lambda^{(2)}_a\}_{a=1}^{N^{(2)}_{\Lambda}}\right)\left[\widehat{\xi}^{(2)}_{\Upsilon},\widehat{\xi}^{(2)}_{\Lambda}\right], \tag{3.20}$$

where the functions $\mathcal{D}$ are as defined in (1.55).

3.2. Form factors of local operators

In this section, we discuss the form taken by the form factors of the three local operators E_1^{22}, E_1^{23} and E_1^{21}. The moduli squared of the E_1^{22} form factor follows directly from (3.3). The form factors of the operators E_1^{23} and E_1^{21} can be deduced from those of E_1^{22} by using certain identities that were discovered in [60] and which relate to each other the various form factors. We will show that the decomposition for the β-twisted scalar products given in the previous section does indeed lead to the thermodynamic limit of form factors as claimed in Conjecture 1.51.

First, we remind some of the results of [60].

Lemma 3.4. *Let Υ and Λ be two collections of Bethe roots. Then it holds*

$$\left|\frac{\langle\,\Upsilon\,|\mathbf{E}_1^{21}|\,\Lambda\,\rangle}{\|\Upsilon\|\cdot\|\Lambda\|}\right|^2 = \big(L+N_\Lambda^{(2)}-2N_\Lambda^{(1)}\big)\lim_{\omega\to+\infty}\left\{\left|\frac{\langle\,\Upsilon\,|\mathbf{E}_1^{22}|\,\Lambda_\omega^{(1)}\,\rangle}{\|\Upsilon\|\cdot\|\Lambda_\omega^{(1)}\|}\right|^2\right\} \tag{3.21}$$

as well as

$$\left|\frac{\langle\,\Upsilon\,|\mathbf{E}_1^{23}|\,\Lambda\,\rangle}{\|\Upsilon\|\cdot\|\Lambda\|}\right|^2 = \big(N_\Upsilon^{(1)}-2N_\Upsilon^{(2)}\big)\lim_{\omega\to+\infty}\left\{\left|\frac{\langle\,\Upsilon_\omega^{(2)}\,|\mathbf{E}_1^{22}|\,\Lambda\,\rangle}{\|\Upsilon_\omega^{(2)}\|\cdot\|\Lambda\|}\right|^2\right\}. \tag{3.22}$$

Above, we have introduced the following convention for the "augmented" sets of Bethe roots

$$\Lambda_\omega^{(1)} = \Big\{\{\lambda_a^{(1)}\}_1^{N_\Lambda^{(1)}+1}\,;\,\{\lambda_a^{(2)}\}_1^{N_\Lambda^{(2)}}\Big\} \qquad \text{and} \qquad \Upsilon_\omega^{(2)} = \Big\{\{\mu_a^{(1)}\}_1^{N_\Upsilon^{(1)}}\,;\,\{\mu_a^{(2)}\}_1^{N_\Upsilon^{(2)}+1}\Big\} \tag{3.23}$$

where respectively, $\lambda^{(1)}_{N_\Lambda^{(1)}+1}=\omega$ or $\mu^{(2)}_{N_\Upsilon^{(2)}+1}=\omega$.

Proof. Recall that the matrix elements of local operators are reconstructed as [50]:

$$\mathbf{E}_1^{k\ell} = \lim_{z\to \mathrm{i}c/2}\left\{\mathbf{T}_{\ell k}(z)\cdot\Big\{\mathrm{tr}_0\big[\mathbf{T}^{(0)}_{0;1,\dots,L}(z)\big]\Big\}^{-1}\right\}. \tag{3.24}$$

The action of the inverse transfer matrix is easily computed on Bethe eigenvectors. Then, it solely remains to use the following identities (see [60] for more details):

$$\begin{aligned}\langle\,\Upsilon\,|\mathbf{T}_{32}(z)|\,\Lambda\,\rangle &= -\lim_{\omega\to+\infty}\left\{\frac{\omega}{\mathrm{i}c}\langle\,\Upsilon_\omega^{(2)}\,|\mathbf{T}_{22}(z)|\,\Lambda\,\rangle\right\} \quad \text{and}\\ \langle\,\Upsilon\,|\mathbf{T}_{12}(z)|\,\Lambda\,\rangle &= \lim_{\omega\to+\infty}\left\{\frac{\omega}{\mathrm{i}c}\langle\,\Upsilon\,|\mathbf{T}_{22}(z)|\,\Lambda_\omega^{(1)}\,\rangle\right\}\end{aligned} \tag{3.25}$$

as well as

$$\|\Lambda\|^2 = \lim_{\omega\to+\infty}\left\{\frac{\omega^2\cdot\|\Lambda_\omega^{(1)}\|^2/c^2}{L+N_\Lambda^{(2)}-2N_\Lambda^{(1)}}\right\} \quad \text{and} \quad \|\Upsilon\|^2 = \lim_{\omega\to+\infty}\left\{\frac{\omega^2\cdot\|\Upsilon_\omega^{(2)}\|^2/c^2}{N_\Upsilon^{(1)}-2N_\Upsilon^{(2)}}\right\} \tag{3.26}$$

and the limit $\tau_0\big(z\mid\Upsilon\big)=\lim_{\omega\to+\infty}\Big\{\tau_0\big(z\mid\Upsilon_\omega^{(a)}\big)\Big\}$. $\square$

Lemma 3.5. *Let Υ and Λ be two collections of Bethe roots. Then it holds*

$$\left|\frac{\langle\,\Upsilon\,|\mathbf{E}_1^{12}|\,\Lambda\,\rangle}{\|\Upsilon\|\cdot\|\Lambda\|}\right|^2=\left|\frac{\langle\,\Lambda\,|\mathbf{E}_1^{21}|\,\Upsilon\,\rangle}{\|\Upsilon\|\cdot\|\Lambda\|}\right|^2,\qquad\left|\frac{\langle\,\Upsilon\,|\mathbf{E}_1^{32}|\,\Lambda\,\rangle}{\|\Upsilon\|\cdot\|\Lambda\|}\right|^2=\left|\frac{\langle\,\Lambda\,|\mathbf{E}_1^{23}|\,\Upsilon\,\rangle}{\|\Upsilon\|\cdot\|\Lambda\|}\right|^2 \tag{3.27}$$

and

$$\left|\frac{\langle\,\Upsilon\,|\mathbf{E}_1^{13}|\,\Lambda\,\rangle}{\|\Upsilon\|\cdot\|\Lambda\|}\right|^2=\left|\frac{\langle\,\Lambda\,|\mathbf{E}_1^{31}|\,\Upsilon\,\rangle}{\|\Upsilon\|\cdot\|\Lambda\|}\right|^2. \tag{3.28}$$

Moreover, one has

$$\left|\frac{\langle\,\Upsilon\,|\mathbf{E}_1^{31}|\,\Lambda\,\rangle}{\|\Upsilon\|\cdot\|\Lambda\|}\right|^2=\big(N_\Upsilon^{(1)}-2N_\Upsilon^{(2)}\big)\lim_{\omega\to+\infty}\left\{\left|\frac{\langle\,\Upsilon_\omega^{(2)}\,|\mathbf{E}_1^{12}|\,\Lambda\,\rangle}{\|\Upsilon_\omega^{(2)}\|\cdot\|\Lambda\|}\right|^2\right\}. \tag{3.29}$$

Finally, for the diagonal amplitudes, one has:

$$\left|\frac{\langle\,\Upsilon\,|\mathbf{E}_1^{11}-\mathbf{E}_1^{22}|\,\Lambda\,\rangle}{\|\Upsilon\|\cdot\|\Lambda\|}\right|^2=\big(L+N_\Lambda^{(2)}-2N_\Lambda^{(1)}\big)\lim_{\omega\to+\infty}\left\{\left|\frac{\langle\,\Upsilon\,|\mathbf{E}_1^{12}|\,\Lambda_\omega^{(1)}\,\rangle}{\|\Upsilon\|\cdot\|\Lambda_\omega^{(1)}\|}\right|^2\right\} \tag{3.30}$$

$$\left|\frac{\langle\,\Upsilon\,|\mathbf{E}_1^{22}-\mathbf{E}_1^{33}|\,\Lambda\,\rangle}{\|\Upsilon\|\cdot\|\Lambda\|}\right|^2=\big(N_\Upsilon^{(1)}-2N_\Upsilon^{(2)}\big)\lim_{\omega\to+\infty}\left\{\left|\frac{\langle\,\Upsilon_\omega^{(2)}\,|\mathbf{E}_1^{32}|\,\Lambda\,\rangle}{\|\Upsilon_\omega^{(2)}\|\cdot\|\Lambda\|}\right|^2\right\}. \tag{3.31}$$

Proof. Relations (3.29), (3.30) and (3.31) are obtained using the limits

$$\langle\,\Upsilon\,|\mathbf{T}_{31}(z)|\,\Lambda\,\rangle=-\lim_{\omega\to+\infty}\left\{\frac{\omega}{\mathrm{i}c}\langle\,\Upsilon_\omega^{(2)}\,|\mathbf{T}_{21}(z)|\,\Lambda\,\rangle\right\}, \tag{3.32}$$

$$\langle\,\Upsilon\,|\mathbf{T}_{11}(z)-\mathbf{T}_{22}(z)|\,\Lambda\,\rangle=\lim_{\omega\to+\infty}\left\{\frac{\omega}{\mathrm{i}c}\langle\,\Upsilon\,|\mathbf{T}_{12}(z)|\,\Lambda_\omega^{(1)}\,\rangle\right\}, \tag{3.33}$$

$$\langle\,\Upsilon\,|\mathbf{T}_{22}(z)-\mathbf{T}_{33}(z)|\,\Lambda\,\rangle=\lim_{\omega\to+\infty}\left\{\frac{\omega}{\mathrm{i}c}\langle\,\Upsilon_\omega^{(2)}\,|\mathbf{T}_{23}(z)|\,\Lambda\,\rangle\right\}. \tag{3.34}$$

Then, to establish relations (3.27) and (3.28), we use the antimorphism ψ that acts as $\psi(T_{jk}(z))=T_{kj}(z)$ and $\psi(|\,\Lambda\,\rangle)=\langle\,\Lambda\,|$, see [60] for details. □

In the following, owing to Lemma 3.5, a good deal of amplitudes can be deduced from the three that are described in Lemma 3.4. We shall therefore only focus our attention the latter.

In order to state the formulae for the moduli squared of the form factors, we need to introduce some notations.

Proposition 3.6. *Let Λ and Υ denote two sets of Bethe roots. Then, the amplitudes factorise as follows.*

For the diagonal operator

$$\left|\frac{\langle\,\Upsilon\,|\mathbf{E}_1^{22}|\,\Lambda\,\rangle}{\|\Upsilon\|\cdot\|\Lambda\|}\right|^2=-\frac{1}{2}\frac{\partial^2}{\partial\beta^2}\left\{\widehat{\mathcal{D}}\big(\Upsilon_\beta,\Lambda\big)\cdot\widehat{\mathcal{A}}^{(22)}\big(\Upsilon_\beta,\Lambda\big)\right\}_{|\beta=0} \tag{3.35}$$

the function $\widehat{\mathcal{D}}\big(\Upsilon,\Lambda\big)$ being defined as in (3.20), while

$$\widehat{\mathcal{A}}^{(22)}\big(\Upsilon_\beta,\Lambda\big)=\left|\frac{\tau_0\big(\mathrm{i}c/2\mid\Upsilon_\beta\big)}{\tau_0\big(\mathrm{i}c/2\mid\Lambda\big)}-1\right|^2\cdot\big|1-\mathrm{e}^\beta\big|^2\cdot\widehat{\mathcal{W}}\big(\Upsilon_\beta;\Lambda\big)\cdot\widehat{\mathcal{R}}^{(22)}\big(\Upsilon_\beta;\Lambda\big) \tag{3.36}$$

where the two functions $\widehat{\mathcal{W}}$ and $\widehat{\mathcal{R}}^{(22)}$ are, respectively, defined in (3.17) and (3.19).

For the off-diagonal terms, one has

$$\Big|\frac{\langle\,\Upsilon\,|\mathbf{E}_1^{21}|\,\Lambda\,\rangle}{\|\Upsilon\|\cdot\|\Lambda\|}\Big|^2 = \widehat{\mathcal{D}}\big(\Upsilon,\Lambda\big)\cdot\widehat{\mathcal{A}}^{(21)}\big(\Upsilon,\Lambda\big) \quad \textit{and}$$

$$\Big|\frac{\langle\,\Upsilon\,|\mathbf{E}_1^{23}|\,\Lambda\,\rangle}{\|\Upsilon\|\cdot\|\Lambda\|}\Big|^2 = \widehat{\mathcal{D}}\big(\Upsilon,\Lambda\big)\cdot\widehat{\mathcal{A}}^{(23)}\big(\Upsilon,\Lambda\big)\,, \tag{3.37}$$

with

$$\widehat{\mathcal{A}}^{(21)}\big(\Upsilon,\Lambda\big)=\Big|\frac{\tau_0\big(\mathrm{i}c/2\mid\Upsilon\big)}{\tau_0\big(\mathrm{i}c/2\mid\Lambda\big)}-1\Big|^2\cdot\widehat{\mathcal{W}}\big(\Upsilon;\Lambda\big)\cdot\widehat{\mathcal{R}}^{(21)}\big(\Upsilon;\Lambda\big)\,, \tag{3.38}$$

$$\widehat{\mathcal{A}}^{(23)}\big(\Upsilon,\Lambda\big)=\Big|\frac{\tau_0\big(\mathrm{i}c/2\mid\Upsilon\big)}{\tau_0\big(\mathrm{i}c/2\mid\Lambda\big)}-1\Big|^2\cdot\widehat{\mathcal{W}}\big(\Upsilon;\Lambda\big)\cdot\widehat{\mathcal{R}}^{(23)}\big(\Upsilon;\Lambda\big)\,, \tag{3.39}$$

where

$$\widehat{\mathcal{R}}^{(23)}\big(\Upsilon;\Lambda\big)=\frac{1}{8c}\,\frac{\sin^2\Big[\pi\widehat{F}^{(2)}_{\Upsilon;\Lambda}\big(\overline{\mu}^{(2)}\big)\Big]}{\sin^2\Big[\pi\widehat{F}^{(2)}_{\Upsilon;\Lambda}\big(\overline{\lambda}^{(2)}\big)\Big]}\cdot\frac{\big|\det\Big[\mathrm{id}+\widehat{\mathbf{U}}_{\infty}\big(\Upsilon;\Lambda\big)\Big]\big|^2}{\det\Big[I_{|\Upsilon|}+\mathcal{K}_{\Upsilon}\Big]\cdot\det\Big[I_{|\Lambda|}+\mathcal{K}_{\Lambda}\Big]}$$

$$\widehat{\mathcal{R}}^{(21)}\big(\Upsilon;\Lambda\big)=\frac{1}{8c}\,\frac{\sin^2\Big[\pi\widehat{F}^{(2)}_{\Upsilon;\Lambda}\big(\overline{\mu}^{(2)}\big)\Big]}{\sin^2\Big[\pi\widehat{F}^{(2)}_{\Upsilon;\Lambda}\big(\overline{\lambda}^{(2)}\big)\Big]}\cdot\Big|\frac{h\big(\overline{\mu}^{(2)},\theta\big)}{h\big(\overline{\lambda}^{(2)},\theta\big)}\Big|^2$$
$$\times\frac{\big|1-k\big(\theta,\overline{\mu}^{(1)}\big)/k\big(\theta,\overline{\lambda}^{(1)}\big)\big|^2\cdot\big|\det\Big[\mathrm{id}+\widehat{\mathbf{V}}_{\theta}\big(\Upsilon;\Lambda\big)\Big]\big|^2}{\sin^2\Big[\pi\widehat{F}^{(2)}_{\Upsilon;\Lambda}\big(\theta\big)\Big]\cdot\det\Big[I_{|\Upsilon|}+\mathcal{K}_{\Upsilon}\Big]\cdot\det\Big[I_{|\Lambda|}+\mathcal{K}_{\Lambda}\Big]}\,.$$

The parameter θ is any of the Bethe roots $\{\mu_a^{(2)}\}_1^{N_\Upsilon^{(2)}}$ and the operator $\widehat{\mathbf{V}}_\theta\big(\Upsilon;\Lambda\big)$ is an integral operator on the contour $\mathcal{C}_{\Upsilon;\Lambda}$ defined by (3.6). It corresponds to a rank one perturbation of $\widehat{\mathbf{U}}_\theta\big(\Upsilon;\Lambda\big)$:

$$\widehat{\mathbf{V}}_\theta^{(ab)}\big(\Upsilon;\Lambda\big)(z,z') = \widehat{\mathbf{U}}_\theta^{(ab)}\big(\Upsilon;\Lambda\big)(z,z') - \widehat{G}^{(a)}_{\theta;L}\big(\Upsilon;\Lambda\big)(z)\cdot\widehat{G}^{(b)}_{\theta;R}\big(\Upsilon;\Lambda\big)(z') \tag{3.40}$$

where the functions $\widehat{G}^{(a)}_{\theta;L/R}(\Upsilon,\Lambda)(z)$ read

$$\widehat{G}^{(1)}_{\theta;L}\big(\Upsilon;\Lambda\big)(z)=\frac{\mathrm{i}c}{2}\Bigg\{\frac{1}{\theta-z+\mathrm{i}c/2}-\frac{1}{\theta-z-\mathrm{i}c/2}\cdot\frac{k\big(\theta,\overline{\mu}^{(1)}\big)}{k\big(\theta,\overline{\lambda}^{(1)}\big)}\Bigg\}\,, \tag{3.41}$$

$$\widehat{G}^{(2)}_{\theta;L}\big(\Upsilon;\Lambda\big)(z)=\frac{\mathrm{i}c}{2}\Bigg\{1-\frac{1}{f(\theta,z)}\cdot\frac{k\big(\theta,\overline{\mu}^{(1)}\big)}{k\big(\theta,\overline{\lambda}^{(1)}\big)}+\frac{\mathrm{i}c}{z-\theta+\mathrm{i}c}\Bigg\}\,, \tag{3.42}$$

$$\widehat{G}^{(1)}_{\theta;R}\big(\Upsilon;\Lambda\big)(z)=\frac{(c\pi)^{-1}}{1-\exp\Big\{2\mathrm{i}\pi\widehat{F}^{(1)}_{\Upsilon;\Lambda}\big(z\big)\Big\}}\cdot\frac{f\big(z,\overline{\lambda}^{(1)}\big)}{f\big(z,\overline{\mu}^{(1)}\big)}\cdot\Bigg\{\frac{k\big(\theta,\overline{\mu}^{(1)}\big)}{k\big(\theta,\overline{\lambda}^{(1)}\big)}-\frac{k\big(\overline{\lambda}^{(2)},z\big)}{k\big(\overline{\mu}^{(2)},z\big)}\Bigg\}$$
$$\cdot\Bigg\{1-\frac{k\big(\theta,\overline{\mu}^{(1)}\big)}{k\big(\theta,\overline{\lambda}^{(1)}\big)}\Bigg\}^{-1}\,, \tag{3.43}$$

$$\widehat{G}^{(2)}_{\theta;R}\big(\Upsilon;\Lambda\big)(z)=\frac{(-c\pi)^{-1}}{1-\exp\Big\{2\mathrm{i}\pi \widehat{F}^{(2)}_{\Upsilon;\Lambda}\big(z\big)\Big\}}\cdot\frac{f\big(\overline{\mu}^{(2)},z\big)}{f\big(\overline{\lambda}^{(2)},z\big)}\cdot\bigg\{\frac{k\big(z,\overline{\mu}^{(1)}\big)}{k\big(z,\overline{\lambda}^{(1)}\big)}-\frac{k\big(\theta,\overline{\mu}^{(1)}\big)}{k\big(\theta,\overline{\lambda}^{(1)}\big)}\bigg\} \times\bigg\{1-\frac{k\big(\theta,\overline{\mu}^{(1)}\big)}{k\big(\theta,\overline{\lambda}^{(1)}\big)}\bigg\}^{-1}. \tag{3.44}$$

Proof. Formula (3.35) is a direct consequence of the reconstruction (3.3) and (3.24), the decomposition of the scalar product (3.16) and the fact that $\beta\in\mathrm{i}\mathbb{R}$.

• **The amplitudes for $\mathbf{E}^{21}_1$**

The starting point to establish (3.37) and (3.38) is the identity (3.21). Since the two sets Λ, Υ are different, $\widehat{\mathcal{R}}^{(22)}\big(\Upsilon_\beta,\Lambda\big)$ is regular at $\beta=0$, so that the β-derivative has to act on $|1-\mathrm{e}^{\beta}|^2$. Thus, due to $\tau_0\big(z\mid\Lambda\big)=\lim_{\omega\to+\infty}\Big\{\tau_0\big(z\mid\Lambda^{(1)}_\omega\big)\Big\}$, one gets

$$\bigg|\frac{\langle\,\Upsilon\,|\mathbf{E}^{21}_1|\,\Lambda\,\rangle}{\|\Upsilon\|\cdot\|\Lambda\|}\bigg|^2=\bigg|\frac{\tau_0\big(\mathrm{i}c/2\mid\Upsilon\big)}{\tau_0\big(\mathrm{i}c/2\mid\Lambda\big)}-1\bigg|^2\lim_{\omega\to+\infty}\Big\{\widehat{\mathcal{W}}\big(\Upsilon,\Lambda^{(1)}_\omega\big)\cdot\widehat{\mathcal{D}}\big(\Upsilon,\Lambda^{(1)}_\omega\big)\cdot\widehat{\mathcal{R}}^{(22)}\big(\Upsilon,\Lambda^{(1)}_\omega\big)\Big\}. \tag{3.45}$$

It follows from straightforward algebra that

$$\widehat{\mathcal{W}}\big(\Upsilon,\Lambda^{(1)}_\omega\big)\;=\;\frac{\omega^2}{c^2}\cdot\widehat{\mathcal{W}}\big(\Upsilon,\Lambda\big)\cdot\Big(1+\mathrm{O}\big(\omega^{-1}\big)\Big). \tag{3.46}$$

Furthermore, we observe that, for any fixed z and $a=1,2$, it holds

$$\big(\xi^{(a)}_{\Lambda^{(1)}_\omega}\big)'(z)\;=\;\big(\xi^{(a)}_{\Lambda}\big)'(z)\;+\;\mathrm{O}\big(\omega^{-2}\big) \tag{3.47}$$

as well as

$$\widehat{F}^{(1)}_{\Upsilon,\Lambda^{(1)}_\omega}(z)\;=\;\widehat{F}^{(1)}_{\Upsilon,\Lambda}(z)\;+\;\underbrace{1/2\;+\;\vartheta_1(z-\omega)}_{1+\mathrm{O}(\omega^{-1})}\qquad\text{and}$$

$$\widehat{F}^{(2)}_{\Upsilon,\Lambda^{(1)}_\omega}(z)\;=\;\widehat{F}^{(2)}_{\Upsilon,\Lambda}(z)\;+\;\underbrace{1/2\;-\;\vartheta_2(z-\omega)}_{\mathrm{O}(\omega^{-1})}-\delta_{n^{(1)}_\Lambda,0}\,. \tag{3.48}$$

The above identities also imply that $\widehat{F}^{(1)}_{\Upsilon,\Lambda^{(1)}_\omega}(\omega)=1/2-c/(\pi\omega)+\mathrm{O}\big(\omega^{-2}\big)$. As a consequence, one gets that

$$\widehat{\mathcal{D}}\big(\Upsilon,\Lambda^{(1)}_\omega\big)\;=\;\frac{\widehat{\mathcal{D}}\big(\Upsilon,\Lambda\big)}{\pi L\omega^2\big(\xi^{(1)}_{\Lambda^{(1)}_\omega}\big)'(\omega)}\cdot\Big(1+\mathrm{O}\big(\omega^{-1}\big)\Big)\,. \tag{3.49}$$

It remains to deal with the limit of $\widehat{\mathcal{R}}^{(22)}\big(\Upsilon,\Lambda^{(1)}_\omega\big)$. The $\omega\to+\infty$ behaviour of the pre-factors in $\widehat{\mathcal{R}}^{(22)}\big(\Upsilon;\Lambda^{(1)}_\omega\big)$ is easily computed, leading to

$$\widehat{\mathcal{R}}^{(22)}\big(\Upsilon,\Lambda^{(1)}_\omega\big)=\frac{\det\big[I_{|\Lambda|}\,+\,\mathcal{K}_{|\Lambda|}\big]}{\det\big[I_{|\Lambda^{(1)}_\omega|}\,+\,\mathcal{K}_{|\Lambda^{(1)}_\omega|}\big]}\cdot\bigg|\frac{\det\big[\mathrm{id}\,+\,\widehat{\mathbf{U}}_\theta\big(\Upsilon;\Lambda^{(1)}_\omega\big)\big]}{\det\big[\mathrm{id}\,+\,\widehat{\mathbf{U}}_\theta\big(\Upsilon;\Lambda\big)\big]}\bigg|^2_{|\beta=0} \times\widehat{\mathcal{R}}^{(22)}\big(\Upsilon,\Lambda^{(1)}_\omega\big)\cdot\Big(1\,+\,\mathrm{O}\big(\omega^{-1}\,\big)\Big)\,. \tag{3.50}$$

The treatment of the ratio of determinants is a bit more involved since one has to extract out of the determinants the part which vanishes when $\omega \to +\infty$. We first focus on the norm determinant. One has the $N_{\Lambda}^{(1)} \times 1 \times N_{\Lambda}^{(2)}$ block form decomposition:

$$I_{|\Lambda_{\omega}^{(1)}|} + \mathcal{K}_{|\Lambda_{\omega}^{(1)}|} = \begin{pmatrix} \delta_{ab} + \widehat{K}^{(11)}_{\Lambda_{\omega}^{(1)}}\big(\lambda_a^{(1)}, \lambda_b^{(1)}\big) & \widehat{K}^{(11)}_{\Lambda_{\omega}^{(1)}}\big(\lambda_a^{(1)}, \omega\big) & \widehat{K}^{(12)}_{\Lambda_{\omega}^{(1)}}\big(\lambda_a^{(1)}, \lambda_b^{(2)}\big) \\ \widehat{K}^{(11)}_{\Lambda_{\omega}^{(1)}}\big(\omega, \lambda_b^{(1)}\big) & 1 + \widehat{K}^{(11)}_{\Lambda_{\omega}^{(1)}}\big(\omega, \omega\big) & \widehat{K}^{(12)}_{\Lambda_{\omega}^{(1)}}\big(\omega, \lambda_b^{(2)}\big) \\ \widehat{K}^{(21)}_{\Lambda_{\omega}^{(1)}}\big(\lambda_a^{(2)}, \lambda_b^{(1)}\big) & \widehat{K}^{(21)}_{\Lambda_{\omega}^{(1)}}\big(\lambda_a^{(2)}, \omega\big) & \delta_{ab} + \widehat{K}^{(2)}_{\Lambda_{\omega}^{(1)}}\big(\lambda_a^{(2)}, \lambda_b^{(2)}\big) \end{pmatrix} \tag{3.51}$$

where $\widehat{K}^{(ab)}_{\Lambda}(y,z)$ has been defined in (3.13). In virtue of (3.47), it follows that for any fixed y, z

$$\widehat{K}^{(ab)}_{\Lambda_{\omega}^{(1)}}(y,z) = \widehat{K}^{(ab)}_{\Lambda}(y,z) \cdot \Big(1 + \mathrm{O}\big(\omega^{-1}\big)\Big)$$
$$\text{while} \quad \Big|\widehat{K}^{(a1)}_{\Lambda_{\omega}^{(1)}}(y,\omega)\Big| + \Big|\widehat{K}^{(1a)}_{\Lambda_{\omega}^{(1)}}(\omega,y)\Big| = \mathrm{O}\big(\omega^{-2}\big)\,. \tag{3.52}$$

Note that the last bounds follow from the large-ω asymptotics

$$\big(\widehat{\xi}^{(1)}_{\Lambda_{\omega}^{(1)}}\big)'(\omega) = -\frac{1}{L}K_1(0) + \frac{c}{2\pi L\omega^2}\big(L - 2N_{\Lambda}^{(1)} + N_{\Lambda}^{(2)}\big) + \mathrm{O}\big(\omega^{-3}\big) \tag{3.53}$$

and the fact that $K_1(0) = 1/(\pi c) \neq 0$. These asymptotics also imply that

$$1 + \widehat{K}^{(11)}_{\Lambda_{\omega}^{(1)}}\big(\omega,\omega\big) = c \cdot \frac{L - 2N_{\Lambda}^{(1)} + N_{\Lambda}^{(2)}}{2\pi L\omega^2\big(\widehat{\xi}^{(a)}_{\Lambda_{\omega}^{(1)}}\big)'(\omega)} \cdot \big(1 + \mathrm{O}(\omega^{-2})\big)\,. \tag{3.54}$$

In order to extract the ω^{-2} decay out of $\det\big[I_{|\Lambda_{\omega}^{(1)}|} + \mathcal{K}_{|\Lambda_{\omega}^{(1)}|}\big]$, it is enough to "kill" the off-diagonal entries of the column associated with the root ω by making linear combinations of lines. Since the line associated with the root ω is of order $\mathrm{O}\big(\omega^{-2}\big)$, in doing so, we only modify the other entries of the matrix by a $\mathrm{O}\big(\omega^{-2}\big)$ quantity. Therefore, all-in-all,

$$\det\big[I_{|\Lambda_{\omega}^{(1)}|} + \mathcal{K}_{|\Lambda_{\omega}^{(1)}|}\big] = \det\big[I_{|\Lambda|} + \mathcal{K}_{|\Lambda|}\big] \cdot c \cdot \frac{L - 2N_{\Lambda}^{(1)} + N_{\Lambda}^{(2)}}{2\pi L\omega^2\big(\widehat{\xi}^{(1)}_{\Lambda_{\omega}^{(1)}}\big)'(\omega)} \cdot \big(1 + \mathrm{O}(\omega^{-2})\big)\,. \tag{3.55}$$

The strategy to extract the decaying term out of $\det\big[\mathrm{id} + \mathfrak{V}_{\theta}\big(\Upsilon; \Lambda_{\omega}^{(1)}\big)\big]$ is roughly similar. By doing contour deformations so as to explicitly evaluate the residue at ω, one recasts the determinant as

$$\det\big[\mathrm{id} + \widehat{\mathfrak{V}}_{\theta}\big(\Upsilon; \Lambda_{\omega}^{(1)}\big)\big] =$$
$$\det\begin{bmatrix} \mathrm{id} + \widehat{\mathfrak{V}}^{(11)}_{\theta}\big(\Upsilon; \Lambda_{\omega}^{(1)}\big)(z,y) & 2\mathrm{i}\pi\,\mathrm{Res}_{s=\omega}\Big(\widehat{\mathfrak{V}}^{(11)}_{\theta}\big(\Upsilon; \Lambda_{\omega}^{(1)}\big)(z,s)\Big) & \widehat{\mathfrak{V}}^{(12)}_{\theta}\big(\Upsilon; \Lambda_{\omega}^{(1)}\big)(z,y') \\ \widehat{\mathfrak{V}}^{(11)}_{\theta}\big(\Upsilon; \Lambda_{\omega}^{(1)}\big)(\omega,y) & 1 + 2\mathrm{i}\pi\,\mathrm{Res}_{s=\omega}\Big(\widehat{\mathfrak{V}}^{(11)}_{\theta}\big(\Upsilon; \Lambda_{\omega}^{(1)}\big)(\omega,s)\Big) & \widehat{\mathfrak{V}}^{(12)}_{\theta}\big(\Upsilon; \Lambda_{\omega}^{(1)}\big)(\omega,y') \\ \widehat{\mathfrak{V}}^{(21)}_{\theta}\big(\Upsilon; \Lambda_{\omega}^{(1)}\big)(z',y) & 2\mathrm{i}\pi\,\mathrm{Res}_{s=\omega}\Big(\widehat{\mathfrak{V}}^{(21)}_{\theta}\big(\Upsilon; \Lambda_{\omega}^{(1)}\big)(z',s)\Big) & \mathrm{id} + \widehat{\mathfrak{V}}^{(22)}_{\theta}\big(\Upsilon; \Lambda_{\omega}^{(1)}\big)(z',y') \end{bmatrix}. \tag{3.56}$$

The entries arising in the block determinant correspond to integral kernels where the unprimed variables belong to $\Gamma\big(\{\lambda_a^{(1)}\}_1^{N_{\Lambda}^{(1)}}\big)$ while the primed ones belong to $\Gamma\big(\{\mu_a^{(2)}\}_1^{N_{\Upsilon}^{(1)}}\big)$. After some algebra, one finds

$$\widehat{\mathbf{U}}_{\theta}^{(1a)}\big(\Upsilon;\Lambda_{\omega}^{(1)}\big)(\omega,y)=-\frac{\mathrm{i}c}{2\omega}\widehat{G}_{\theta;R}^{(a)}\big(\Upsilon;\Lambda\big)(y)\cdot\left\{1-\frac{k\big(\theta,\overline{\mu}^{(1)}\big)}{k\big(\theta,\overline{\lambda}^{(1)}\big)}\right\}\cdot\Big(1+\mathrm{O}\big(\omega^{-1}\big)\Big) \tag{3.57}$$

$$2\mathrm{i}\pi\,\mathrm{Res}_{s=\omega}\Big(\widehat{\mathbf{U}}_{\theta}^{(a1)}\big(\Upsilon;\Lambda_{\omega}^{(1)}\big)(z,s)\Big)=\widehat{G}_{\theta;L}^{(a)}\big(\Upsilon;\Lambda\big)(z)\cdot\Big(1+\mathrm{O}\big(\omega^{-1}\big)\Big)\,. \tag{3.58}$$

Finally, one has

$$1+2\mathrm{i}\pi\,\mathrm{Res}_{s=\omega}\Big(\widehat{\mathbf{U}}_{\theta}^{(11)}\big(\Upsilon;\Lambda_{\omega}^{(1)}\big)(\omega,s)\Big)=-\frac{\mathrm{i}c}{2\omega}\left\{1-\frac{k\big(\theta,\overline{\mu}^{(1)}\big)}{k\big(\theta,\overline{\lambda}^{(1)}\big)}\right\}\cdot\Big(1+\mathrm{O}\big(\omega^{-1}\big)\Big)\,. \tag{3.59}$$

Once these expansions are established, it is enough to factor out the diagonal term associated with the ω-column and then cancel the off-diagonal entries of the ω-column by linear combinations. Since, for bounded z, y one has

$$\widehat{\mathbf{U}}_{\theta}^{(ab)}\big(\Upsilon;\Lambda_{\omega}^{(1)}\big)(z,y)=\widehat{\mathbf{U}}_{\theta}^{(ab)}\big(\Upsilon;\Lambda\big)(z,y)\cdot\Big(1+\mathrm{O}\big(\omega^{-1}\big)\Big) \tag{3.60}$$

one gets

$$\det\big[\mathrm{id}+\widehat{\mathbf{U}}_{\theta}\big(\Upsilon;\Lambda_{\omega}^{(1)}\big)\big]=-\frac{\mathrm{i}c}{2\omega}\left\{1-\frac{k\big(\theta,\overline{\mu}^{(1)}\big)}{k\big(\theta,\overline{\lambda}^{(1)}\big)}\right\}\cdot\det\big[\mathrm{id}+\widehat{\mathbf{V}}_{\theta}\big(\Upsilon;\Lambda\big)\big]\cdot\Big(1+\mathrm{O}\big(\omega^{-1}\big)\Big)\,. \tag{3.61}$$

The above handlings thus leads to

$$\widehat{\mathcal{R}}^{(22)}\big(\Upsilon;\Lambda_{\omega}^{(1)}\big)=\frac{c^{2}\pi L\big(\widehat{\xi}^{(a)}_{\Lambda_{\omega}^{(1)}}\big)'(\omega)}{L-2N_{\Lambda}^{(1)}+N_{\Lambda}^{(2)}}\cdot\widehat{\mathcal{R}}^{(21)}\big(\Upsilon;\Lambda\big)\cdot\Big(1+\mathrm{O}\big(\omega^{-1}\big)\Big)\,. \tag{3.62}$$

Then, it remains to put the various large-ω asymptotics together.

• **The amplitudes for $\mathbf{E}_1^{23}$**

Likewise, the starting point to establish (3.37) and (3.39) is the identity (3.22). The limits of $\widehat{\mathcal{D}}$ and $\widehat{\mathcal{W}}$ are treated similarly to the above, leading to

$$\widehat{\mathcal{D}}\big(\Upsilon_{\omega}^{(2)},\Lambda\big)=\frac{\widehat{\mathcal{D}}\big(\Upsilon,\Lambda\big)}{\pi L\omega^{2}\big(\widehat{\xi}^{(2)}_{\Upsilon_{\omega}^{(2)}}\big)'(\omega)}\cdot\Big(1+\mathrm{O}\big(\omega^{-1}\big)\Big)\qquad\text{and}$$

$$\widehat{\mathcal{W}}\big(\Upsilon_{\omega}^{(2)},\Lambda\big)=\frac{\omega^{2}}{c^{2}}\cdot\widehat{\mathcal{W}}\big(\Upsilon,\Lambda\big)\cdot\Big(1+\mathrm{O}\big(\omega^{-2}\big)\Big)\,. \tag{3.63}$$

In order to compute the large-ω behaviour of $\widehat{\mathcal{R}}^{(23)}\big(\Upsilon_{\omega}^{(2)};\Lambda\big)$ it is convenient to set the arbitrary parameter to $\theta=\omega$. Then, the Fredholm determinant occurring in the numerator can be recast as

$$\begin{aligned}&\det\big[\mathrm{id}+\widehat{\mathbf{U}}_{\omega}\big(\Upsilon;\Lambda_{\omega}^{(1)}\big)\big]\\&\quad=\det\begin{bmatrix}\mathrm{id}+\widehat{\mathbf{U}}_{\omega}^{(11)}\big(\Upsilon;\Lambda_{\omega}^{(1)}\big)(z,y)&\widehat{\mathbf{U}}_{\omega}^{(12)}\big(\Upsilon;\Lambda_{\omega}^{(1)}\big)(z,y')&0\\\widehat{\mathbf{U}}_{\omega}^{(21)}\big(\Upsilon;\Lambda_{\omega}^{(1)}\big)(z',y)&\mathrm{id}+\widehat{\mathbf{U}}_{\omega}^{(22)}\big(\Upsilon;\Lambda_{\omega}^{(1)}\big)(z',y')&0\\ *&*&1\end{bmatrix}\\&\quad=\det\big[\mathrm{id}+\widehat{\mathbf{U}}_{\infty}\big(\Upsilon;\Lambda\big)\big]\cdot\Big(1+\mathrm{O}\big(\omega^{-1}\big)\Big)\,,\end{aligned} \tag{3.64}$$

where $*$ are some coefficients.

The $\omega \to +\infty$ behaviour of the norm determinant $\det\big[I_{|\Upsilon^{(2)}_{\omega}|} + \mathcal{K}_{\Upsilon^{(2)}_{\omega}}\big]$ is calculated as previously and one gets

$$\det\big[I_{|\Upsilon^{(2)}_{\omega}|} + \mathcal{K}_{\Upsilon^{(2)}_{\omega}}\big] = \frac{c \cdot \big(N^{(1)}_{\Upsilon} - 2N^{(2)}_{\Upsilon}\big)}{2\pi L\omega^2 \cdot \big(\widehat{\xi}^{(2)}_{\Upsilon^{(2)}_{\omega}}\big)'(\omega)} \cdot \det\big[I_{|\Upsilon|} + \mathcal{K}_{\Upsilon}\big] \cdot \Big(1 + \mathrm{O}\big(\omega^{-1}\big)\Big) . \quad (3.65)$$

Then, upon extracting the large-ω behaviour of all the pre-factors in $\widehat{\mathcal{R}}^{(22)}\big(\Upsilon^{(2)}_{\omega}; \Lambda\big)$ one obtains

$$\widehat{\mathcal{R}}^{(22)}\big(\Upsilon^{(2)}_{\omega}; \Lambda\big) = \frac{c^2 \pi L \big(\widehat{\xi}^{(a)}_{\Lambda^{(1)}_{\omega}}\big)'(\omega)}{N^{(1)}_{\Upsilon} - 2N^{(2)}_{\Upsilon}} \cdot \widehat{\mathcal{R}}^{(23)}(\Upsilon; \Lambda) \cdot \Big(1 + \mathrm{O}\big(\omega^{-1}\big)\Big) , \quad (3.66)$$

so that it solely remains to put all the pieces together to end the proof. □

Corollary 3.7. *The factorisations given in Proposition 3.6 is precisely of the form stated in the main conjecture.*

Proof. The statement relatively to the smooth parts $\widehat{\mathcal{A}}^{(ab)}$ is readily obtained by repeating the handlings of [36]. We refer the reader to that paper for more details. □

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Appendix A. Thermodynamic limit of the model

A.1. Non-linear integral equations and large-L expansion

In this sub-section we establish that the counting function (1.28) associated with a particle–hole excited state admits the large-L asymptotic behaviour

$$\widehat{\xi}^{(k)}_{\Lambda}(\omega) = \widehat{\sigma}^{(k)}(\omega) + \frac{1}{2L}\big(N^{(k)}_{\Omega} + 1\big) + \frac{1}{L}\widehat{\Psi}^{(k)}\Big(\omega \mid \big\{\hat{\lambda}^{(s)}_{\widetilde{p}^{(s)}_a}\big\} ; \big\{\hat{\lambda}^{(s)}_{\widetilde{h}^{(s)}_a}\big\}\Big) + \frac{1}{2L}\Big[\widehat{\boldsymbol{Z}}(\omega) \cdot (\boldsymbol{\kappa} - \boldsymbol{\mathfrak{n}})\Big]^{(k)} + \mathrm{O}\Big(\frac{1}{L^2}\Big) . \quad (A.1)$$

Above, the integers $\widetilde{h}^{(s)}_a$ and $\widetilde{p}^{(s)}_a$ are defined as

$$\big\{\widetilde{p}^{(s)}_a\big\}_1^{\widetilde{n}^{(s)}_p} = \begin{cases} \big\{p^{(s)}_a\big\}_1^{n^{(s)}} \cup \big\{N^{(s)}_{\Omega} + a\big\}_1^{\kappa^{(s)}} & \kappa^{(s)} \geq 0 \\ \big\{p^{(s)}_a\big\}_1^{n^{(s)}} & \kappa^{(s)} < 0 \end{cases} ,$$

$$\big\{\widetilde{h}^{(s)}_a\big\}_1^{\widetilde{n}^{(s)}_h} = \begin{cases} \big\{h^{(s)}_a\big\}_1^{n^{(s)}} & \kappa^{(s)} \geq 0 \\ \big\{h^{(s)}_a\big\}_1^{n^{(s)}} \cup \big\{N^{(s)}_{\Omega} + a\big\}_1^{|\kappa^{(s)}|} & \kappa^{(s)} < 0 \end{cases} .$$

The asymptotic expansion (A.1) involves the auxiliary function

$$\widehat{\Psi}^{(k)}\Big(\omega \mid \{\mu_a^{(s)}\}\,;\,\{\nu_a^{(s)}\}\Big) = \sum_{s=1}^{r}\Big\{\sum_{a=1}^{n_\mu^{(s)}}\widehat{\Phi}_{ks}\Big(\omega,\mu_a^{(s)}\Big) - \sum_{a=1}^{n_\nu^{(s)}}\widehat{\Phi}_{ks}\Big(\omega,\nu_a^{(s)}\Big)\Big\} \tag{A.2}$$

which is expressed in terms of the finite-size dressed phase matrix $\widehat{\Phi}_{ks} = \widehat{\boldsymbol{\Phi}}_s^{(k)}$. The finite-size counterparts $\widehat{\boldsymbol{\Phi}}_s^{(k)}$, $\widehat{\boldsymbol{\sigma}}$ and $\widehat{\boldsymbol{Z}}$ of the functions introduced in (1.25), (1.26) and (1.27) are defined as the solutions to the linear integral equations

$$\Big(\mathrm{id} + \widehat{\mathbf{K}}\Big)[\widehat{\boldsymbol{\sigma}}](\omega) = \boldsymbol{\sigma}_0(\omega) + \sum_{s=1}^{r}\frac{N_\Omega^{(s)}}{2L}\Big(\Xi_s\big(\omega,\widehat{q}^{\,(s)}\big) + \Xi_s\big(\omega,-\widehat{q}^{\,(s)}\big)\Big) \tag{A.3}$$

$$\Big(\mathrm{id} + \widehat{\mathbf{K}}\Big)[\widehat{\boldsymbol{\Phi}}_s(*,z)](\omega) = \boldsymbol{\Xi}_s(\omega,z) \quad \text{and} \quad \Big(\mathrm{id} + \widehat{\mathbf{K}}\Big)[\widehat{\boldsymbol{Z}}](\omega) = I_r \tag{A.4}$$

where the finite-size kernel acts as

$$\Big[\Big(\mathrm{id} + \widehat{\mathbf{K}}\Big)[\boldsymbol{f}](\omega)\Big]^{(k)} = \boldsymbol{f}^{(k)}(\omega) + \sum_{\ell=1}^{r}\int_{-\widehat{q}^{\,(\ell)}}^{\widehat{q}^{\,(\ell)}} \partial_\mu\theta_{k\ell}(\omega,\mu)\cdot\boldsymbol{f}^{(\ell)}(\mu)\cdot\mathrm{d}\mu\,. \tag{A.5}$$

Finally, $\widehat{q}^{\,(\ell)}$ is defined as the positive solution to

$$\widehat{\sigma}^{(k)}\big(\widehat{q}^{\,(k)}\big) = \frac{N_\Omega^{(k)}}{2L}\,. \tag{A.6}$$

We assume that this solution does exist and is furthermore unique. The condition $D^{(s)} - N_\Omega^{(s)}/L = \mathrm{O}\big(L^{-2}\big)$ ensures that $q^{(s)} - \widehat{q}^{\,(s)} = \mathrm{O}\big(L^{-2}\big)$ and thus $\widehat{\sigma}^{\,(s)} - \sigma^{(s)} = \mathrm{O}\big(L^{-2}\big)$. This being settled, the very definition of the roots $\hat{\lambda}_a^{(s)}$ then ensures that

$$\hat{\lambda}_a^{(s)} - \mu_a^{(s)} = \mathrm{O}\Big(L^{-1}\Big) \qquad \text{and} \qquad \mu^{(s)}_{N_\Omega^{(s)}+p} - \widehat{q}^{\,(s)} = \mathrm{O}\Big(\frac{p}{L}\Big)\,. \tag{A.7}$$

Given these pieces of information, one readily deduces the asymptotic expansion (1.32).

Under the hypothesis that the counting functions are strictly increasing on $\mathbb{R}$, following [15] one obtains the non-linear integral equation

$$\begin{aligned}
\widehat{\xi}^{(k)}_{\Lambda;\mathrm{sym}}(\omega) = {} & \sigma_0^{(k)}(\omega) + \frac{1}{2L}\big(\kappa^{(k)} - n_\Lambda^{(k)}\big) + \frac{1}{L}\Theta^{(k)}\Big(\omega \mid \{\hat{\lambda}^{(s)}_{\widetilde{p}_a^{(s)}}\}\,;\,\{\hat{\lambda}^{(s)}_{\widetilde{h}_a^{(s)}}\}\Big) \\
& - \sum_{s=1}^{r}\int_{\widehat{q}_-^{(s)}}^{\widehat{q}_+^{(s)}} \partial_\tau\theta_{ks}\big(\omega,\tau\big)\widehat{\xi}^{(s)}_{\Lambda;\mathrm{sym}}(\tau)\cdot\mathrm{d}\tau \\
& + \sum_{s=1}^{r}\frac{N_\Omega^{(s)}}{2L}\Big(\theta_{ks}\big(\omega,\widehat{q}_+^{(s)}\big) + \theta_{ks}\big(\omega,\widehat{q}_-^{(s)}\big)\Big) \\
& - \sum_{s=1}^{r}\sum_{\epsilon_s=\pm}\int_{(\widehat{\xi}_\Lambda^{(s)})^{-1}(\Gamma^{(s)}_{\epsilon_s})} \partial_\tau\theta_{ks}(\lambda,\tau)\ln\Big[1 - \mathrm{e}^{2\mathrm{i}\pi\epsilon_s L\widehat{\xi}_\Lambda^{(s)}(\tau)}\Big]\cdot\frac{\mathrm{d}\tau}{2\mathrm{i}\pi L}
\end{aligned} \tag{A.8}$$

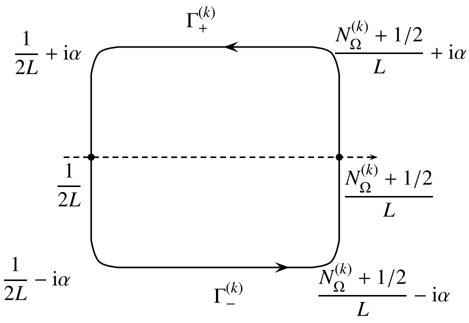

Fig. 1. The contours $\Gamma_+^{(k)} \cup \Gamma_-^{(k)}$.

where the endpoints of integration are defined as

$$\widehat{\xi}^{(k)}_{\Lambda;\mathrm{sym}}\big(\widehat{q}^{(s)}_{\pm}\big) = \pm\frac{N_\Omega^{(s)}}{2L} \qquad \text{with} \qquad \widehat{\xi}^{(k)}_{\Lambda;\mathrm{sym}}(\omega) = \widehat{\xi}^{(k)}_{\Lambda}(\omega) - \frac{1}{2L}\big(N_\Omega^{(k)}+1\big)\,. \tag{A.9}$$

Note that we have also introduced

$$\Theta^{(k)}\Big(\omega \mid \{\mu_a^{(s)}\}\,;\,\{\nu_a^{(s)}\}\Big) = \sum_{s=1}^{r}\Bigg\{\sum_{a=1}^{n_\mu^{(s)}}\theta_{ks}\Big(\omega,\mu_a^{(s)}\Big) - \sum_{a=1}^{n_\nu^{(s)}}\theta_{ks}\Big(\omega,\nu_a^{(s)}\Big)\Bigg\}\,. \tag{A.10}$$

Finally, the integration in the non-linear term runs through the contours depicted in Fig. 1.

The above can be recast as

$$\begin{aligned}\Big[\Big(\mathrm{id}+\widehat{\mathbf{K}}\Big)\big[\widehat{\boldsymbol{\xi}}_{\Lambda;\mathrm{sym}}\big](\omega)\Big]^{(k)} &= \sigma_0^{(k)}(\omega) + \sum_{s=1}^{r}\frac{N_\Omega^{(s)}}{2L}\Big(\theta_{ks}\big(\omega,\widehat{q}^{(s)}\big)+\theta_{ks}\big(\omega,-\widehat{q}^{(s)}\big)\Big) \\ &\quad + \frac{1}{2L}\big(\kappa^{(k)}-n_\Lambda^{(k)}\big) + \frac{1}{L}\Theta^{(k)}\Big(\omega \mid \{\hat{\lambda}^{(s)}_{\widetilde{p}_a^{(s)}}\}\,;\,\{\hat{\lambda}^{(s)}_{\widetilde{h}_a^{(s)}}\}\Big) \\ &\quad + \mathcal{R}^{(k)}\big[\boldsymbol{\xi}_\Lambda\big]\,,\end{aligned} \tag{A.11}$$

where

$$\begin{aligned}\mathcal{R}^{(k)}\big[\boldsymbol{\xi}_\Lambda\big] = &-\sum_{s=1}^{r}\sum_{\epsilon_s=\pm}\epsilon_s\int_{\epsilon_s\widehat{q}^{(s)}}^{\widehat{q}^{(s)}_{\epsilon_s}}\partial_\tau\theta_{ks}\big(\omega,\tau\big)\cdot\Big[\widehat{\xi}^{(s)}_\Lambda(\tau)-\widehat{\xi}^{(s)}_\Lambda\big(\widehat{q}^{(s)}_{\epsilon_s}\big)\Big]\cdot\mathrm{d}\tau \\ &-\sum_{s=1}^{r}\sum_{\epsilon_s=\pm}\int_{(\widehat{\xi}^{(s)}_\Lambda)^{-1}(\Gamma^{(s)}_{\epsilon_s})}\partial_\tau\theta_{ks}(\lambda,\tau)\ln\Big[1-\mathrm{e}^{2\mathrm{i}\pi\epsilon_s L\widehat{\xi}^{(s)}_\Lambda(\tau)}\Big]\cdot\frac{\mathrm{d}\tau}{2\mathrm{i}\pi L}\,.\end{aligned}$$

It then remains to invert the operator. The remainder is easily seen to be a $\mathrm{O}\big(L^{-2}\big)$ by means of Watson's lemma, which allows one to recover the expansion (A.1).

Finally, going back to the definition (A.9) of $\widehat{q}^{(s)}_{\pm}$ and (A.6) of $\widehat{q}^{(s)}$, one can deduce from the asymptotic expansion (A.1) that

$$\begin{aligned}\widehat{q}^{(k)}_{\pm} \pm \widehat{q}^{(k)} = \frac{-1}{L\cdot\big(\widehat{\sigma}^{(k)}\big)'\big(\pm\widehat{q}^{(k)}\big)}\cdot\Big\{&\widehat{\Psi}\Big(\pm\widehat{q}^{(k)} \mid \{\hat{\lambda}^{(s)}_{\widetilde{p}_a^{(s)}}\}\,;\,\{\hat{\lambda}^{(s)}_{\widetilde{h}_a^{(s)}}\}\Big) \\ &+\frac{1}{2}\widehat{\mathbf{Z}}\big(\pm\widehat{q}^{(k)}\big)\cdot(\boldsymbol{\kappa}-\mathfrak{n})\Big\}^{(k)} + \mathrm{O}\Big(\frac{1}{L^2}\Big)\,.\end{aligned} \tag{A.12}$$

A.2. Several identities satisfied by the dressed phase

In this subsection we derive various useful identities that are satisfied by the dressed phase.

Lemma A.1. *The dressed charge and phase matrices satisfy the differential identities*

$$\partial_\mu \Phi_{ks}(\lambda,\mu) = R_{ks}(\lambda,\mu)\,;$$
$$\partial_\lambda \Phi_{ks}(\lambda,\mu) = -R_{ks}(\lambda,\mu) + \sum_{\ell=1}^{r}\sum_{\epsilon_\ell=\pm}\epsilon_\ell R_{k\ell}\big(\lambda,\epsilon_\ell q^{(\ell)}\big)\cdot\Phi_{\ell s}\big(\epsilon_\ell q^{(\ell)},\mu\big) \tag{A.13}$$

and

$$\partial_\lambda Z_{ks}(\lambda) = R_{ks}\big(\lambda,q^{(s)}\big) - R_{ks}\big(\lambda,-q^{(s)}\big) - \sum_{\ell=1}^{r}\sum_{\epsilon_s,\epsilon_\ell=\pm}\epsilon_s\epsilon_\ell\, R_{k\ell}\big(\lambda,\epsilon_\ell q^{(\ell)}\big)\Phi_{\ell s}\big(\epsilon_\ell q^{(\ell)},\epsilon_s q^{(s)}\big). \tag{A.14}$$

Moreover, the dressed charge Φ *and the resolvent* R *enjoy the reflection properties*

$$\Phi_{ks}(\lambda,\mu) + \Phi_{sk}(\mu,\lambda) = \sum_{\ell=1}^{r}\sum_{\epsilon_\ell=\pm}\epsilon_\ell\Phi_{\ell k}\big(\epsilon_\ell q^{(\ell)},\lambda\big)\Phi_{\ell s}\big(\epsilon_\ell q^{(\ell)},\mu\big) \tag{A.15}$$
$$R_{k\ell}(\lambda,\mu) = R_{\ell k}(\mu,\lambda), \quad \Phi_{ks}(\lambda,\mu) = -\Phi_{ks}(-\lambda,-\mu)\,. \tag{A.16}$$

Proof. The two differential identities satisfied by the dressed phase matrix are obtained by differentiating the linear integral equations fulfilled by the matrix entries and integrating by parts, using the fact that the bare phases are functions of difference of arguments only. The derivative of the dressed charge matrix (A.14) can be computed by means of (A.13) and (1.46). The identity (A.16) relative to the resolvent is obtained by a direct inspection of the Neumann series representation for the resolvent and by using that $\partial_\mu\theta_{k\ell}(\lambda,\mu) = \partial_\lambda\theta_{\ell k}(\mu,\lambda)$. As for the second identity, one invokes the integral representation

$$\Phi_{k\ell}\big(\lambda,\mu\big) = \theta_{k\ell}\big(\lambda,\mu\big) - \sum_{s=1}^{r}\int_{-q^{(s)}}^{q^{(s)}} R_{ks}(\lambda,\nu)\theta_{s\ell}\big(\nu,\mu\big)\cdot \mathrm{d}\nu \tag{A.17}$$

and uses the symmetry properties of the bare phase. Finally, the identity (A.15) is obtained by introducing the function

$$f_{ks}(\lambda,\mu) = \Phi_{ks}(\lambda,\mu) + \Phi_{sk}(\mu,\lambda) \tag{A.18}$$

and computing its partial λ and μ derivatives by means of (A.13). This shows that

$$f_{ks}(\lambda,\mu) = \sum_{\ell=1}^{r}\sum_{\epsilon_\ell=\pm}\epsilon_\ell\Phi_{\ell k}\big(\epsilon_\ell q^{(\ell)},\lambda\big)\Phi_{\ell s}\big(\epsilon_\ell q^{(\ell)},\mu\big) + c_{ks}\,. \tag{A.19}$$

However, owing to (A.16), it holds $f_{ks}(\lambda,\mu) = -f_{ks}(-\lambda,-\mu)$ which implies that $c_{ks}=0$. □

In the case of rank one models, Slavnov [69] identified the mechanism which gives rise to a quadratic identity satisfied by the dressed phase. The latter allows one to find a closed expression for the inverse of the dressed charge in terms of the dressed phase. We establish a generalisation of this identity below, hence giving rise to the

Proposition A.2. *The inverse of the matrix*

$$\mathcal{Z}_{k\ell} \equiv Z_{k\ell}\big(q^{(k)}\big) = \delta_{k\ell} + \Phi_{k\ell}\big(q^{(k)}, -q^{(\ell)}\big) - \Phi_{k\ell}\big(q^{(k)}, q^{(\ell)}\big) \tag{A.20}$$

takes the form

$$\Big[\mathcal{Z}^{-1}\Big]_{k\ell} = \delta_{k\ell} - \Phi_{k\ell}\big(q^{(k)}, -q^{(\ell)}\big) - \Phi_{k\ell}\big(q^{(k)}, q^{(\ell)}\big)\,. \tag{A.21}$$

Proof. Starting from the integral representation (A.17) for the dressed phase matrix and using the symmetry properties of the bare phase $\theta_{kp}(\lambda,\mu) = \theta_{pk}(-\mu,-\lambda) = -\theta_{pk}(\mu,\lambda)$, one gets

$$\begin{aligned}
\delta &= \Phi_{k\ell}\big(\lambda,-\mu\big) - \Phi_{\ell k}\big(\mu,-\lambda\big) + \Phi_{k\ell}\big(\lambda,\mu\big) + \Phi_{\ell k}\big(\mu,\lambda\big) \\
&= \sum_{s=1}^{r} \int_{-q^{(s)}}^{q^{(s)}} \Big\{ -R_{ks}(\lambda,\nu)\theta_{sl}(\nu,-\mu) + R_{\ell s}(\mu,\nu)\theta_{sk}(\nu,-\lambda) - R_{ks}(\lambda,\nu)\theta_{s\ell}(\nu,\mu) \\
&\qquad - R_{\ell s}(\mu,\nu)\theta_{sk}(\nu,\lambda) \Big\} \\
&= \sum_{s=1}^{r} \int_{-q^{(s)}}^{q^{(s)}} \Big\{ -R_{ks}(\lambda,\nu)\theta_{\ell s}(\mu,-\nu) + R_{\ell s}(\mu,-\nu)\theta_{ks}(\lambda,\nu) \\
&\qquad + R_{ks}(\lambda,\nu)\theta_{\ell s}(\mu,\nu) + R_{\ell s}(\mu,\nu)\theta_{ks}(\lambda,\nu) \Big\}\,.
\end{aligned} \tag{A.22}$$

Re-expressing the bare phase in terms of the dressed phase and of the resolvent by means of (A.17) one observes that the quadratic terms in the resolvent kernel cancel out so that one obtains

$$\begin{aligned}
\delta &= \sum_{s=1}^{r} \int_{-q^{(s)}}^{q^{(s)}} \Big\{ -R_{ks}(\lambda,\nu)\Phi_{\ell s}(\mu,-\nu) + R_{\ell s}(\mu,-\nu)\Phi_{ks}(\lambda,\nu) + R_{ks}(\lambda,\nu)\Phi_{\ell s}(\mu,\nu) \\
&\qquad + R_{\ell s}(\mu,\nu)\Phi_{ks}(\lambda,\nu) \Big\} \\
&= \sum_{s=1}^{r} \int_{-q^{(s)}}^{q^{(s)}} \partial_\nu \Big\{ \Phi_{ks}(\lambda,\nu)\Phi_{\ell s}(\mu,\nu) - \Phi_{ks}(\lambda,\nu)\Phi_{\ell s}(\mu,-\nu) \Big\} \cdot \mathrm{d}\nu \\
&= \sum_{s=1}^{r} \sum_{\epsilon_s=\pm} \epsilon_s \Big\{ \Phi_{ks}\big(\lambda,\epsilon_s q^{(s)}\big)\Phi_{\ell s}\big(\mu,\epsilon_s q^{(s)}\big) + \Phi_{ks}\big(\lambda,-\epsilon_s q^{(s)}\big)\Phi_{\ell s}\big(\mu,\epsilon_s q^{(s)}\big) \Big\}
\end{aligned} \tag{A.23}$$

where we used the first equation given in (A.13). Specialising to $\lambda = q^{(k)}$ and $\mu = q^{(\ell)}$ one gets the identity

$$\Phi_{k\ell}\big(q^{(k)},-q^{(\ell)}\big) - \Phi_{\ell k}\big(q^{(\ell)},-q^{(k)}\big) + \Phi_{k\ell}\big(q^{(k)},q^{(\ell)}\big) + \Phi_{\ell k}\big(q^{(\ell)},q^{(k)}\big)$$
$$= \sum_{s=1}^{r}\Big\{ \Phi_{ks}\big(q^{(k)},q^{(s)}\big)\Phi_{\ell s}\big(q^{(\ell)},q^{(s)}\big) - \Phi_{ks}\big(q^{(k)},-q^{(s)}\big)\Phi_{\ell s}\big(q^{(\ell)},-q^{(s)}\big)$$
$$+ \Phi_{ks}\big(q^{(k)},-q^{(s)}\big)\Phi_{\ell s}\big(q^{(\ell)},q^{(s)}\big) - \Phi_{ks}\big(q^{(k)},q^{(s)}\big)\Phi_{\ell s}\big(q^{(\ell)},-q^{(s)}\big)\Big\} \tag{A.24}$$

which is all that one needs to check that the matrix

$$\mathcal{M}_{k\ell} = \delta_{k\ell} - \Phi_{k\ell}\big(q^{(k)},-q^{(\ell)}\big) - \Phi_{k\ell}\big(q^{(k)},q^{(\ell)}\big) \tag{A.25}$$

satisfies $\mathcal{M}\cdot\mathcal{Z} = I_r$. □

A.3. The large-L expansion of energies and momenta

Given a functions $f^{(k)}$ holomorphic in a neighbourhood of $[-q^{(k)};q^{(k)}]$ consider the sums

$$S_\Lambda[f] = \sum_{s=1}^{r}\sum_{a=1}^{N_\Lambda^{(k)}} f^{(s)}\big(\lambda_a^{(s)}\big)\,. \tag{A.26}$$

By repeating the steps outlined in the derivation of the non-linear integral equation satisfied by the counting function one recasts $S_\Lambda[f]$ in the form

$$S_\Lambda[f] = L\sum_{s=1}^{r}\int_{-\widehat{q}^{(s)}}^{\widehat{q}^{(s)}} f^{(s)}(\tau)\Big(\widehat{\sigma}^{(s)}(\tau)\Big)'\mathrm{d}\tau + \mathcal{S}^{(0)}[f] + \delta S_\Lambda[f] \tag{A.27}$$

where $\delta S_\Lambda[f] = \mathrm{O}\big(L^{-1}\big)$ and

$$\mathcal{S}^{(0)}[f] = \sum_{s=1}^{r}\int_{-\widehat{q}^{(s)}}^{\widehat{q}^{(s)}} f^{(s)}(\tau)\Big\{\partial_\tau\widehat{\Psi}^{(s)}\Big(\tau \mid \big\{\hat{\lambda}^{(\ell)}_{\widetilde{p}_a^{(\ell)}}\big\};\ \big\{\hat{\lambda}^{(\ell)}_{\widetilde{h}_a^{(\ell)}}\big\}\Big) + \frac{1}{2}\Big[\widehat{\boldsymbol{Z}}'(\tau)\cdot(\boldsymbol{\kappa}-\mathfrak{n})\Big]^{(s)}\Big\}\cdot\mathrm{d}\tau$$
$$+ \sum_{s=1}^{r}\epsilon_s\int_{\epsilon_s\widehat{q}^{(s)}}^{\widehat{q}^{(s)}_{\epsilon_s}} f^{(s)}(\tau)\partial_\tau\widehat{\sigma}^{(s)}(\tau)\cdot\mathrm{d}\tau + \sum_{s=1}^{r}\Big\{\sum_{a=1}^{\widetilde{n}_p^{(s)}} f^{(s)}\big(\hat{\lambda}^{(s)}_{\widetilde{p}_a^{(s)}}\big) - \sum_{a=1}^{\widetilde{n}_h^{(s)}} f^{(s)}\big(\hat{\lambda}^{(s)}_{\widetilde{h}_a^{(s)}}\big)\Big\}. \tag{A.28}$$

Owing to $\big|\widehat{q}^{(s)} - q^{(s)}\big| = \mathrm{O}\big(L^{-2}\big)$, one can replace $\widehat{q}^{(s)}$ by its $L\to+\infty$ limit without altering the form of the asymptotic expansion up to $\mathrm{O}\big(L^{-1}\big)$. After some algebra, one recasts $\mathcal{S}^{(0)}[f]$ in the form

$$\mathcal{S}^{(0)}[f] = -\sum_{s=1}^{r}\sum_{\epsilon_s=\pm}\epsilon_s F^{(s)}[f]\big(\epsilon_s q^{(s)}\big)\,\big[\mathcal{Z}^{\mathrm{t}}\cdot\frac{\boldsymbol{\kappa}-\mathfrak{n}}{2}\big]^{(s)}$$
$$+ \sum_{s=1}^{r}\Big\{\sum_{a=1}^{\widetilde{n}_p^{(s)}}\mathcal{U}^{(s)}[f]\big(\hat{\lambda}^{(s)}_{\widetilde{p}_a^{(s)}}\big) - \sum_{a=1}^{\widetilde{n}_h^{(s)}}\mathcal{U}^{(s)}[f]\big(\hat{\lambda}^{(s)}_{\widetilde{h}_a^{(s)}}\big)\Big\} + \mathrm{O}\Big(L^{-1}\Big)$$

where

$$F^{(k)}[f](\lambda) = f^{(k)}(\lambda) - \sum_{s=1}^{r} \int_{-q^{(s)}}^{q^{(s)}} R_{ks}(\lambda, \nu) f^{(s)}(\nu) \cdot \mathrm{d}\nu \tag{A.29}$$

and

$$\mathcal{U}^{(s)}[f](\lambda) = F^{(k)}[f](\lambda) - \sum_{\ell=1}^{r} \sum_{\epsilon_\ell=\pm} \epsilon_\ell \, F^{(\ell)}[f](\epsilon_\ell q^{(\ell)}) \Phi_{\ell s}(\epsilon_\ell q^{(\ell)}, \lambda) . \tag{A.30}$$

In order to obtain the first terms in the large-L expansion of the momenta and energies of the eigenstates one should observe that

$$\sum_{\epsilon_s=\pm} \epsilon_s F^{(s)}[\sigma_0](\epsilon_s q^{(s)}) = 2\sum_{\ell=1}^{r} \sigma^{(\ell)}(\widehat{q}^{(\ell)}) [\mathcal{Z}^{-1}]_{s\ell} \tag{A.31}$$

while

$$F^{(k)}[\sigma_0](\lambda) = \sigma^{(k)}(\lambda) - \sum_{\ell=1}^{r} \sum_{\epsilon_\ell=\pm} \sigma^{(\ell)}(\widehat{q}^{(\ell)}) \Phi_{k\ell}(\lambda, \epsilon_\ell \widehat{q}^{\ell}) \tag{A.32}$$

so that, owing to (A.15), $\mathcal{U}^{(s)}[\sigma_0](\lambda) = \sigma^{(s)}(\lambda)$. The situation with the dressed energy is even simpler since, by construction $\varepsilon(\pm q^{(s)}) = 0$ so that $\mathcal{U}^{(s)}[\varepsilon_0](\lambda) = F^{(s)}[\varepsilon_0](\lambda) = \varepsilon^{(s)}(\lambda)$. In order to obtain (1.35)–(1.36), it remains to invoke (A.7).

Appendix B. Proof of Proposition 3.1

The work [8] provided a determinant based representation for the scalar product $\mathcal{S}_\beta(\Upsilon_\beta, \Lambda)$ that was introduced in (3.1). For the intermediate calculations that we ought to carry out below, it is convenient to agree upon the following re-parametrisation of the sets $\Lambda = \{\{\lambda_k^{(a)}\}_{k=1}^{N_\Lambda^{(a)}}, a = 1, 2\}$ and $\Upsilon_\beta = \{\{\mu_k^{(a)}\}_{k=1}^{N_\Upsilon^{(a)}}, a = 1, 2\}$. In what concerns the variables of the first type we set

$$\lambda_a^{(1)} - \mathrm{i}\frac{c}{2} = u_a^B \qquad \text{and} \qquad \mu_a^{(1)} - \mathrm{i}\frac{c}{2} = u_a^C \qquad \text{for} \quad a = 1, \dots, N \equiv N_\Lambda^{(1)} = N_\Upsilon^{(1)} , \tag{B.1}$$

whereas, in what concerns the variables of the second type,

$$\lambda_a^{(2)} - \mathrm{i}c = v_a^B \qquad \text{and} \qquad \mu_a^{(2)} - \mathrm{i}c = v_a^C \qquad \text{for} \quad a = 1, \dots, M \equiv N_\Lambda^{(2)} = N_\Upsilon^{(2)} . \tag{B.2}$$

Within such a re-parametrisation, we are precisely in the normalisation adopted in [8]. The latter is slightly more adapted for dealing with the large-size formulae that ought to be handled in the course of the proof. We start with the representation

$$\begin{aligned} &\mathcal{S}_\beta(\{u_a^C\}_1^N, \{v_a^C\}_1^M \mid \{u_a^B\}_1^N, \{v_a^B\}_1^M) \\ &\quad = f(\overline{v}^C, \overline{u}^C) \cdot f(\overline{v}^B, \overline{u}^B) \cdot t(\overline{v}^C, \overline{u}^B) \cdot \Delta_N'(\overline{u}^C)\Delta_N(\overline{u}^B) \\ &\qquad \times \Delta_M'(\overline{v}^C)\Delta_M(\overline{v}^B) \cdot h^2(\overline{v}^C, \overline{u}^B) \cdot h(\overline{u}^B, \overline{u}^C) \cdot h(\overline{v}^B, \overline{v}^C) \cdot \det_{N+M}[\mathcal{N}] . \end{aligned} \tag{B.3}$$

In addition to the functions that were already introduced in the proposition, we also need to define

$$t(x,y) = \frac{-c^2}{(x-y)\cdot(x-y+\mathrm{i}c)}, \quad \Delta_N(\overline{u}^C) = \prod_{j<k}^{N} g(u_j^C, u_k^C) \qquad \text{and}$$

$$\Delta'_N(\overline{u}^C) = \prod_{j>k}^{N} g(u_j^C, u_k^C). \tag{B.4}$$

The function g appearing above reads

$$g(x,y) = \frac{\mathrm{i}c}{x-y}. \tag{B.5}$$

Finally the matrix $\mathcal{N}$ reads, in its block decomposition subordinate to a splitting in respect to the two types of roots u and v,

$$\mathcal{N} = \begin{pmatrix} \mathrm{e}^{\beta} t\big(u_k^B, u_j^C\big) + t\big(u_j^C, u_k^B\big) V_{11}\big(u_k^B\big) & \mathrm{e}^{\beta} t\big(v_k^C, u_j^C\big) V_{12}\big(v_k^C\big) \\ t\big(v_j^B, u_k^B\big) V_{21}\big(u_k^B\big) & t\big(v_j^B, v_k^C\big) + \kappa t\big(v_k^C, v_j^B\big) V_{22}\big(v_k^C\big) \end{pmatrix}$$
$$= \begin{pmatrix} \mathcal{N}^{(11)}\big(u_j^C, u_k^B\big) & \mathcal{N}^{(12)}\big(u_j^C, v_k^C\big) \\ \mathcal{N}^{(21)}\big(v_j^B, u_k^B\big) & \mathcal{N}^{(22)}\big(v_j^B, v_k^C\big) \end{pmatrix}. \tag{B.6}$$

Above, we have introduced four shorthand notations

$$V_{11}(\lambda) = \frac{f\big(\overline{v}^B, \lambda\big) h\big(\overline{u}^C, \lambda\big) h\big(\lambda, \overline{u}^B\big)}{f\big(\overline{v}^C, \lambda\big) h\big(\overline{u}^B, \lambda\big) h\big(\lambda, \overline{u}^C\big)} \qquad V_{12}(\lambda) = \frac{h\big(\lambda, \overline{u}^C\big) h\big(\overline{v}^C, \lambda\big)}{h\big(\lambda, \overline{u}^B\big) h\big(\overline{v}^B, \lambda\big)} \tag{B.7}$$

$$V_{21}(\lambda) = \big[V_{12}(\lambda)\big]^{-1} \qquad V_{22}(\lambda) = \frac{f\big(\lambda, \overline{u}^C\big) h\big(\overline{v}^C, \lambda\big) h\big(\lambda, \overline{v}^B\big)}{f\big(\lambda, \overline{u}^B\big) h\big(\overline{v}^B, \lambda\big) h\big(\lambda, \overline{v}^C\big)}. \tag{B.8}$$

Let A be the $(N+M)\times(N+M)$ matrix written in block form

$$A = \begin{pmatrix} A^{(11)} & 0_{N\times M} \\ A^{(21)} & A^{(22)} \end{pmatrix} \tag{B.9}$$

where we agree upon

$$A_{kj}^{(11)} = \frac{1}{u_j^C - u_k^B} \frac{\prod_{s=1}^{N}\big(u_j^C - u_s^B\big)}{\prod_{s=1, \neq j}^{N}\big(u_j^C - u_s^C\big)}, \qquad A_{kj}^{(21)} = \frac{\prod_{s=1}^{N}\big(u_j^C - u_s^B\big)}{\prod_{s=1, \neq j}^{N}\big(u_j^C - u_s^C\big)} \tag{B.10}$$

and finally

$$A_{kj}^{(22)} = \frac{\prod_{s=1}^{N}\big(v_j^B - v_s^C\big)}{\prod_{s=1, \neq j}^{N}\big(v_j^B - v_s^B\big)} \cdot \times \begin{cases} \mathrm{i}c \cdot \big(v_j^B - v_k^C\big)^{-1} & k \neq M \\ 1 & k = M \end{cases}. \tag{B.11}$$

It is readily seen that

$$\operatorname*{det}_{N+M}\big[A\big] = \big(\mathrm{i}c\big)^{M} \frac{\Delta_N\big(\overline{u}^{C}\big)\Delta_M\big(\overline{v}^{B}\big)}{\Delta_N\big(\overline{u}^{B}\big)\Delta_M\big(\overline{v}^{C}\big)} \cdot \frac{\not{g}\big(\overline{v}^{C}, v_M^{C}\big)}{g\big(\overline{v}^{B}, v_M^{C}\big)} . \tag{B.12}$$

Above, we agree that in the product $\not{g}\big(\overline{v}^{C}, v_k^{C}\big)$ one should omit the index $a=k$ which produces a singular term, namely

$$\not{g}\big(\overline{v}^{C}, v_k^{C}\big) = \prod_{\substack{a=1\\ \neq k}}^{N} g\big(v_a^{B}, v_k^{B}\big) . \tag{B.13}$$

It is easy to derive, with the help of contour integrals, the sum identities

$$\begin{aligned}
&\sum_{j=1}^{N} \frac{-c^2}{\big(x-u_j^{C}\big)\cdot\big(x-u_j^{C}\pm \mathrm{i}c\big)} \cdot \frac{\prod_{s=1}^{N}\big(u_j^{C}-u_s^{B}\big)}{\prod_{\substack{s=1\\ \neq j}}^{N}\big(u_j^{C}-u_s^{C}\big)} \cdot \frac{1}{\big(u_j^{C}-u_k^{B}\big)^{\alpha}} \\
&= \frac{\mp \mathrm{i}c}{\big(x-u_k^{B}\pm \mathrm{i}c\big)^{\alpha}} \prod_{s=1}^{N}\left\{\frac{x-u_s^{B}\pm \mathrm{i}c}{x-u_s^{C}\pm \mathrm{i}c}\right\} \pm \frac{\mathrm{i}c\delta_{\alpha,1}\delta_{x,u_k^{B}}}{\prod_{s=1}^{N}\big(u_k^{B}-u_s^{C}\big)} \cdot \prod_{\substack{s=1\\ \neq k}}^{N}\big(u_k^{B}-u_s^{B}\big) \\
&\quad \pm \mathrm{i}c \frac{1-\delta_{x,u_k^{B}}}{\big(x-u_k^{B}\big)^{\alpha}} \cdot \prod_{s=1}^{N}\left\{\frac{x-u_s^{B}}{x-u_s^{C}}\right\} .
\end{aligned} \tag{B.14}$$

There, we assume that $\alpha \in \{0, 1\}$ and that the parameters u_a^{B} and u_a^{C} are all generic. In particular, setting $\alpha = 0$ yields

$$\sum_{j=1}^{N} \frac{-c^2}{\big(x-u_j^{C}\big)\cdot\big(x-u_j^{C}\pm \mathrm{i}c\big)} \frac{\prod_{s=1}^{N}\big(u_j^{C}-u_s^{B}\big)}{\prod_{\substack{s=1\\ \neq j}}^{N}\big(u_j^{C}-u_s^{C}\big)} = \pm \mathrm{i}c \prod_{s=1}^{N}\left\{\frac{x-u_s^{B}}{x-u_s^{C}}\right\} \mp \mathrm{i}c \prod_{s=1}^{N}\frac{x-u_s^{B}\pm \mathrm{i}c}{x-u_s^{C}\pm \mathrm{i}c} . \tag{B.15}$$

Finally, a similar identity involving the v-like variables reads

$$\begin{aligned}
&\sum_{j=1}^{M} \frac{-c^2}{\big(x-v_j^{B}\big)\cdot\big(x-v_j^{B}\pm \mathrm{i}c\big)} \cdot \frac{\prod_{s=1}^{M}\big(v_j^{B}-v_s^{C}\big)}{\prod_{\substack{s=1\\ \neq j}}^{M}\big(v_j^{B}-v_s^{B}\big)} \cdot \frac{1}{\big(v_j^{B}-v_k^{C}\big)^{\alpha}} \\
&= \frac{\mp \mathrm{i}c}{\big(x-v_k^{C}\pm \mathrm{i}c\big)^{\alpha}} \prod_{s=1}^{M}\left\{\frac{x-v_s^{C}\pm \mathrm{i}c}{x-v_s^{B}\pm \mathrm{i}c}\right\} \pm \frac{\mathrm{i}c\delta_{\alpha,1}\delta_{x,v_k^{C}}}{\prod_{s=1}^{M}\big(v_k^{C}-v_s^{B}\big)} \cdot \prod_{\substack{s=1\\ \neq k}}^{M}\big(v_k^{C}-v_s^{C}\big) \\
&\quad \pm \mathrm{i}c \frac{1-\delta_{x,v_k^{C}}}{\big(x-v_k^{C}\big)^{\alpha}} \cdot \prod_{s=1}^{M}\left\{\frac{x-v_s^{C}}{x-v_s^{B}}\right\} .
\end{aligned} \tag{B.16}$$

A little algebra yields the identities necessary for computing the matrix products relative to the lines arising in the upper block:

$$\sum_{j=1}^{N} A_{kj}^{(11)} \mathcal{N}^{(11)}\big(u_j^C, u_\ell^B\big) = +\Big[\mathrm{e}^{\beta} - V_{11}\big(u_\ell^B\big)\Big]\delta_{k\ell} \cdot \frac{g\big(u_k^B, \overline{u}^C\big)}{\not{g}\big(u_k^B, \overline{u}^B\big)} - \mathrm{i}c \cdot \frac{h\big(u_\ell^B, \overline{u}^B\big)}{h\big(u_\ell^B, \overline{u}^C\big)} \cdot \left\{ \frac{\mathrm{e}^{\beta}}{u_\ell^B - u_k^B + \mathrm{i}c} + \frac{1}{u_k^B - u_\ell^B + \mathrm{i}c} \cdot \frac{f\big(\overline{v}^B, u_\ell^B\big)}{f\big(\overline{v}^C, u_\ell^B\big)} \right\} \tag{B.17}$$

and similarly, using the convention given in (B.13),

$$\sum_{j=1}^{N} A_{kj}^{(11)} \mathcal{N}^{(12)}\big(u_j^C, v_\ell^C\big) = \mathrm{e}^{\beta}\mathrm{i}c \frac{h(\overline{v}^C, v_\ell^C)}{h(\overline{v}^B, v_\ell^C)} \left\{ \frac{1}{\big(v_\ell^C - u_k^B\big)} \cdot \frac{f\big(v_\ell^C, \overline{u}^C\big)}{f\big(v_\ell^C, \overline{u}^B\big)} - \frac{1}{\big(v_\ell^C - u_k^B + \mathrm{i}c\big)} \right\}. \tag{B.18}$$

Very similar manipulations yield the identities necessary for computing the matrix products relative to the lines arising in the bottom block:

$$\sum_{j=1}^{N} A_{kj}^{(21)} \mathcal{N}^{(11)}\big(u_j^C, u_\ell^B\big) = \mathrm{i}c \frac{h\big(u_\ell^B, \overline{u}^B\big)}{h\big(u_\ell^B, \overline{u}^C\big)} \cdot \left\{ \frac{f\big(\overline{v}^B, u_\ell^B\big)}{f\big(\overline{v}^C, u_\ell^B\big)} - \mathrm{e}^{\beta} \right\} \tag{B.19}$$

and

$$\sum_{j=1}^{N} A_{kj}^{(21)} \mathcal{N}^{(12)}\big(u_j^C, v_\ell^C\big) = \mathrm{e}^{\beta}\mathrm{i}c \frac{h\big(\overline{v}^C, v_\ell^C\big)}{h\big(\overline{v}^B, v_\ell^C\big)} \cdot \left\{ \frac{f\big(v_\ell^C, \overline{u}^C\big)}{f\big(v_\ell^C, \overline{u}^B\big)} - 1 \right\}. \tag{B.20}$$

Finally, one gets that

$$\sum_{j=1}^{M} A_{kj}^{(22)} \mathcal{N}^{(21)}\big(v_j^B, u_\ell^B\big) = \mathrm{i}c \frac{h(u_\ell^B, \overline{u}^B)}{h(u_\ell^B, \overline{u}^C)} \left\{ \Big(\frac{\mathrm{i}c}{u_\ell^B - v_k^C - \mathrm{i}c}\Big)^{1-\delta_{kM}} - \Big(\frac{\mathrm{i}c}{u_\ell^B - v_k^C}\Big)^{1-\delta_{kM}} \cdot \frac{f\big(\overline{v}^B, u_\ell^B\big)}{f\big(\overline{v}^C, u_\ell^B\big)} \right\} \tag{B.21}$$

and similarly, for $k = 1, \dots, M-1$

$$\sum_{j=1}^{M} A_{kj}^{(22)} \mathcal{N}^{(22)}\big(v_j^B, v_\ell^C\big) = -\Big[1 - \mathrm{e}^{\beta} V_{22}\big(v_\ell^C\big)\Big]\delta_{k\ell} \cdot \mathrm{i}c \cdot \frac{g\big(v_k^C, \overline{v}^B\big)}{\not{g}\big(v_k^C, \overline{v}^C\big)} - \mathrm{i}c \cdot \frac{h\big(\overline{v}^C, v_\ell^C\big)}{h\big(\overline{v}^B, v_\ell^C\big)} \cdot \left\{ \frac{\mathrm{i}c}{v_k^C - v_\ell^C + \mathrm{i}c} + \frac{\mathrm{e}^{\beta}\mathrm{i}c}{v_\ell^C - v_k^C + \mathrm{i}c} \cdot \frac{f\big(v_\ell^C, \overline{u}^C\big)}{f\big(v_\ell^C, \overline{u}^B\big)} \right\} \tag{B.22}$$

whereas, for $k = M$, one has

$$\sum_{j=1}^{M} A_{Mj}^{(22)} \mathcal{N}^{(22)}\big(v_j^B, v_\ell^C\big) = \mathrm{i}c \cdot \frac{h\big(\overline{v}^C, v_\ell^C\big)}{h\big(\overline{v}^B, v_\ell^C\big)} \cdot \left\{ 1 - \mathrm{e}^{\beta} \cdot \frac{f\big(v_\ell^C, \overline{u}^C\big)}{f\big(v_\ell^B, \overline{u}^B\big)} \right\}. \tag{B.23}$$

With the help of these identities, we simplify the determinant of the matrix $A\mathcal{N}$ by factoring

$$\mathrm{i}c \cdot \frac{h\big(u_\ell^B, \overline{u}^B\big)}{h\big(u_\ell^B, \overline{u}^C\big)} \tag{B.24}$$

out of all u-type columns and the term

$$\mathrm{i}c \cdot \frac{h\big(\overline{v}^C, v_\ell^C\big)}{h\big(\overline{v}^B, v_\ell^C\big)} \tag{B.25}$$

out of all v-type columns. This leads to

$$\operatorname*{det}_{N+M}\big[A\mathcal{N}\big] = \big(\mathrm{i}c\big)^{N+M} \cdot \frac{h\big(\overline{u}^B, \overline{u}^B, \big)h\big(\overline{v}^C, \overline{v}^C\big)}{h\big(\overline{u}^B, \overline{u}^C\big)h\big(\overline{v}^B, \overline{v}^C\big)} \cdot \operatorname*{det}_{N+M}\big[\mathcal{B}\big] \tag{B.26}$$

in which the matrix $\mathcal{B}$ given in block form

$$\mathcal{B} = \begin{pmatrix} \mathcal{B}^{(11)}\big(u_k^B, u_\ell^B\big) & \mathcal{B}^{(12)}\big(u_k^B, v_\ell^C\big) \\ \mathcal{B}^{(21)}\big(v_k^C, u_\ell^B\big) & \mathcal{B}^{(22)}\big(v_k^C, v_\ell^C\big) \end{pmatrix} \tag{B.27}$$

has its block entries on the first line given by

$$\begin{aligned} \mathcal{B}^{(11)}\big(u_k^B, u_\ell^B\big) = (\mathrm{i}c)^{-1} \cdot \Big[\mathrm{e}^\beta - V_{11}\big(u_\ell^B\big)\Big]\delta_{k\ell} \cdot \frac{f\big(u_\ell^B, \overline{u}^C\big)}{\mathfrak{f}\big(u_\ell^B, \overline{u}^B\big)} - \frac{\mathrm{e}^\beta}{u_\ell^B - u_k^B + \mathrm{i}c} \\ - \frac{1}{u_k^B - u_\ell^B + \mathrm{i}c} \cdot \frac{f\big(\overline{v}^B, u_\ell^B\big)}{f\big(\overline{v}^C, u_\ell^B\big)} \end{aligned} \tag{B.28}$$

$$\mathcal{B}^{(12)}\big(u_k^B, v_\ell^C\big) = \mathrm{e}^\beta \cdot \left\{ \frac{1}{v_\ell^C - u_k^B} \cdot \frac{f\big(v_\ell^C, \overline{u}^C\big)}{f\big(v_\ell^B, \overline{u}^B\big)} - \frac{1}{v_\ell^C - u_k^B + \mathrm{i}c} \right\}. \tag{B.29}$$

Above, we made use of the same convention as in (B.13) relatively to $\mathfrak{f}\big(u_\ell^B, \overline{u}^B\big)$.

Finally, for $k = 1, \dots, M-1$, the (v, u) and (v, v) blocks are given by

$$\mathcal{B}^{(21)}\big(v_k^C, u_\ell^B\big) = \frac{\mathrm{i}c}{u_\ell^B - v_k^C + \mathrm{i}c} - \mathrm{e}^\beta + f\big(v_k^C, u_\ell^B\big) \cdot \frac{f\big(\overline{v}^B, u_\ell^B\big)}{f\big(\overline{v}^C, u_\ell^B\big)} \tag{B.30}$$

$$\begin{aligned} \mathcal{B}^{(22)}\big(v_k^C, v_\ell^C\big) = \Big[1 - \mathrm{e}^\beta V_{22}\big(v_\ell^C\big)\Big]\delta_{k\ell} \cdot \frac{f\big(\overline{v}^B, v_\ell^C\big)}{\mathfrak{f}\big(\overline{v}^C, v_\ell^C\big)} + \frac{\mathrm{e}^\beta}{f\big(v_\ell^C, v_k^C\big)} \cdot \frac{f\big(v_\ell^C, \overline{u}^C\big)}{f\big(v_\ell^C, \overline{u}^B\big)} \\ - \left\{ \frac{\mathrm{i}c}{v_k^C - v_\ell^C + \mathrm{i}c} + \mathrm{e}^\beta \right\} \end{aligned} \tag{B.31}$$

whereas, at $k = M$ we have

$$\mathcal{B}^{(21)}\big(v_M^C, u_\ell^B\big) = \mathcal{B}^{(22)}\big(v_M^C, v_\ell^C\big) = 1 - \mathrm{e}^\beta . \tag{B.32}$$

We then subtract the last column of $\mathcal{B}$ from all the others. Upon factorising the diagonal elements, we get

$$\begin{aligned} \operatorname*{det}_{N+M}\big[\mathcal{B}\big] = \frac{1 - \mathrm{e}^\beta}{(\mathrm{i}c)^N} \cdot \frac{f\big(\overline{v}^B, \overline{v}^C \setminus \{v_M^C\}\big) f\big(\overline{u}^B, \overline{u}^C\big)}{\mathfrak{f}\big(\overline{v}^C, \overline{v}^C \setminus \{v_M^C\}\big) \cdot \mathfrak{f}\big(\overline{u}^B, \overline{u}^B\big)} \cdot \Big(\mathrm{e}^\beta - V_{11}\big(\overline{u}^B\big)\Big) \\ \times \Big(1 - \mathrm{e}^\beta V_{22}\big(\overline{v}^C \setminus \{v_M^C\}\big)\Big) \operatorname*{det}_{N+M}\big[I_{N+M} + \widehat{\mathcal{U}}_{v_M^C}\big] \end{aligned} \tag{B.33}$$

where the block decomposition of the matrix $\widehat{\mathcal{U}}_\theta$ takes the form

$$\widehat{\mathcal{U}}_\theta = \begin{pmatrix} 2\mathrm{i}\pi \operatorname{Res}_{\omega'=u^B_\ell}\Big(\widehat{\mathcal{U}}^{(11)}_\theta\big(u^B_k,\omega'\big)\Big) & 2\mathrm{i}\pi \operatorname{Res}_{\omega'=v^C_\ell}\Big(\widehat{\mathcal{U}}^{(12)}_\theta\big(u^B_k,\omega'\big)\Big) \\ 2\mathrm{i}\pi \operatorname{Res}_{\omega'=u^B_\ell}\Big(\widehat{\mathcal{U}}^{(21)}_\theta\big(v^C_k,\omega'\big)\Big) & 2\mathrm{i}\pi \operatorname{Res}_{\omega'=v^C_\ell}\Big(\widehat{\mathcal{U}}^{(22)}_\theta\big(v^C_k,\omega'\big)\Big) \end{pmatrix} \tag{B.34}$$

and $\not{f}$ means the product where *all* coinciding elements between the first and second argument of f are omitted.

The block entries of the matrix $\widehat{\mathcal{U}}_\theta$ are given in terms of four auxiliary kernels. The ones of the first block column read

$$\widehat{\mathcal{U}}^{(11)}_\theta\big(\omega,\omega'\big) = \frac{(2\mathrm{i}\pi)^{-1}}{1-\mathrm{e}^{-\beta}V_{11}\big(\omega'\big)} \frac{f\big(\omega',\overline{u}^B\big)}{f\big(\omega',\overline{u}^C\big)} \Bigg\{ \Bigg[\frac{1}{\theta-\omega+\mathrm{i}c} - \frac{1}{\omega'-\omega+\mathrm{i}c}\Bigg] - \Bigg[\frac{\mathrm{e}^{-\beta}}{\omega-\omega'+\mathrm{i}c}\cdot\frac{f\big(\overline{v}^B,\omega'\big)}{f\big(\overline{v}^C,\omega'\big)} + \frac{1}{\theta-\omega}\cdot\frac{f\big(\theta,\overline{u}^C\big)}{f\big(\theta,\overline{u}^B\big)}\Bigg]\Bigg\} \tag{B.35}$$

$$\widehat{\mathcal{U}}^{(21)}_\theta\big(\omega,\omega'\big) = \frac{(2\mathrm{i}\pi)^{-1}}{\mathrm{e}^{\beta}-V_{11}\big(\omega'\big)} \frac{f\big(\omega',\overline{u}^B\big)}{f\big(\omega',\overline{u}^C\big)} \Bigg\{ f(\omega,\omega')\cdot\frac{f\big(\overline{v}^B,\omega'\big)}{f\big(\overline{v}^C,\omega'\big)} - \frac{\mathrm{e}^{\beta}}{f(\theta,\omega)}\cdot\frac{f\big(\theta,\overline{u}^C\big)}{f\big(\theta,\overline{u}^B\big)} + \frac{\mathrm{i}c}{\omega'-\omega+\mathrm{i}c} + \frac{\mathrm{i}c}{\omega-\theta+\mathrm{i}c}\Bigg\} \tag{B.36}$$

whereas those of the second block column are given by

$$\widehat{\mathcal{U}}^{(12)}_\theta\big(\omega,\omega'\big) = \frac{\mathrm{e}^{\beta}\cdot(-2\pi c)^{-1}}{1-\mathrm{e}^{\beta}V_{22}\big(\omega'\big)} \frac{f\big(\overline{v}^C,\omega'\big)}{f\big(\overline{v}^B,\omega'\big)}\cdot \Bigg\{ \frac{1}{\omega'-\omega}\cdot\frac{f\big(\omega',\overline{u}^C\big)}{f\big(\omega',\overline{u}^B\big)} - \frac{1}{\theta-\omega}\cdot\frac{f\big(\theta,\overline{u}^C\big)}{f\big(\theta,\overline{u}^B\big)} + \frac{1}{\theta-\omega+\mathrm{i}c} - \frac{1}{\omega'-\omega+\mathrm{i}c}\Bigg\} \tag{B.37}$$

$$\widehat{\mathcal{U}}^{(22)}_\theta\big(\omega,\omega'\big) = \frac{(-2\pi c)^{-1}}{1-\mathrm{e}^{\beta}V_{22}\big(\omega'\big)} \frac{f\big(\overline{v}^C,\omega'\big)}{f\big(\overline{v}^B,\omega'\big)}\cdot \Bigg\{ \frac{\mathrm{e}^{\beta}}{f\big(\omega',\omega\big)}\cdot\frac{f\big(\omega',\overline{u}^C\big)}{f\big(\omega',\overline{u}^B\big)} - \frac{\mathrm{e}^{\beta}}{f\big(\theta,\omega\big)}\cdot\frac{f\big(\theta,\overline{u}^C\big)}{f\big(\theta,\overline{u}^B\big)} + \frac{\mathrm{i}c}{\omega-\theta+\mathrm{i}c} - \frac{\mathrm{i}c}{\omega-\omega'+\mathrm{i}c}\Bigg\}. \tag{B.38}$$

The determinant occurring in the *rhs* of (B.33) is readily recast into the Fredholm determinant of the operator $\mathrm{id}+\widehat{\mathbf{U}}_\theta$, with $\widehat{\mathbf{U}}_\theta$ being of finite rank and acting on the contour

$$\mathscr{C}_{u,v} = \Gamma\Big(\{u^B_a\}_1^N\Big)\cup\Gamma\Big(\{v^C_a\}_1^M\Big). \tag{B.39}$$

More precisely, one has that

$$\det_{N+M}\Big[I_{N+M} + \mathcal{U}_{v^C_M}\Big] = \det_{\mathscr{C}_{u,v}}\Big[\mathrm{id}+\widehat{\mathbf{U}}_{v^C_M}\Big] \tag{B.40}$$

where the block decomposition of the kernel $\widehat{U}_\theta(\omega,\omega')$ corresponds to the splitting of the integration contour given in (B.39)

$$\widehat{U}_\theta = \begin{pmatrix} \widehat{\mathcal{U}}^{(11)}_\theta(\omega,\omega') & \widehat{\mathcal{U}}^{(12)}_\theta(\omega,\omega') \\ \widehat{\mathcal{U}}^{(21)}_\theta(\omega,\omega') & \widehat{\mathcal{U}}^{(22)}_\theta(\omega,\omega') \end{pmatrix}. \tag{B.41}$$

Hence, by putting together all the terms, one is led to

$$\mathcal{S}_{\beta}\big(\{u^C\},\{v^C\}\mid\{u^B\},\{v^B\}\big)=(1-\mathrm{e}^{\beta})\cdot\frac{h\big(\overline{v}^C,v_M^C\big)}{h\big(\overline{v}^B,v_M^C\big)}\cdot f\big(\overline{v}^C,\overline{u}^B\cup\overline{u}^C\big)$$
$$\times f\big(\overline{v}^B,\overline{u}^B\cup\overline{v}^C\big)\cdot f\big(\overline{u}^B,\overline{u}^C\big)\cdot\Big(\mathrm{e}^{\beta}-V_{11}\big(\overline{u}^B\big)\Big)\cdot\Big(1-\mathrm{e}^{\beta}V_{22}\big(\overline{v}^C\setminus\{v_M^C\}\big)\Big)$$
$$\times\det_{\mathscr{C}_{u,v}}\Big[\mathrm{id}+\widehat{\mathfrak{V}}_{v_M^C}\Big]. \tag{B.42}$$

Thus, upon implementing the afore-discussed correspondence between the variables (u,v) and those corresponding to real valued solutions to the Bethe equations, one is led to the claim. Finally, observe that instead of choosing the Mth line in the second block, one could have chosen any other. This entails that one can, in fact, do the substitution $v_M^C\hookrightarrow v_a^C$ in the above formula. □

Appendix C. Main definitions for $SU(3)$ invariant XXX model

For the $SU(3)$ invariant XXX model, we use the following functions

$$g(x,y)=\frac{\mathrm{i}c}{x-y},\quad f(x,y)=\frac{x-y+\mathrm{i}c}{x-y},\quad k(x,y)=\frac{x-y+\mathrm{i}c/2}{x-y-\mathrm{i}c/2}, \tag{C.1}$$

$$t(x,y)=\frac{-c^2}{(x-y+\mathrm{i}c)(x-y)},\quad h(x,y)=\frac{x-y+\mathrm{i}c}{\mathrm{i}c}. \tag{C.2}$$

We also introduce the short hand notation for $\overline{u}=\{u_j,\ j=1,\dots,N\}$, $\overline{v}=\{v_j,\ j=1,\dots,M\}$ and any function f of two variables

$$f(x,\overline{u})=\prod_{j=1}^{N}f(x,u_j),\quad f(\overline{u},\overline{v})=\prod_{j=1}^{N}\prod_{k=1}^{M}f(u_j,v_k),\quad \not f(\overline{u},\overline{u})=\prod_{j\neq k}^{N}f(u_j,u_k) \tag{C.3}$$

as well as for g given in (C.1)

$$\Delta_N(\overline{u})=\prod_{j<k}^{N}g(u_j,u_k)\quad\text{and}\quad\Delta'_N(\overline{u})=\prod_{j>k}^{N}g(u_j,u_k). \tag{C.4}$$

We also define the XXX bare phase and momentum

$$\vartheta_n(\omega)=\frac{1}{2\mathrm{i}\pi}\ln\left(\frac{\mathrm{i}c/n+\omega}{\mathrm{i}c/n-\omega}\right)\quad\text{and}\quad\sigma^{(1)}_{0;XXX}(\omega)=\frac{\mathrm{i}}{2\pi}\ln\left(\frac{\mathrm{i}c/2+\omega}{\mathrm{i}c/2-\omega}\right). \tag{C.5}$$

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