European Journal of Combinatorics 31 (2010) 129–144

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Cycle factorizations and 1-faced graph embeddings

Yurii Burman [a,b], Dimitri Zvonkine [c,d]

[a] Independent University of Moscow, B. Vlassievsky per. 11, Russia

[b] Higher School of Economics, Moscow, Myasnitskaya, 20, Russia

[c] CNRS, Institut Mathématique de Jussieu, 175 rue du Chevaleret, 75013 Paris, France

[d] Stanford University, Palo Alto 94305 CA, USA

ARTICLE INFO

Article history:
Received 26 October 2008
Accepted 10 April 2009
Available online 7 July 2009

ABSTRACT

Consider factorizations into transpositions of an n-cycle in the symmetric group S_n. To every such factorization we assign a monomial in variables w_{ij} that retains the transpositions used, but forgets their order. Summing over all possible factorizations of n-cycles we obtain a polynomial that happens to admit a closed expression. From this expression we deduce a formula for the number of 1-faced embeddings of a given graph.

1. Introduction and main results

The Hurwitz problem is the problem of counting ramified coverings of surfaces with prescribed ramification types. It is a classical problem presently enjoying a regain of interest due to its discovered relations with the Young–Mills models, matrix integrals, and intersection theory on moduli spaces [2,7,1].

1.1. Hurwitz numbers and Hurwitz polynomials

In this paper we not only count ramified coverings, but actually also retain an important part of the structure of each ramified covering. As a consequence, the answer to the Hurwitz problem will be a polynomial in many variables rather than a number.

We concentrate on the particular case of *pseudo-polynomial Morse* coverings of the sphere by an arbitrary surface. In other words, we consider degree n ramified coverings of the sphere by a genus g surface with full ramification over one point labelled ∞ (*i.e.*, ∞ has a unique preimage) and simple ramifications over $n+2g-1$ other points (*i.e.*, each of them has one double and $n-2$ simple preimages).

E-mail addresses: burman@mccme.ru (Y. Burman), dimitri.zvonkine@gmail.com (D. Zvonkine).

0195-6698/$ – see front matter
doi:10.1016/j.ejc.2009.04.005

The problem of counting pseudo-polynomial Morse coverings arises as a particular case in [14,15,2] and is completely solved. Our approach allows us, however, not only to count the coverings, but also to obtain new information on the structure of the set of these coverings. In particular, we draw some nontrivial consequences on the number of 1-faced graph embeddings. In the Appendix and partly in Section 3 we also deal with coverings that are Morse, but not necessarily pseudo-polynomial.

Choosing a base point on the sphere (different from the branch points) and numbering the preimages of the base point, we obtain a description of every ramified covering in terms of permutations. The monodromy of a pseudo-polynomial Morse covering over ∞ is an n-cycle in S_n, while its monodromies over the other branch points are transpositions. The monodromies determine the covering uniquely up to isomorphism. Therefore we are actually interested in describing the set of factorizations of n-cycles into $n+2g-1$ transpositions.

Definition 1.1. A list of transpositions $\tau_1, \ldots, \tau_{n+2g-1} \in S_n$ such that the product $\tau_{n+2g-1} \cdots \tau_1$ is an n-cycle is called a *genus g cycle factorization*.

Denote by $\mathbb{C}_n[\mathbf{w}]$ the ring of polynomials in (commuting) variables w_{ij}, $1 \le i, j \le n$, $i \ne j$, modulo the relations $w_{ij} = w_{ji}$. (We could have restricted ourselves to the variables w_{ij} with $i < j$, but sometimes it is convenient to use w_{ij} without bothering to know if i is smaller or greater than j.)

To every transposition $\tau \in S_n$ we assign the variable $w(\tau) = w_{ij}$, where i and j are the elements permuted by τ.

Definition 1.2. The *Hurwitz polynomial* $P_{g,n}(\mathbf{w})$ is defined by

$$P_{g,n}(\mathbf{w}) = \sum w(\tau_1) \cdots w(\tau_{n+2g-1}),$$

where the sum is taken over all cycle factorizations of genus g.

The *Hurwitz number* $h_{g,n}$ is defined by

$$h_{g,n} = \frac{1}{n!} P_{g,n}(\mathbf{1}),$$

where $P_{g,n}(\mathbf{1})$ is the result of the substitution $w_{ij} = 1$ for all i, j in $P_{g,n}$.

The Hurwitz number is the number of all pseudo-polynomial Morse coverings, each covering being counted with weight 1/(number of its symmetries). Equivalently, it is the number of all pseudo-polynomial Morse coverings with numbered sheets, divided by $n!$.

Example 1.3. For $n = 2$ we have $P_{g,n} = w_{12}^{2g+1}$, $h_{g,n} = 1/2$.

The Hurwitz polynomial $P_{g,n}$ is the main object of our study. From each cycle factorization $P_{g,n}$ retains the transpositions that compose it, but forgets their order.

The set of cycle factorizations is invariant under the action of S_n. Hence $P_{g,n}$ is S_n-invariant under the renumberings of indices of the variables w_{ij}. Therefore it is a natural idea to combine similar terms and represent their sum as a graph.

Recall that an automorphism of a graph is a permutation of its half-edges preserving the relations "to belong to the same edge" and "to have a common vertex".

Definition 1.4. Let G be a graph with no loops. We denote by $G_n(\mathbf{w}) \in \mathbb{C}_n[\mathbf{w}]$ the polynomial in variables w_{ij}, obtained as follows. Label the vertices of G with distinct numbers from 1 to n in all possible ways (if G has $v \le n$ vertices there are $n!/(n-v)!$ labellings; if $v > n$ there are no possible labellings). To each labelling assign the product of the variables w_{ij} corresponding to the edges. Sum the obtained monomials over all labellings. Divide by the number $|\mathrm{Aut}\, G|$ of automorphisms of G.

Remark 1.5. The same graph G represents a polynomial $G_n \in C_n[\mathbf{w}]$ for each n. If G has more than n vertices, then $G_n(\mathbf{w}) = 0$.

Example 1.6.

For $G =$ we have $G_n(\mathbf{w}) = \frac{1}{2}\sum_{i\neq j} w_{ij} = \sum_{i<j} w_{ij}.$

For $G =$  we have $G_n(\mathbf{w}) = \frac{1}{12}\sum_{i\neq j} w_{ij}^3 = \frac{1}{6}\sum_{i<j} w_{ij}^3.$

For $G =$ we have $G_n(\mathbf{w}) = \sum_{\substack{i\neq j\neq k\\ i<k}} w_{ij}w_{jk}.$

Remark 1.7. The factor $1/\,|\mathrm{Aut}\,G|$ is, so far, just an encoding convention that tells us, for instance, to assign to the graph the polynomial $\sum_{i<j} w_{ij}$ rather than $\sum_{i\neq j} w_{ij}$. However this convention is not random; indeed it will avoid us the appearance of the factor $|\mathrm{Aut}\,G|$ in other formulas. See, for instance, the proof of Corollary 3.7.

Let $\pi_n : \mathbb{C}_n[\mathbf{w}] \to \mathbb{C}_{n-1}[\mathbf{w}]$ be the substitution $w_{1n} = \cdots = w_{n-1,n} = 0$.

Definition 1.8. The algebra $\mathcal{W}$ is the projective limit of spaces of S_n-invariants in $\mathbb{C}_n[\mathbf{w}]$ with respect to the projections π_n.

A family of S_n-invariant polynomials $P_n(\mathbf{w}) \in \mathbb{C}_n[\mathbf{w}]$ is an element of $\mathcal{W}$ if their degrees are uniformly bounded and

$$P_n|_{w_{1n}=\cdots=w_{n-1,n}=0} = P_{n-1}$$

for all n. For example, for any graph G the sequence $G_n(\mathbf{w})$ defines an element of $\mathcal{W}$, to be denoted by $G(\mathbf{w})$ or just by G if it does not lead to ambiguity. Moreover, it is clear that the elements $G(\mathbf{w})$ for all G span $\mathcal{W}$.

Example 1.9. We have

 $^2 = \left(\sum_{i<j} w_{ij}\right)^2 = \sum_{i<j} w_{ij}^2 + 2\sum_{\substack{i\neq j\neq k\\ i<k}} w_{ij}w_{jk} + 2\sum_{\substack{i<j,i<k<l\\ j\neq l,j\neq k}} w_{ij}w_{kl}$

$= 2\Big($  $+$ $+$  $\Big).$

Our aim is now to give an explicit expression for $P_{g,n}(\mathbf{w})$. To do this we need to introduce some notation.

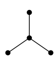
Definition 1.10. We let $T_n(\mathbf{w}) = \sum_t t_n(\mathbf{w})$, where the sum is taken over all trees t with n vertices.

Example 1.11. We have

$T_1(\mathbf{w}) \;=\; 1$ (by convention),

$T_2(\mathbf{w}) \;=$  $= w_{12},$

$T_3(\mathbf{w}) \;=$ $= w_{12}w_{13} + w_{12}w_{23} + w_{13}w_{23},$

$T_4(\mathbf{w}) \;=$ $+$  $=$ 16 terms.

Notation 1.12. We will repeatedly use the function

$$\phi(t) = \frac{\sinh(t/2)}{t/2}$$

and its logarithm

$$\ln \phi = \sum_{g \geq 0} \frac{B_{2g}}{(2g)!\,(2g)} t^{2g},$$

where B_{2g} are the Bernoulli numbers.

Notation 1.13. Denote by A_n the following $n \times n$ matrix:

$$A_n = \begin{pmatrix} \sum w_{1i} & -w_{12} & -w_{13} & \cdots & -w_{1n} \\ -w_{12} & \sum w_{2i} & -w_{23} & \cdots & -w_{2n} \\ -w_{13} & -w_{23} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -w_{n-1,n} \\ -w_{1n} & -w_{2n} & \cdots & -w_{n-1,n} & \sum w_{ni} \end{pmatrix}.$$

Definition 1.14. Introduce the power series

$$r_n(\mathbf{w}) = \operatorname{Tr}\, \ln \phi(A_n) = \sum_{g \geq 0} \frac{B_{2g}}{(2g)!\,(2g)} \operatorname{Tr} A_n^{2g} \tag{1}$$

and

$$R_n(\mathbf{w}) = \exp(r_n(\mathbf{w})) = \det \phi(A_n). \tag{2}$$

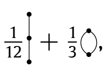
Both r_n and R_n are even. We denote by $r_{g,n}$ and $R_{g,n}$, respectively, their homogeneous parts of degree $2g$.

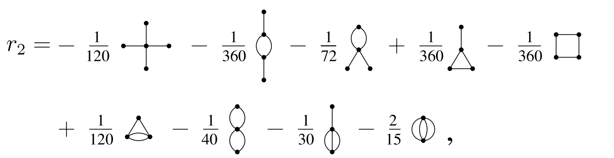
Proposition 1.15. *For any given g, $R_g = (R_{g,n})_{n \geq 1}$ and $r_g = (r_{g,n})_{n \geq 1}$ define elements of $\mathcal{W}$.*

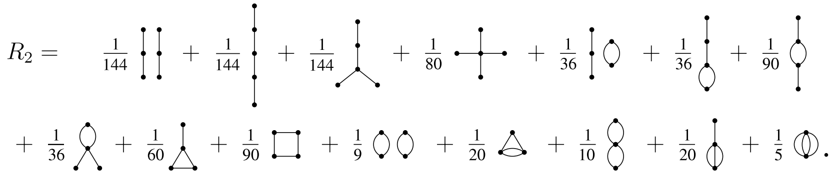
Proof. We have

$$A_n^{2g}\big|_{w_{1n}=\cdots=w_{n-1,n}=0} = \left(\begin{array}{c|c} A_{n-1}^{2g} & 0 \\ \hline 0 & 0 \end{array}\right).$$

The second expression for r_n in (1) implies then $r_n|_{w_{1n}=\cdots=w_{n-1,n}=0} = r_{n-1}$. The result about R_n follows. ◇

Example 1.16. We have $r_0 = 0$, $R_0 = 1$, $r_1 = R_1 =$ 





Conjecture 1.17. *$R_{g,n}$ is a polynomial with positive coefficients for all g, n.*

The importance of R_g is due to their role in the following theorem:

Theorem 1. *We have*

$$P_{g,n}(\mathbf{w}) = (n + 2g - 1)!\; T_n(\mathbf{w})\; R_{g,n}(\mathbf{w}).$$

Example 1.18. We have $P_{0,n} = (n-1)!\; T_n$. The latter equality is interpreted as follows: for every tree with n vertices, the product of the $n-1$ transpositions corresponding to its edges is an n-cycle *whatever the ordering of the transpositions*. The coefficient $(n-1)!$ is simply the number of possible orderings.

Corollary 1.19 ([15, Theorem 6]). *The Hurwitz number $h_{g,n}$ equals*

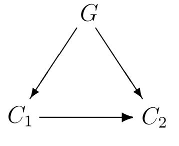
$$h_{g,n} = \frac{(n+2g-1)!}{n!}\; n^{n-2}\; \rho_{g,n},$$

where $\rho_{g,n}$ is the coefficient of w^{2g} in the power series $(\phi(nw))^{n-1}$.

1.2. Graph embeddings

An important application of our results is the counting of 1-faced embeddings of arbitrary graphs.

A connected graph is *embedded* into a closed oriented surface of genus g if it is drawn on the surface with no intersections between the edges (except at their endpoints) and if the edges cut the surface into topological discs. These discs are called *faces*. Two embeddings of a graph $G \to C_1$ and $G \to C_2$ are considered isomorphic if there exists an orientation-preserving homeomorphism between C_1 and C_2 that makes the triangular diagram commute:



Up to this isomorphism, giving an embedding of G is the same thing as giving a *cyclic order* of half-edges issuing from every vertex of G. Indeed, if the embedding is given then the cyclic order is just the counterclockwise order of the half-edges on the surface. Conversely, given a cyclic order of half-edges at every vertex, it is possible to reconstruct the faces of the covering and to glue a disc into each face thus obtaining the surface of embedding.

If the valencies of the vertices of a connected graph G are $k_1, \ldots, k_n$ then it has $\mathrm{Emb}\, G = \prod (k_i - 1)!$ different embeddings.

For a detailed introduction into graph embeddings see [11].

Definition 1.20. Let G be a connected graph with $2g$ independent cycles, *i.e.*, with the first Betti number equal to $2g$. A *decoration* of G is the choice of several vertices of G and, for each of these vertices v, the choice of a positive *even* number k_v of half-edges adjacent to it, such that $\sum k_v = 2g$ and that if we erase the chosen half-edges the remaining part of G is contractible.[1]

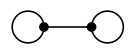
The *weight* of a decoration equals

$$\frac{1}{2^{2g}} \prod_v \frac{1}{k_v + 1}.$$

[1] We could have said that the remaining part is a tree, except that this "tree" has $2g$ dangling half-edges.

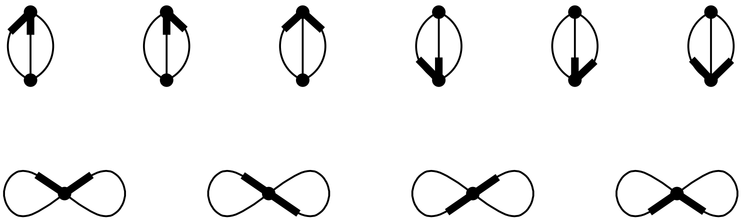
Theorem 2. *Let G be a connected graph with 2g independent cycles. The number of 1-faced (that is, genus g) embeddings of G divided by the total number of its embeddings is equal to the sum of weights of all decorations of G.*

Example 1.21. Consider the following graphs:

$G_1 =$ , $G_2 =$ , $G_3 =$ .

They have, respectively, $(3-1)!\,(3-1)! = 4$, $(4-1)! = 6$, and $(3-1)!\,(3-1)! = 4$ embeddings, out of which, respectively 2, 2, and none are 1-faced.

The decorations of G_1 and G_2 are shown in the figure. Each of them has weight $1/12$. The graph G_3 has no decorations.



For G_1 we have $6/12 = 2/4$, for G_2 we have $4/12 = 2/6$, and for G_3 we have $0 = 0/4$.

Example 1.22. Let G be the graph with 1 vertex and $2g$ loops. Then it has 2^{2g} decorations, each of weight $1/2^{2g}(2g+1)$. It also has $(4g-1)!$ embeddings, of which $(4g-1)!/(2g+1)$ are 1-faced: this is a particular case of the Harer–Zagier formula [8]. We have

$$\frac{2^{2g}}{2^{2g}\,(2g+1)} = \frac{(4g-1)!/(2g+1)}{(4g-1)!}.$$

Example 1.23. The graph in the figure has 96 decorations. One of them is shown in the figure, while the others are obtained from it by graph automorphisms. Every decoration has weight $\frac{1}{2^6 \cdot 3 \cdot 5} = \frac{1}{960}$.

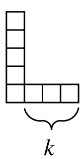
Therefore this graph has $6!^2 \cdot \frac{96}{960} = 51\,840$ embeddings into a genus 3 surface.

2. Representations of S_n and Frobenius formula

The main goal of this section is to prove Theorem 1.

Throughout the section we will be using the following representations of the symmetric group S_n.

- $\mathbb{C}^n$, where S_n acts by permutations of coordinates (the "permutation representation").
- $V := \{(x_1, \ldots, x_n) \mid \sum_i x_i = 0\} \subset \mathbb{C}^n$ – an $(n-1)$-dimensional irreducible representation ("geometric" or "defining" representation of S_n as a Coxeter group).
- The exterior powers $\bigwedge^k V$, $0 \le k \le n-1$. These are also irreducible representations; $\bigwedge^k V$ corresponds to the "hook" Young diagram (see [3, page 48])



.

- The group algebra $\mathbb{C}S_n$ where S_n acts by left multiplication. A classical theorem says that it is a direct sum of all the irreducible representations of S_n containing every representation λ exactly $\dim\lambda$ times.

  Let $C \in \mathbb{C}S_n$ be the sum of all n-cycles. Let $B(\mathbf{w}) \in \mathbb{C}_n[\mathbf{w}]S_n$ be the sum

$$B(\mathbf{w}) = \sum_{i<j} w_{ij}\tau_{ij},$$

where τ_{ij} is the transposition interchanging i and j.

Proposition 2.1.

$$P_{g,n}(\mathbf{w}) = \frac{1}{n}\sum_{k=0}^{n-1}(-1)^k \operatorname{Tr}_{\bigwedge^k V}\big(B(\mathbf{w})^m\big),$$

where $m = n + 2g - 1$.

Proof. This is a particular case of the well-known Frobenius formula (see [11], Section A.1.3). For reader's convenience we summarize the proof here.

For a permutation $\sigma \in S_n$, the trace $\operatorname{Tr}_{\mathbb{C}S_n}\sigma$ equals $n!$ if σ is the identity and 0 otherwise. In other words, $\frac{1}{n!}\operatorname{Tr}_{\mathbb{C}S_n}x$ is the coefficient of the identity permutation in x. It follows that

$$P_{g,n}(\mathbf{w}) = \frac{1}{n!}\operatorname{Tr}_{\mathbb{C}S_n}\big(B(\mathbf{w})^m C\big) = \frac{1}{n!}\sum_{\lambda}\dim\lambda\cdot\operatorname{Tr}_{\lambda}\big(B(\mathbf{w})^m C\big),$$

where the sum is taken over the irreducible representations λ. Now, C belongs to the center of $\mathbb{C}S_n$ and therefore, by Schur's lemma, acts in every irreducible representation λ by a scalar. Using the ribbon rule [12, Example 5 Section 3] for the character of an n-cycle, we see that this scalar equals $(-1)^k(n-1)!/(\dim\lambda)$ for $\lambda = \bigwedge^k V$ and vanishes otherwise. Substituting this in the above formula we obtain the equality of the proposition. ◇

Proposition 2.2. *Let* $\tau \in S_n$ *be a transposition. Then the action of* $\tau - 1$ *in* $\bigwedge^k \mathbb{C}^n = \mathbb{C}^n \wedge \mathbb{C}^n \wedge \cdots \wedge \mathbb{C}^n$ *is given by*

$$\begin{array}{ccccccc} & (\tau-1) & \wedge & \mathrm{id} & \wedge & \cdots & \wedge & \mathrm{id} \\ + & \mathrm{id} & \wedge & (\tau-1) & \wedge & \cdots & \wedge & \mathrm{id} \\ + & & & & & \cdots & & \\ + & \mathrm{id} & \wedge & \mathrm{id} & \wedge & \cdots & \wedge & (\tau-1). \end{array}$$

The same formula is true for the exterior powers of the representation V.

In other words, taking the action of τ in the exterior power of $\mathbb{C}^n$ or of V and then subtracting the identity operator is the same as taking the sum of actions of $\tau - 1$ in every exterior factor.

Proof. Assume τ is the transposition $(1, 2)$. Let $X = x_{i_1} \wedge \cdots \wedge x_{i_p} \in \bigwedge^p \mathbb{C}^n$ be a wedge product that does not contain x_1 and x_2. It is easy to check that both $\tau - 1$ and the sum given in the proposition act as follows (we have $p = k-2, k-1$, or k):

$$\begin{array}{rcl} X & \mapsto & 0, \\ x_1 \wedge X & \mapsto & (x_2 - x_1)\wedge X, \\ x_2 \wedge X & \mapsto & (x_1 - x_2)\wedge X, \\ x_1 \wedge x_2 \wedge X & \mapsto & 2\,x_2 \wedge x_1 \wedge X. \end{array}$$

The last statement follows from the fact that V is a subrepresentation of $\mathbb{C}^n$. ◇

Remark 2.3. Proposition 2.2 is crucial, although its proof is a trivial check. One can formulate it by saying that $\tau - 1$ is "Lie-algebra-like" for exterior powers of the representation $\mathbb{C}^n$. Lie-algebra-like elements form a vector subspace of $\mathbb{C}S_n$, and we believe they deserve a special study. Examples include the difference of 3-cycles $(123) - (132)$ and two linear combinations of 4-cycles: $(1234) - (1324) - (1423) + (1432)$ and $(1243) - (1324) + (1342) - (1423)$.

Denote by $A(\mathbf{w})$ the Lie-algebra-like element of $\mathbb{C}S_n$

$$A(\mathbf{w}) = \sum_{1\le i<j\le n} w_{ij}(1-\tau_{ij}) = \sum_{1\le i<j\le n} w_{ij} \; - B(\mathbf{w}).$$

The matrix of the action of $A(\mathbf{w})$ in $\mathbb{C}^n$ is A_n of Notation 1.13.

Proposition 2.4. *Let $\sigma_1, \dots, \sigma_{n-1}$ be the eigenvalues of $A(\mathbf{w})$ in V. Then the eigenvalues of $B(\mathbf{w})$ in $\bigwedge^k V$ are*

$$\frac{\pm\sigma_1 \pm \sigma_2 \pm \dots \pm \sigma_{n-1}}{2}$$

with k signs "$-$" and $n-1-k$ signs "$+$".

Proof. Since $A(\mathbf{w})$ is Lie-algebra-like, its eigenvalues in $\bigwedge^k V$ are all sums $\sigma_{i_1} + \dots + \sigma_{i_k}$ for distinct $i_1, \dots, i_k$. In particular, $A(\mathbf{w})$ obviously acts by zero in the trivial representation and by $\sum_{i=1}^{n-1} \sigma_i$ in the (1-dimensional) sign representation $\bigwedge^{n-1} V$. On the other hand, in the sign representation every transposition acts by -1, so that $A(\mathbf{w})$ acts by $2\sum_{i<j} w_{ij}$. Thus we have

$$\sum_{i<j} w_{ij} = \frac{1}{2}\sum_{i=1}^{n-1} \sigma_i.$$

This equality allows us to write the eigenvalue of $B(\mathbf{w})$ in terms of $\sigma_1, \dots, \sigma_{n-1}$ only, which gives the formula of the proposition. $\diamond$

Proposition 2.5. *We have $\sigma_1 \cdots \sigma_{n-1} = nT_n(\mathbf{w})$.*

Proof. The product $\sigma_1 \cdots \sigma_{n-1}$ is the determinant of the action of $A(\mathbf{w})$ in V. Let us compute this determinant in a different way.

Let $v_1, \dots, v_n$ be the standard basis in $\mathbb{C}^n$. Then A_n is the matrix of the action of $A(\mathbf{w})$ in $\mathbb{C}^n$. Denote $v = \frac{1}{n}\sum v_i$. Then the vectors $u_i = v_i - v$ for $2 \le i \le n$ form a basis of the subrepresentation $V \subset \mathbb{C}^n$.

Denote by $\widetilde{A}_n$ the matrix obtained from A_n by erasing the first column and the first row. Denote by M_{n-1} the $(n-1)\times(n-1)$ matrix of the form

$$M_{n-1} = \begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \cdots & 2 \end{pmatrix}.$$

We claim that the matrix of the operator $A(\mathbf{w})$ in the basis $(u_2, \dots, u_n)$ of V is $M_{n-1}\widetilde{A}_n$.

Indeed, the sum of coefficients of A_n in every row and in every column vanishes. Therefore

$$\begin{aligned} A(\mathbf{w})(u_j) &= A(\mathbf{w})(v_j - v) = A(\mathbf{w})(v_j) = \sum_{i=1}^n (A_n)_{ij} v_i \\ &= (A_n)_{1j}\left(v - \sum_{i=2}^n u_i\right) + \sum_{i=2}^n (A_n)_{ij}(u_i + v) = -(A_n)_{1j}\sum_{i=2}^n u_i + \sum_{i=2}^n (A_n)_{ij} u_i \\ &= \sum_{k=2}^n (A_n)_{kj} \cdot \sum_{i=2}^n u_i + \sum_{i=2}^n (A_n)_{ij} u_i = \sum_{i=2}^n (M_{n-1}\widetilde{A}_n)_{ij} u_i. \end{aligned}$$

The Kirchhoff formula [10] (a.k.a. the matrix-tree theorem) implies that $\det \widetilde{A}_n = T_n(\mathbf{w})$. As $\det M_{n-1} = n$, we obtain that the determinant of the action of $A(\mathbf{w})$ in V equals $nT_n(\mathbf{w})$. $\diamond$

Now we are ready to prove two results announced in the introduction.

Proof of Theorem 1. According to Proposition 2.4, we have

$$P_{g,n}(\mathbf{w}) = \frac{1}{n} \sum_{\varepsilon_1,\ldots,\varepsilon_{n-1}\in\{-1,1\}} \varepsilon_1 \cdots \varepsilon_{n-1} \, (\varepsilon_1\sigma_1/2 + \cdots + \varepsilon_{n-1}\sigma_{n-1}/2)^{n+2g-1}. \quad (3)$$

This formula implies that $P_{g,n}$ is odd as a function of every σ_i (all the other σ_j being fixed). Therefore, if we expand the expression for $P_{g,n}$, all monomials $\sigma_1^{m_1} \cdots \sigma_{n-1}^{m_{n-1}}$ with at least one even m_i cancel out. On the other hand, if all the m_i are odd, the contribution of every term in (3) to the coefficient of this monomial is the same and equals

$$\frac{(n+2g-1)!}{2^{n+2g-1} \cdot n \cdot m_1! \cdots m_{n-1}!}.$$

The total number of terms is 2^{n-1}, so we finally get

$$\begin{aligned} P_{g,n} &= \sum_{\substack{m_1+\cdots+m_{n-1}=n+2g-1 \\ m_1,\ldots,m_{n-1} \text{ odd}}} \frac{1}{2^{2g} \cdot n} \, \frac{(n+2g-1)!}{m_1! \cdots m_{n-1}!} \, \sigma_1^{m_1} \cdots \sigma_{n-1}^{m_{n-1}}, \\ &= (n+2g-1)! \; \frac{1}{n} \, \sigma_1 \cdots \sigma_{n-1} \sum_{p_1+\cdots+p_{n-1}=g} \frac{(\sigma_1/2)^{2p_1}}{(2p_1+1)!} \cdots \frac{(\sigma_{n-1}/2)^{2p_{n-1}}}{(2p_{n-1}+1)!}. \end{aligned}$$

By Proposition 2.5, we have $\frac{1}{n}\,\sigma_1 \cdots \sigma_{n-1} = T_n(\mathbf{w})$, so it remains to identify the last sum as $R_{g,n}$. Recall that

$$\phi(\sigma) = \frac{\sinh(\sigma/2)}{\sigma/2} = \sum_{p\geq 0} \frac{(\sigma/2)^{2p}}{(2p+1)!}.$$

Thus we see that

$$\sum_{p_1,\ldots,p_{n-1}} \frac{(\sigma_1/2)^{2p_1}}{(2p_1+1)!} \cdots \frac{(\sigma_{n-1}/2)^{2p_{n-1}}}{(2p_{n-1}+1)!} = \prod_{i=1}^{n-1} \phi(\sigma_i) = \det_V \phi(A_n).$$

In the last determinant we can replace V by $\mathbb{C}^n$. Indeed, the difference between these two representations is the trivial representation, where $A(\mathbf{w})$ acts by 0. It remains to recall that $R_{g,n}$ was defined as the degree $2g$ part of $\det_{\mathbb{C}^n} \phi(A_n)$. ◇

Proof of Corollary 1.19. After the substitution $w_{ij} = w$, the operator $A(\mathbf{w})$ acts in V by the scalar nw. In other words, all the σ_i are equal to nw, so we have

$$\sum_{g\geq 0} \rho_{g,n} w^{2g} = \det_V \phi(A(\mathbf{w})) = (\phi(nw))^{n-1}. \quad ◇$$

3. Graph embeddings

The aim of this section is to deduce several corollaries of Theorem 1, in particular to prove Theorem 2. To prove Theorem 2 we relate the coefficient of a graph in $P_{g,n}$ with the number of 1-faced embeddings of this graph and then compute this coefficient using the right-hand side of the equality of Theorem 1.

First, explain the relation between the polynomials $P_{g,n}$ and graph embeddings.

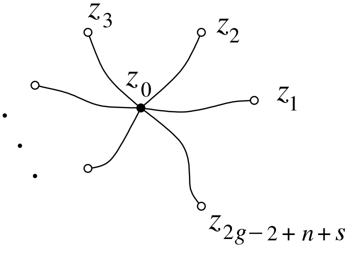
Recall that an embedding of a graph into a surface is defined, up to equivalence, by the cyclic order of its half-edges at every vertex.

Definition 3.1. Let G be a connected graph without loops. To a numbering of the edges of G, we assign the *embedding of the numbering* obtained in the following way: the cyclic order of the half-edges at each vertex is simply the increasing order of the numbers of edges.

Remark 3.2. In general, not every embedding can be obtained in this way: for instance, a graph with 2 vertices joined by k edges has $(k-1)!^2$ embeddings and only $k!$ numberings. However, in Definition 3.9 we introduce "long graphs" which have the property that every embedding can be obtained from a numbering of edges.

Remark 3.3. Consider a Morse covering of $\mathbb{C}P^1$ of degree n with s preimages of ∞. Let $\sigma \in S_n$ be its monodromy at ∞ (a permutation with s cycles). As we have seen in the Introduction, the covering determines, once we have chosen a set of generators of $\pi_1(\mathbb{C} \setminus \{\text{branch points}\})$, a factorization of σ into transpositions. This factorization, in turn, determines a graph with numbered vertices and edges, endowed with a natural embedding that we have just defined (see Definition 3.1).

There exists a more concise way to obtain the embedded graph from a ramified covering. Let $z_1, \ldots, z_{2g-2+n+s}$ be the branch points of the covering and z_0 a base point (distinct from the branch points). On the plane draw a star as follows:



This is equivalent to choosing a set of generators of $\pi_1(\mathbb{C} \setminus \{z_1, \ldots, z_{2g-2+n+s}\})$. The arrows of the star will be considered as half-edges. Now the procedure to obtain an embedded graph from the ramified covering is simply the following: take the preimage of the star and erase the dangling half-edges. (Every z_k has $n-2$ simple preimages and one double preimage. Out of a simple preimage issues only one half-edge; these half-edges are called dangling. Out of a double preimage issue two half-edges that form a complete edge.)

We leave it to the reader to check that the embedded graph with numbered edges that we have obtained coincides with the construction via factorizations into transpositions.

Let G be a connected graph without loops, with n vertices and with numbered edges. Consider the corresponding embedding and attribute a color to each face. Follow the boundary of a face in the clockwise direction (recall that the surface of embedding is oriented). For each vertex of the boundary, if the edge preceding it has a larger number than the edge following it, or the vertex is of valency one, then mark the vertex with the color of the face.

Proposition 3.4. *In the situation above, consider the product $\sigma \in S_n$ of the transpositions corresponding to the edges in the order specified by the numbering. Then the cycles of σ are in a one-to-one correspondence with the faces of the embedding. The cycle corresponding to a face is composed of the vertices bearing the color of the face.*

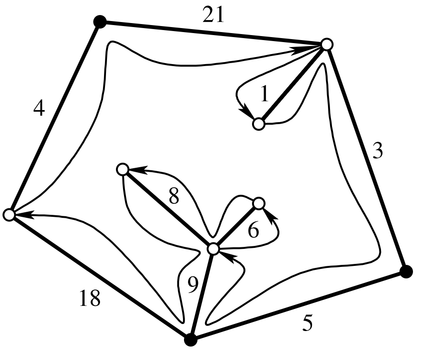
Proof. Introduce the following cyclic order on the vertices of a given color. Go around the face of this color in the clockwise direction. As we meet two successive edges whose numbers go in a nonincreasing order, the vertex between them is declared the next with respect to the order.

Let us prove that (i) each vertex of G is colored exactly once, and appears exactly once in the order corresponding to this color, (ii) every color is present, and (iii) the vertices of a given color taken in the corresponding order form a cycle of σ.

(i) The color of a vertex is simply the color of the face that lies after its largest and before its smallest edge. It appears in the order when we meet this pair of edges.

(ii) If some color was absent it would mean that the numbers of edges around the corresponding face increase indefinitely.

(iii) As we apply one by one the transpositions corresponding to the edges, a vertex of a given color moves along the face of the same color following a sequence of edges with increasing numbers.



In the end it arrives at the next vertex of the same color (with respect to the order). The figure shows an example of a face, the vertices of the corresponding color being represented in white. ◇

Remark 3.5. Using the construction of Remark 3.3 we can immediately conclude that the faces of the embedding of G correspond to the cycles of σ : indeed both are in a natural one-to-one correspondence with the poles of the covering.

Remark 3.6. In [6] the relation between embeddings and numberings of edges is studied in detail for the case when G is a tree.

Corollary 3.7. *Let G be a connected graph without loops, with n vertices, and with $2g$ independent cycles, (i.e., its first Betti number equals $2g$). The coefficient of G in $P_{g,n}$ is equal to the number of numberings of the edges of G such that the corresponding embedding is 1-faced.*

Proof. This is almost obvious; the only subtlety in the proof is to take into account the automorphisms of G.

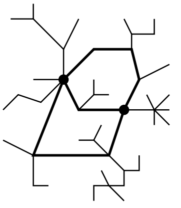
Let V be a numbering of the vertices of G. Let $|\mathrm{Aut}\,(G, V)|$ be the number of automorphisms of G that preserve the numbers of the vertices. (This number is greater then 1 iff G has multiple edges.) According to the definition of the polynomial $G(\mathbf{w})$, it contains the monomial $\prod_{e \in \text{edges of } G} w_e$ with coefficient $1/\,|\mathrm{Aut}\,(G, V)|$.

Now let us determine the coefficient of this monomial in $P_{g,n}$. It is equal to the number of pairwise non-isomorphic *fully marked graphs*, that is graphs with numbered vertices and numbered edges, satisfying two conditions: (i) forgetting the numbering of the edges we obtain a graph isomorphic to (G, V), and (ii) the product of the transpositions corresponding to the order of the edges is a long cycle. According to Proposition 3.4, the second condition is equivalent to the embedding of the numbering being 1-faced. Thus we need to count the pairwise non-isomorphic 1-faced fully marked graphs that give (G, V) if we forget the numbers of the edges.

This number is equal to the number of all possible 1-faced edge numberings of G divided by $|\mathrm{Aut}\,(G, V)|$ (because distinct edge numberings of (G, V) can be isomorphic as fully marked graphs if (G, V) has nontrivial automorphisms).

Comparing the two coefficients, we see that the coefficient of $G(\mathbf{w})$ in $P_{g,n}$ is precisely the number of 1-faced edge numberings of G. ◇

Definition 3.8. Let G be a connected graph with at least 2 independent cycles. The subgraph $\widehat{G}$ obtained from G by a repeated cutting off of valency 1 vertices is called the *skeleton* of G. The vertices of G that have valencies ≥ 3 in $\widehat{G}$ are called the *essential vertices* of G.



The skeleton and the essential vertices of a graph.

Definition 3.9. A connected graph G with no loops and at least 2 independent cycles is called *long* if its essential vertices are never connected by an edge.

Proposition 3.10. *Let G be a long graph with n vertices and $2g$ independent cycles. The coefficient of G in $P_{g,n}/(n+2g-1)!$ equals the number of 1-faced embeddings of G divided by the number of all its embeddings.*

Proof. Denote by $\operatorname{Emb} G$, $\operatorname{Emb}\widehat{G}$, $\operatorname{Emb}_1 G$, and $\operatorname{Emb}_1\widehat{G}$ the number of all embeddings and of 1-faced embeddings of G and of its skeleton $\widehat{G}$.

The number of faces in the embedding of a long graph G depends only on the cyclic order α of the half-edges of the skeleton adjacent to the essential vertices. Therefore we have

$$\frac{\operatorname{Emb}_1 G}{\operatorname{Emb} G} = \frac{\operatorname{Emb}_1 \widehat{G}}{\operatorname{Emb} \widehat{G}}.$$

Recall that by Corollary 3.7 we have to count the numberings of the edges of G that give rise to 1-faced embeddings. To do this, choose a cyclic order α corresponding to a 1-faced embedding.

Let $k_1, \ldots, k_m$ be the valencies *in the skeleton* $\widehat{G}$ of the essential vertices. We can enumerate the numberings of the edges of G that lead to the cyclic order α in the following way. First choose a set of $k = k_1 + \cdots + k_m$ numbers for the skeleton edges adjacent to the essential vertices. Second, attribute these numbers to the edges so as to obtain the desired cyclic order. Third, number the remaining edges with the remaining numbers (in an arbitrary way). We obtain

$$\frac{(n+2g-1)!}{k!(n+2g-1-k)!} \cdot \frac{k!}{(k_1-1)!\cdots(k_m-1)!} \cdot (n+2g-1-k)! = \frac{(n+2g-1)!}{\operatorname{Emb}\widehat{G}}.$$

Thus every cyclic order α contributes by $1/(\operatorname{Emb}\widehat{G})$ to the coefficient of G in $P_{g,n}/(n+2g-1)!$. Therefore the coefficient of G equals $\operatorname{Emb}_1\widehat{G}/\operatorname{Emb}\widehat{G}$. $\diamond$

Now we will evaluate the coefficient of a long graph in $P_{g,n}$ in another way using Theorem 1.

Definition 3.11. A *k-spider X_k* is the tree with k vertices of valency 1 and one vertex of valency k.

Lemma 3.12. *Disjoint unions of spiders are the only terms of R_g that give nonzero contributions to the coefficient of a long graph in the product $T_n R_{g,n}$.*

Proof. Let G be a long graph with $n+2g-1$ edges and $2g$ independent cycles. On this graph we can mark the $n-1$ edges that come from the factor T_n (these edges form a tree) and the $2g$ edges that come from the factor $R_{g,n}$. Since G has $2g$ independent cycles, erasing each of the $2g$ edges of the second group should disrupt exactly one cycle of G. It follows that if two edges of the second group are adjacent, then their common vertex is an essential vertex of G. Thus the graph formed by the second group of edges is composed of isolated edges (which are 1-spiders) and spiders centered at essential vertices of G. $\diamond$

Remark 3.13. Actually we will soon see that graphs with isolated edges do not occur in $R_{g,n}$, so the group of $2g$ edges will never contain isolated edges.

Lemma 3.14. *The coefficient of* X_{2g} *in* r_g *and* R_g *is equal to* $\frac{B_{2g}}{2g}$ *and* $\frac{1}{2^{2g}(2g+1)}$*, respectively.*

Proof. According to Notation 1.13, we have

$$r_g = \frac{B_{2g}}{(2g)!(2g)} \operatorname{Tr} A^{2g} = \frac{B_{2g}}{(2g)!(2g)} \sum_{i_1,\ldots,i_{2g}} A_{i_1 i_2} A_{i_2 i_3} \cdots A_{i_{2g-1} i_{2g}} A_{i_{2g} i_1}. \tag{4}$$

Obviously, X_{2g} appears only in the terms with $i_1 = \cdots = i_{2g}$. The contribution of these terms is $(2g)!$ because the "legs" of the spider can appear in any order. This proves the first statement.

To find the coefficient of X_{2g} in R_g we note that a subgraph of a spider is always a spider. In other words, the linear span $\mathcal{N} = \langle G \mid G \neq X_1, X_2, \ldots \rangle \subset \mathcal{W}$ of all the graphs except spiders is an ideal in $\mathcal{W}$. We can make our computation modulo this ideal, *i.e.*, neglecting all graphs except the spiders. We have

$$\mathrm{I}^{2g} = (2g)!\, X_{2g} \mod \mathcal{N}, \tag{5}$$

and therefore

$$\sum_g r_g = \sum_g \frac{B_{2g}}{2g} X_{2g} = \sum_g \frac{B_{2g}}{(2g)!(2g)}\, \mathrm{I}^{2g} = \ln \phi(\mathrm{I}) \mod \mathcal{N}, \tag{6}$$

where, as usual, $\phi(t) = \frac{\sinh(t/2)}{t/2}$. Eqs. (5) and (6) imply

$$\sum_g R_g = \phi(\mathrm{I}) = \sum_g \frac{1}{2^{2g}(2g+1)!}\, \mathrm{I}^{2g} = \sum_g \frac{1}{2^{2g}(2g+1)} X_{2g} \mod \mathcal{N}.$$

This proves the second statement. ◇

Lemma 3.15. *Let G be a disjoint union of several (more than one) connected graphs. The coefficient of G in* $r = \sum r_g$ *equals 0. The coefficient of G in* $R = \sum R_g$ *is the product of coefficients of its connected components.*

Proof. The first statement follows from Eq. (4). Indeed, we have $(A_n)_{ij} = -w_{ij}$ if $i \neq j$, while $(A_n)_{ii} = \sum_j w_{ij}$. Thus every term in the sum of Eq. (4) corresponds to a (possibly self-intersecting) cycle and several edges sticking out of it (possibly leading to vertices that are already in the cycle). Such a graph is always connected.

The second statement is an elementary corollary of the first. ◇

Remark 3.16. The last lemma allows us, in particular, to find the coefficient of a disjoint union of spiders in R_g: it is just the product of the coefficients of the individual spiders. Note that since r contains only even degree terms, it follows that R_g only contains graphs whose each connected component has an even degree of edges. In particular, graphs in R_g do not contain isolated edges. Similarly, the coefficient of the disjoint union of two 3-spiders in R_6 vanishes.

Proof of Theorem 2. Let G be a connected graph with $2g$ independent cycles. Insert a large enough number of vertices into the edges of G to obtain a long graph $\widetilde{G}$. This operation changes neither the number of 1-faced embeddings, nor the total number of embeddings.

Let n be the number of vertices of $\widetilde{G}$ and let us compute the coefficient of $\widetilde{G}$ in $P_{g,n}/(n+2g-1)!$ using the right-hand side of the equality of Theorem 1. We claim that the answer is precisely the sum of weights of the decorations of G. Indeed, the decorations of G and the ways to erase $2g$ edges of $\widetilde{G}$ so as to obtain a tree are in a natural one-to-one correspondence. Moreover, the weight of a decoration was defined to coincide with the coefficient of the corresponding disjoint union of spiders in R_g.

On the other hand, the coefficient of $\widetilde{G}$ in $P_{g,n}/(n+2g-1)!$ equals $\operatorname{Emb}_1 \widetilde{G} / \operatorname{Emb} \widetilde{G}$ according to Proposition 3.10.

Thus the sum of weights of the decorations of G equals

$$\frac{\operatorname{Emb}_1 \widetilde{G}}{\operatorname{Emb} \widetilde{G}} = \frac{\operatorname{Emb}_1 G}{\operatorname{Emb} G}. \quad \diamond$$

Acknowledgements

The first author was supported by the CRDF grant RUM1-2895-MO-07, INTAS grant 05-7805, RFBR grants 08-01-00110-a and NSh-709.2008.1, and the HSE Scientific Foundation grant 08-01-0019. The second author was partially supported by the ANR-05-BLAN-0029-01 grant (Geometry and Integrability in Mathematical Physics).

Appendix. Hurwitz generating function and the cut-and-join equation

In this Appendix we consider coverings that are not pseudo-polynomial (but are still Morse except at infinity). Namely, fix a partition $\lambda = (\lambda_1, \dots, \lambda_s)$ (where $\lambda_1 \leq \dots \leq \lambda_s$) of the number $n = \lambda_1 + \dots + \lambda_s$ into s parts; consider degree n (not necessarily connected) ramified coverings of the sphere by a surface M_g of the Euler characteristic $\chi(M_g) = 2 - 2g$ with simple ramifications over $n+2g-2+s$ points and a ramification of type λ over one point labelled ∞; the latter condition means that ∞ has s preimages with multiplicities $\lambda_1, \dots, \lambda_s$. Since M_g is not assumed connected, it is possible to have $g < 0$.

The monodromy of the covering in question over ∞ is an element of the conjugacy class $C_\lambda \subset S_n$ that consists of permutations that are products of s independent cycles of lengths $\lambda_1, \dots, \lambda_s$. The monodromies over the other branch points are transpositions. Similarly to Definitions 1.1 and 1.2 a list of transpositions $\tau_1, \dots, \tau_{n+2g-2+s} \in S_n$ is called a *genus g factorization of* λ if $\tau_{n+2g-2+s} \cdots \tau_1 \in C_\lambda$. Consider the *Hurwitz polynomial*

$$P_{g,\lambda} := \sum w(\tau_1) \cdots w(\tau_{n+2g-2+s}), \tag{7}$$

where the sum is taken over all such factorizations; recall that the numbers n and s are determined by λ.

The *Hurwitz number* $h_{g,\lambda}$ is defined by $h_{g,\lambda} = \frac{1}{n!} P_{g,\lambda}(\mathbf{1})$.

The condition $\tau_{n+2g-2+s} \cdots \tau_1 \in C_\lambda$ is invariant under conjugation in S_n, so that $P_{g,\lambda}$ is an S_n-invariant polynomial. Its degree is $m := n + 2g - 2 + s$, so, once λ and $P_{g,\lambda}$ are known, it is possible to determine g. Therefore one can collect all the Hurwitz polynomials into a general *Hurwitz* **w**-*generating function*

$$H(\mathbf{w}; p_1, p_2, \dots) = \sum_{g,\lambda} \frac{1}{m!} \frac{P_{g,\lambda}(\mathbf{w})}{n!} \frac{p_{\lambda_1} \cdots p_{\lambda_s}}{|\mathrm{Aut}\,(\lambda)|} \in \mathbb{C}[[\mathbf{w}, p_1, p_2, \dots]]; \tag{8}$$

here **w** means the infinite collection of variables w_{ij}, $1 \leq i < j$, and $\mathrm{Aut}(\lambda)$ is the set of permutations σ of s elements such that $\lambda_i = \lambda_{\sigma(i)}$ for all i. Taking $w_{ij} = \beta$ for all i, j one obtains the usual Hurwitz generating function

$$h(\beta; p_1, p_2, \dots) = \sum_{g,\lambda} \frac{\beta^m}{m!} h_{g,\lambda} \frac{p_{\lambda_1} \cdots p_{\lambda_s}}{|\mathrm{Aut}\,(\lambda)|} \tag{9}$$

studied intensively during the last decade (see e.g. [4,5,9,13], among many others).

One of the main properties of the Hurwitz generating function (9) is that it satisfies the so-called cut-and-join equation [4]; here we generalize it to the **w**-function (8).

Define the *cut-and-join operator* by

$$L = \frac{1}{2} \sum_{k,l \geq 1} \left((k+l) p_k p_l \frac{\partial}{\partial p_{k+l}} + kl p_{k+l} \frac{\partial^2}{\partial p_k \partial p_l} \right).$$

Theorem 3. *The Hurwitz generating* **w**-*function H satisfies the* cut-and-join differential equation

$$\sum_{1 \leq i < j} \frac{\partial H}{\partial w_{ij}} = LH. \tag{10}$$

Proof of Theorem 3. Let G be a graph without loops with n numbered vertices and m edges. Let λ be a Young diagram with n squares.

Let $S_1(G, \lambda)$ be the set of pairs (e, u) where e is an additional edge joining two vertices of G and u is a numbering of the set {edges of G} $\cup \{e\}$ such that the product ρ of transpositions corresponding to the numbering lies in the conjugacy class C_λ.

Further, let $S_2(G, \lambda)$ be the set of pairs $(v, \{i, j\})$ where v is a numbering of the edges of G and $\{i, j\} \subset \{1, \dots, n\}$ is an unordered pair of elements such that $\sigma\tau_{ij}$ lies in the conjugacy class C_λ (where σ is the product of transpositions corresponding to the numbering).

The coefficient of the monomial $w_G p_\lambda$ (with self-explanatory notation) in the left-hand side of (10) equals $|S_1(G, \lambda)| / n!(m+1)! \, |\mathrm{Aut}\,(\lambda)|$. The coefficient of $w_G p_\lambda$ in the right-hand side equals $|S_2(G, \lambda)| / n!m! \, |\mathrm{Aut}\,(\lambda)|$. (More precisely, if i and j lie in two different cycles of σ of lengths k and l, this corresponds to the term $klp_{k+l}\partial^2/\partial p_k \partial p_l$; if i and j lie in the same cycle of σ of length $k+l$, k and l being the distances from i to j and from j to i along the cycle, this corresponds to the term $(k+l)p_k p_l \partial/\partial p_{k+l}$.)

It remains to establish an $(m+1)$-to-1 correspondence between S_1 and S_2. For a pair $(e, u) \in S_1$ we write $\rho = \rho_2 \tau(e) \rho_1$. Here ρ_1 is the product of the transpositions coming before e, $\tau(e)$ is the transposition corresponding to e, and ρ_2 is the product of transpositions coming after e. Then we let $\sigma = \rho_2 \rho_1$, which also determines a numbering v of the edges of G; $\{i, j\}$ are the elements permuted by $\tau(e)$. The permutation $\sigma\tau_{ij} = \rho_2\rho_1\tau(e)$ is conjugate to ρ and therefore lies in the class C_λ. Finally, the map from S_1 to S_2 we have constructed is invariant under circular permutations of the numbering u, thus it is actually an $(m+1)$-to-1 correspondence. $\diamond$

Remark A.1. Note that the last part of the proof is different from the usual proof of the classical cut-and-join equation, where one can always assume that $\tau(e)$ is the first transposition in the list and there is no need to introduce the circular change in the numbering of transpositions.

Remark A.2. It is also possible to consider Hurwitz polynomials $\widetilde{P}_{g,\lambda}(\mathbf{w})$ and Hurwitz numbers $\widetilde{h}_{g,\lambda} = \frac{1}{n!}\widetilde{P}_{g,\lambda}(\mathbf{1})$ that only count *connected* ramified coverings, and to collect them into generating series $\widetilde{H}$ and $\widetilde{h}$ analogous to (8) and (9), respectively. The relation between h and $\widetilde{h}$ is well-known (see e.g. [4]):

$$h(\beta; p_1, p_2, \dots) = \exp(\widetilde{h}(\beta; p_1, p_2, \dots)). \tag{11}$$

From this formula it is easy to deduce a variant of the cut-and-join equation for the "connected" generating series $\widetilde{h}$:

$$\frac{\partial \widetilde{h}}{\partial \beta} = \frac{1}{2} \sum_{k,l \geq 1} \left((k+l)p_k p_l \frac{\partial \widetilde{h}}{\partial p_{k+l}} + klp_{k+l} \frac{\partial^2 \widetilde{h}}{\partial p_k \partial p_l} + klp_{k+l} \frac{\partial \widetilde{h}}{\partial p_k} \frac{\partial \widetilde{h}}{\partial p_l} \right).$$

On the other hand, the straightforward analog of (11) does not hold for the $\mathbf{w}$-functions: H is not the exponent of $\widetilde{H}$. This problem can be overcome by introducing a different algebra structure on the set $\bigoplus_n \mathbb{C}[\mathbf{w}]^{S_n}$ of all the S_n-invariant polynomials in variables w_{ij}, $1 \leq i < j \leq n$. Namely, for $f_1 \in \mathbb{C}[\mathbf{w}]^{S_k}$ and $f_2 \in \mathbb{C}[\mathbf{w}]^{S_l}$ set, by definition, $f_1 * f_2 \in \mathbb{C}[\mathbf{w}]^{S_{k+l}}$ to be the polynomial obtained by taking the average of $f_1(w_{12}, \dots, w_{k-1,k}) f_2(w_{k+1,k+2}, \dots, w_{k+l-1,k+l})$ over all the permutations $\sigma \in S_{k+l}$ of subscripts. This "shuffle product" is an associative and commutative operation; so one can consider power series with respect to it. Now we have

$$H(\mathbf{w}; p_1, p_2, \dots) = {}^*\exp(\widetilde{H}(\mathbf{w}; p_1, p_2, \dots)),$$

where the right-hand side is the $*$-exponent and the cut-and-join equation for $\widetilde{H}$ reads as follows

$$\sum_{i<j} \frac{\partial \widetilde{H}}{\partial w_{ij}} = \frac{1}{2} \sum_{k,l \geq 1} \left((k+l)p_k p_l \frac{\partial \widetilde{H}}{\partial p_{k+l}} + klp_{k+l} \frac{\partial^2 \widetilde{H}}{\partial p_k \partial p_l} + klp_{k+l} \frac{\partial \widetilde{H}}{\partial p_k} * \frac{\partial \widetilde{H}}{\partial p_l} \right).$$

References

[1] V. Bouchard, M. Mariño, Hurwitz numbers, matrix models and enumerative geometry, 21 pp. arXiv: 0709.1458.
[2] T. Ekedahl, S. Lando, M. Shapiro, A. Vainshtein, Hurwitz numbers and intersections on moduli spaces of curves, Invent. Math. 146 (2) (2001) 463–533. arXiv:math/0004096.

[3] W. Fulton, J. Harris, Representation Theory. A First Course, Springer, Berlin, 1991.
[4] I.P. Goulden, D.M. Jackson, Transitive factorisation into transpositions and holomorphic mappings on the sphere, Proc. Amer. Math. Soc. 125 (1) (1997) 51–60.
[5] I.P. Goulden, D.M. Jackson, R. Vakil, The Gromov–Witten potential of a point, Hurwitz numbers, and Hodge integrals, Proc. London Math. Soc. 83 (3) (2001) 563–581.
[6] I. Goulden, A. Yong, Tree-like properties of cycle factorizations, J. Combin. Theory Ser. A 98 (1) (2002) 106–117.
[7] D. Gross, W. Taylor, Two-dimensional QCD is a string theory, Nuclear Phys. B 400 (1–3) (1993) 181–208.
[8] J. Harer, D. Zagier, The Euler characteristic of the moduli space of curves, Invent. Math. 85 (3) (1986) 457–485.
[9] M.E. Kazarian, S.K. Lando, An algebro-geometric proof of Witten's conjecture, J. Amer. Math. Soc. 20 (4) (2007) 1079–1089.
[10] G. Kirchhoff, Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanischer Ströme gefurht wird, Ann. Phys. Chem. 72 (1847) 497–508.
[11] S.K. Lando, A. Zvonkin, Graphs on Surfaces and their applications. With an appendix by D. Zagier, in: Low-dimensional Topology, II, in: Encyclopaedia of Mathematical Sciences, vol. 141, Springer-Verlag, Berlin, 2004.
[12] I.G. Macdonald, Symmetric functions and Hall polynomials, in: Oxford Mathematical Monographs, Clarendon Press, Oxford, 1979.
[13] D. Panov, D. Zvonkine, Enumeration of almost polynomial rational functions with given critical values, European J. Combin. 29 (2) (2008) 470–479.
[14] D. Poulalhon, G. Schaeffer, Factorizations of large cycles in the symmetric group, Discrete Math. 254 (1–3) (2002) 433–458.
[15] B. Shapiro, M. Shapiro, A. Vainshtein, Ramified coverings of of S^2 with one degenerate branching point and enumeration of edge-ordered graphs, in: Topics in Singularity Theory, vol. 42, in: Amer. Math. Soc. Transl. Ser. 2, vol. 180, 1997, pp. 219–227. www2.math.su.se/~shapiro/Articles.