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ORIGINAL ARTICLE

Approximate analytical solutions of fractional reaction-diffusion equations

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Abstract The homotopy analysis method (HAM) of S.J. Liao has proven useful in obtaining analytical/numerical solutions to various nonlinear differential equations. In this work, the HAM is employed to obtain the analytical/numerical solutions of the nonlinear reaction-diffusion equations with time-fractional derivatives. The fractional derivatives are described in the Caputo sense. This approach transforms the solution of nonlinear reaction-diffusion equations into the solution of a hierarchy of linear equations. The solution is simple yet highly accurate and compare favorably with the solutions obtained early in the literature.

1. Introduction

Nonlinear reaction-diffusion (RD) equations have been widely studied throughout recent years. These equations arise naturally as description models of many evolution problems in the real world, as in chemistry (Slepchenko et al., 2000; Vidal and Pascault, 1986), biology (Murray, 1977), etc. As is well known, complex behavior is a peculiarity of systems modeled by reaction diffusion equations, and the Belousov–Zhabotinskii reaction (Muller et al., 1987; Winfree, 1972) provides a classic example.

The RD equations describes a population of diploid individuals (i.e., the ones that carry two genes) distributed in *a* at two-dimensional habitat. Assuming that a gene occurs in two forms *a* and *A*, called *alleles*, one can divide the population into three genotypes *aa*, *aA*, and *AA*. The RD equations are employed to describe the co-oxidation on Pt(1 1 0) (Bar et al., 1994), the study of temporal and spatial patterns of cytoplasmic Ca^{2+} dynamics under the effects of Ca^{2+}-release activated Ca^{2+} (CRAC) channels in T cells (Cong-Xin et al., 2010), problems in finance (Gorenflo et al., 2001; Mainardi et al., 2000; Raberto et al., 2002), and hydrology (Benson et al., 2000). Burke (Burke et al., 1993) obtained solutions for an enzyme–suicide substrate reaction with an instantaneous point source of substrate. Grimson and Barker (1993)

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introduced a continuum model for the spatio-temporal growth of bacterial colonies on the surface of a solid substrate which utilizes a RD equation for growth. Many cellular and sub-cellular biological processes (Erban and Chapman, 2009) can be described in terms of diffusing and chemically reacting species (e.g. enzymes). A traditional approach to the mathematical modeling of such RD processes is to describe each biochemical species by its (spatially dependent) concentration.

The nature of the diffusion is characterized by the temporal scaling of the mean-square displacement $\langle X^2 \rangle$ which is proportional to a power t^κ of order κ. This type of diffusion is characterized the sub-diffusion, normal diffusion, and super diffusion processes as the cases where $0 < \kappa < 1$, $\kappa = 1$ and $\kappa > 1$ respectively.

In recent time, interest in fractional RD equations (Henry and Wearne, 2000; Grafiychuk et al., 2006, 2007; Saxena et al., 2006) has increased because the equation exhibits self-organization phenomena and introduces a new parameter, the fractional index, into the equation. Additionally, the analysis of fractional RD equations is of great interest from the analytical and numerical point of view. From a mathematics point of view they also offer a rich and promising area of research. RD equations have been investigated for certain boundary and initial conditions and in most cases explicit solutions cannot been found. We present solutions of a more general model of time-fractional RD equations:

$$D_t^\alpha u(x,t) = dD_x^2 u(x,t) + G(u) \quad 0 < \alpha \leqslant 1,\ x \in \Re,\ t > 0 \qquad (1)$$

where d is the diffusion coefficient and $G(u)$ is a nonlinear function representing reaction kinetics.

Let $u(x,t)$ be the concentration of the members of the population with the mutant gene and $r = r(x,t)$ the concentration of the members of the population whose off springs have the mutant gene (x is a position in the habitat). One can assume that $r = 1 - u$. Denote by b the intensity of selection in favor of the mutant gene, which we assume to be independent of u. The nonlinear RD equation (1) is given by

$$D_t^\alpha u(x,t) = dD_x^2 u(x,t) + bu(1-u) \qquad (2)$$

Fitzhugh (1961) suggested a model for the propagation of voltage impulse through a nerve axon which is substantially simpler than the Hodgkin–Huxley model and therefore, is often used in applications. Fitzhugh treated the nerve cell as a nonlinear electric oscillator (Fitzhugh, 1969; Nagumo et al., 1962). The nonlinear RD equation (1) is given by

$$D_t^\alpha u(x,t) = dD_x^2 u(x,t) - au(u-1)(u-\theta) \qquad (3)$$

with $a > 0$ and $\theta \in (0, 0.5]$. Note that the case $\theta \in (0.5, 1)$ can be reduced to the previous one by u with $1 - u$.

The widely applied techniques (i.e., perturbation method) are of great interest to be used in engineering and science. To eliminate the limitation of "small parameter", which is assumed in the perturbation method, a new technique based on the homotopy terminology has been proposed. Accordingly, a nonlinear problem is transformed into an infinite number of simple problems without using the perturbation technique. We used an elegant method which has proved effectiveness and efficiency in solving many type of nonlinear problems (Liao et al., 2002; Sanjay and Wiese, 2003). Homotopy analysis method (HAM) (Liao, 2004) properly overcomes restrictions of perturbation techniques because it does not need any small or large parameters to be contained in the problems. Khan et al. (2010b) studied the fourth-order parabolic partial differential equation via HAM. The validity of the HAM is varied through illustrative examples of Cauchy reaction-diffusion equation (Bataineh et al., 2008) linear and nonlinear fractional diffusion-wave equation (Jafri and Seifi, 2009) and multi-term diffusion-wave (Jafri et al., 2010) equations. Also, it is shown that homotopy perturbation method (HPM) (Khan et al., 2009, 2010c; Mohyud-Din et al., 2010; Yıldırım, 2010), variational iteration method (VIM) (Wang and He, 2008; Mohyud-Din and Yıldırım, 2010), differential transform method (DTM) (Rida et al., 2010), Adomian decomposition method (ADM) (Wazwaz and Gorguis, 2004; Khan et al., 2010a) are just special case of HAM.

2. Basic definitions of fractional calculus

We give some basic definitions and properties of the fractional calculus theory which are used in this paper.

Definition 2.1. Caputo's definition of the fractional-order derivative is defined as

$$D^\mu f(t) = \frac{1}{\Gamma(n-\mu)} \int_a^t \frac{f^{(n)}(\xi)}{(t-\xi)^{\mu+1-n}} d\xi, \quad (n-1 < \mathrm{Re}(\alpha) \leqslant n,\ n \in N),\ t > 0 \qquad (4)$$

where the parameter μ is the order of the derivative and is allowed to be real or even complex, a is the initial value of function f. In the present work only real and positive μ will be considered. For the Caputo's derivative we have

$$D^\mu C = 0, \quad (C \text{ is a constant}) \qquad (5)$$

$$D^\alpha t^\gamma = \begin{cases} 0, & (\gamma \leqslant \alpha - 1) \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha}, & (\gamma > \alpha - 1) \end{cases} \qquad (6)$$

Definition 2.2. For m to be the smallest integer that exceeds a, n to be the smallest integer that exceeds γ, the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as

$$D^\mu u(x,t) = \frac{\partial^\mu u(x,t)}{\partial t^\mu} = \begin{cases} \frac{1}{\Gamma(m-\mu)} \int_a^t \frac{f^{(m)}(\xi)}{(t-\xi)^{\mu+1-m}} \frac{\partial^m u(x,t)}{\partial t^m} d\xi, & m-1 < \beta < m \\ \frac{\partial^m u(x,t)}{\partial t^m}, & \mu = m \in N \end{cases} \qquad (7)$$

For establishing our results, we also necessarily introduce following Riemann–Liouville fractional integral operator.

Definition 2.3. The Riemann–Liouville fractional integral operator of order, of a function f,

$$J^\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^x (t-\xi)^{\mu-1} f(\xi)\, d\xi, \quad \alpha > 0, \quad t > 0 \qquad (8)$$

Properties and applications can be found in references (Kilbas et al., 2006; Mahmood et al., 2009, 2010) and we mention only some following: for $\mu \geqslant -1, \alpha, \beta \geqslant 0, \gamma \geqslant -1$ is defined as

$$J^0 f(t) = f(t), \quad J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t),\ J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\gamma+\alpha},$$
$$J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t) \qquad (9)$$

Also, we need here two of its basic properties. If

$$D^\alpha J^\beta f(t) = f(t), \quad J^\alpha D^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{t^k}{k!}, \ t > 0 \tag{10}$$

3. The HAM

In this paper, we apply the HAM to the nonlinear problems to be discussed. In order to show the basic idea of HAM, consider the following nonlinear fractional differential equation:

$$N_F[z(x,t)] = 0 \tag{11}$$

where N_F is a nonlinear fractional operator, x and t denote the independent variables and $z(x,t)$ is an unknown function. By means of the HAM, we first construct the so-called zeroth-order deformation equation

$$(1-p)\pounds_F[\phi(x,t;p) - z_0(x,t)] = p\hbar N_F[\phi(x,t;p)] \tag{12}$$

Where $p \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ is an auxiliary parameter, $\pounds_F$ is an auxiliary linear operator, $\phi(x,t;p)$ are unknown functions, $z_0(x,t)$ are initial guesses of $\phi(x,t;p)$. It is obvious that when the embedding parameter $p = 0$ and $p = 1$, Eq. (12) becomes

$$\phi(x,t;0) = z_0(x,t;p), \quad \phi(x,t;1) = z(x,t) \tag{13}$$

respectively. Thus as $p = 0$ increases from 0 to 1, the solution $\phi(x,t;p)$ varies from the initial guess $z_0(x,t;p)$ to the exact solution $z(x,t)$. Expanding $\phi(x,t;p)$ in Taylor series with respect to $p = 0$, one has

$$\phi(x,t;p) = z_0(x,t) + \sum_{m=1}^{+\infty} z_m(x,t)p^m \tag{14}$$

where

$$z_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;p)}{\partial p^m}\bigg|_{p=0} \tag{15}$$

The convergence of the solution (14) depends upon the auxiliary parameter $\hbar$. If it is convergent at $p = 1$, one has

$$z(x,t;1) = z_0(x,t) + \sum_{m=1}^{+\infty} z_m(x,t) \tag{16}$$

This must be one of the solutions of the original nonlinear equations, as proved by Liao. Define the vectors

$$\vec{z}_n = \{z_0(x,t), z_1(x,t), z_2(x,t), \ldots, z_n(x,t)\} \tag{17}$$

Differentiating the zeroth-order deformation equation (12) m-times with respect to q and then dividing them by $m!$ and finally setting $p = 0$, we get the following mth-order deformation equation:

$$\pounds_F[z_m(x,t) - \chi_m z_{m-1}(x,t)] = \hbar R_m\left(\vec{z}_{m-1}\right) \tag{18}$$

where

$$R_m(z_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N_F[\phi(x,t;p)]}{\partial p^{m-1}}\bigg|_{p=0} \quad \text{and}$$

$$\chi_m = \begin{cases} 0 & m \leqslant 1 \\ 1, & m > 1 \end{cases}$$

The solution of the mth-order deformation equation (18) is readily found to be

$$z_m(x,t) = \chi_m z_{m-1}(x,t) + \pounds_F^{-1}[\hbar R_m(\vec{z}_{m-1}) \tag{19}$$

It should be emphasized that $z_i(x,t)$ is governed by equation (19) with the initial conditions that come from the original problems, which can be easily solved by symbolic computation software such as *MATHEMATICA*.

4. Time-fractional reaction-diffusion equations

To incorporate our discussion above, we consider nonlinear reaction-diffusion equations with time-fractional derivative which are arising in diverse phenomenon.

4.1. Application 1

Consider the time-fractional RD problem (Wazwaz and Gorguis, 2004)

$$D_t^\alpha u = u_{xx} + u(1-u) \tag{20}$$

Subject to a constant initial condition

$$u(x,0) = c \tag{21}$$

To solve the above problem by HAM, we choose the auxiliary operators as follows

$$\pounds_F[\phi(x,t;p)] = D_t^\alpha[\phi(x,t;p)]$$

with the property $\pounds_F[c_1] = 0$

Using the above definition, we construct the zeroth-order deformation equations

$$(1-p)\pounds_F[\phi(x,t;p) - \chi_m u_0(x,t)] = p\hbar N_F[\phi(x,t;p)] \tag{22}$$

obviously, when $p = 0$ and $p = 1$

$$\phi(X,t;0) = u_0(x,t), \quad \phi(x,t;1) = u(x,t) \tag{23}$$

Differentiating the zeroth-order deformation equation (22) m times with respect to p, and finally dividing by $m!$, we have the mth-order deformation equations

$$\pounds_F[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar R_m[\vec{u}_{m-1}] \tag{24}$$

where

$$R_m[\vec{u}_{m-1}] = D_t^\alpha u_{m-1}(x,t) - u_{xxm-1}(x,t) - u_{m-1}(x,t) + \sum_{i=0}^{m-1} u_i u_{m-1-i} \tag{25}$$

On applying the operator J_t^α both side of equation (24), we get

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar J_t^\alpha R_m[\vec{u}_{m-1}] \tag{26}$$

subsequently solving the mth-order deformation equations one has

$$u_0(x,t) = c \tag{27}$$

$$u_1(x,t) = \frac{\hbar(c^2 - c)t^\alpha}{\Gamma(\alpha+1)} \tag{28}$$

$$u_2(x,t) = (-1+c)c\left(\frac{\hbar(\hbar+1)t^\alpha}{\Gamma(\alpha+1)} + \frac{\hbar^2(-1+2c)t^{2\alpha}}{\Gamma(2\alpha+1)}\right) \tag{29}$$

We use an 11-term approximate solution and set

$$u(x,t) = u_0 + u_1 + \cdots + u_{10} \tag{30}$$

The exact solution of the equation (20) as $\alpha \to 1$ is

$$u(x,t) = \frac{ce^t}{1 - c + ce^t} \tag{31}$$

4.2. Application 2

Let us consider equation the time-fractional Fisher equation (Wazwaz and Gorguis, 2004; Rida et al., 2010)

$$D_t^\alpha u = u_{xx} + 6u(1-u) \tag{32}$$

subject to an initial condition

$$u(x,0) = \frac{1}{(1+e^x)^2} \tag{33}$$

To solve the above problem by HAM, we choose the auxiliary operators as follows

$$\pounds_F[\phi(x,t;p)] = D_t^\alpha \phi(x,t;p)$$

With the properties $\pounds_F[c_1] = 0$

$$N_F[\phi(x,t;q)] = D_t^\alpha \phi(x,t;q) - \phi_{xx}(x,t;q) - 6\phi(x,t;q) + 6\phi^2(x,t;q), \tag{34}$$

By mean of zeroth components u_0 the remaining components $u_m, m \geqslant 1$, can be completely determined such that each terms are determined by using the previous terms, the components of solution for this case are

$$u_0(x,t) = \frac{1}{(1+e^x)^2} \tag{35}$$

$$u_1(x,t) = \frac{-10\hbar e^x}{(1+e^x)^3} \frac{t^\alpha}{\Gamma(\alpha+1)} \tag{36}$$

$$u_2(x,t) = \frac{10e^x\hbar}{(1+e^x)^4}\left(\frac{-(1+\hbar)(1+e^x)t^\alpha}{\Gamma(\alpha+1)} + \frac{5(-1+2e^x)t^{2\alpha}}{\Gamma(2\alpha+1)}\right) \tag{37}$$

The approximate solution of equation (32) by the HAM is

$$u(x,t) = u_0 + u_1 + \cdots + u_{10} \tag{38}$$

Our approximation has one more interesting property, if we set $\hbar = -1$ and expand approximate solution using Taylor's series about $(0,0)$ we obtain the same solution obtained by Rida et al. (Rida et al., 2010). If we set $\hbar = -1, \alpha = 1$ we obtain the same result obtained by Wazwaz and Gorguis (2004).

The exact solution of the equation (32) with condition (33) as $\alpha \to 1$ is

$$u(x,t) = \frac{1}{(1+e^{x-5t})^2} \tag{39}$$

4.3. Application 3

Here, we consider time-fractional Fitzhugh–Nagumo equation (Wazwaz and Gorguis, 2004; Rida et al., 2010)

$$D_t^\alpha u = u_{xx} + u(1-u)(u-\theta), \quad 0 < \theta < 1 \tag{40}$$

subject to initial condition

$$u(x,0) = \frac{1}{1+e^{\frac{-x}{\sqrt{2}}}} \tag{41}$$

In genetics the case $0 < \theta < 1$ is referred to as the heterozygote inferiority. The exact solution of equation (40) for $\alpha = 1$ obtained by Kawahara and Tanaka (1983), which describes the coalescence of two travelling fronts of the same sense into a front connecting two stable constant states is found.

To solve the above equation by HAM, we write

$$\pounds_F[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar R_m[\vec{u}_{m-1}] \tag{42}$$

where

$$R_m = D_t^\alpha u_{m-1} - u_{xxm-1} + \theta u_{m-1} - (1+\theta)\sum_{j=0}^{k} u_{k-l}u_l + \sum_{k=0}^{m-1} u_{m-k-1}\sum_{j=0}^{k} u_{k-l}u_l \tag{43}$$

$$u_m(x,t) = \chi_m u_{m-1} + \hbar J_t^\alpha R_m[\vec{u}_{m-1}] \tag{44}$$

Subsequently solving the mth-order deformation equations one has

$$u_0(x,t) = \frac{1}{1+e^{\frac{-x}{\sqrt{2}}}} \tag{45}$$

$$u_1(x,t) = \frac{\hbar(-1+2\theta)e^{\frac{x}{\sqrt{2}}}}{2(1+e^{\frac{x}{\sqrt{2}}})^2}\frac{t^\alpha}{\Gamma(\alpha+1)} \tag{46}$$

$$u_2(x,t) = \frac{(-1+2\theta)e^{\frac{x}{\sqrt{2}}}}{4(1+e^{\frac{x}{\sqrt{2}}})^4} \times \left(\frac{\hbar^2(-1-2\theta+8e^{\frac{x}{\sqrt{2}}}+(-3+2\theta)e^{\sqrt{2}x})t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\hbar(\hbar+1)(1+e^{\frac{x}{\sqrt{2}}})^2 t^\alpha}{\Gamma(\alpha+1)}\right) \tag{47}$$

The approximate solution of equation (40) by the HAM is

$$u(x,t) = u_0 + u_1 + \cdots + u_{10} \tag{48}$$

If we set $\hbar = -1$ and expand approximate solution using Taylor's series about (0,0) we obtain the same solution obtained by Rida et al. (Rida et al., 2010). If we set $\hbar = -1, \alpha = 1$ we obtain the same result obtained by Wazwaz and Gorguis (2004).

As $\alpha \to 1$ the close form solution is given by

$$u(x,t) = \frac{1}{1+e^{\frac{x}{\sqrt{2}}+\rho t}} \tag{49}$$

Where $\rho = \frac{1}{\sqrt{2}} - \sqrt{2}\theta$. This is an exact solution of the standard form of Fitzhugh–Nagumo equation.

5. Closing remarks

Let us conclude this paper with a summary and discussion of the results presented here. We have studied three important

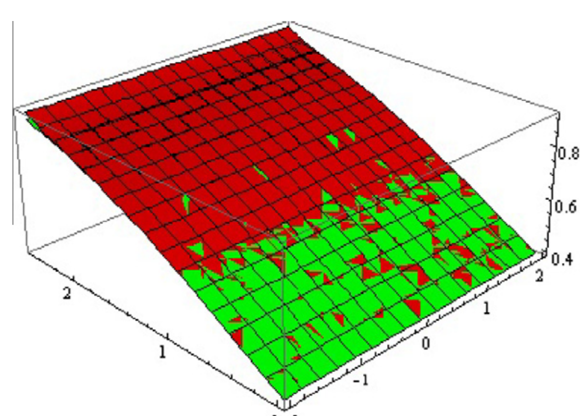

Figure 1 Exact, approx. (red, green) solution surfaces of $u(x,t)$ for $\hbar = -1, c = 0.4$.

applications of reaction-diffusion systems, namely the Fisher equations (with constant and variable initial conditions) and the time-fractional Fitzhugh–Nagumo equation. We are interested in the linearization of these equations, i.e. conversion of the nonlinear problem to linear problems.

We have derived solution for $\alpha \in (0, 1]$, however figures are included here for some particular cases. Plotting the $\hbar$ curves, we have intervals [0, 1.6], [1.6, 2.2] and [−2, 2] as the valid region of $\hbar$ for first, second, and third applications. We test different values of $\hbar$ in the valid region and conclude that $\hbar = -1$

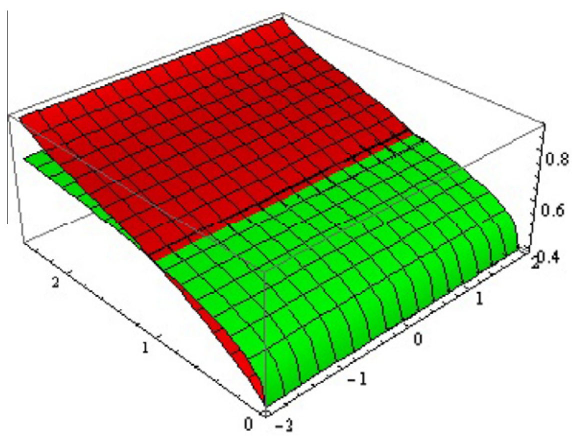

Figure 2 Approx. (red $\alpha = 0.8$, green $\alpha = 0.4$) solutions surfaces of $u(x, t)$ for $\hbar = -1, c = 0.4$.

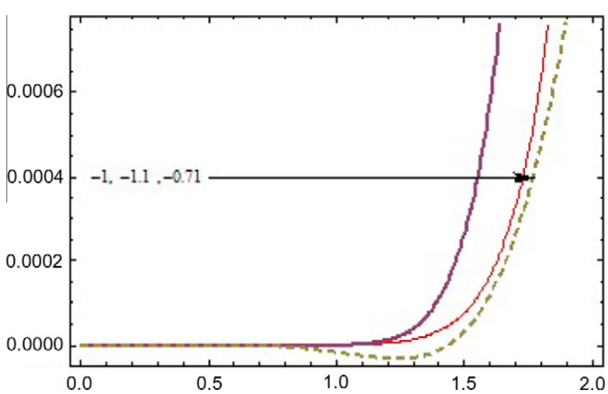

Figure 3 Error curves of $u(x, t)$ for $\alpha = 1, c = 0.4$ $\hbar = -1$ for different $\hbar = -1, -1.1, -0.71$.

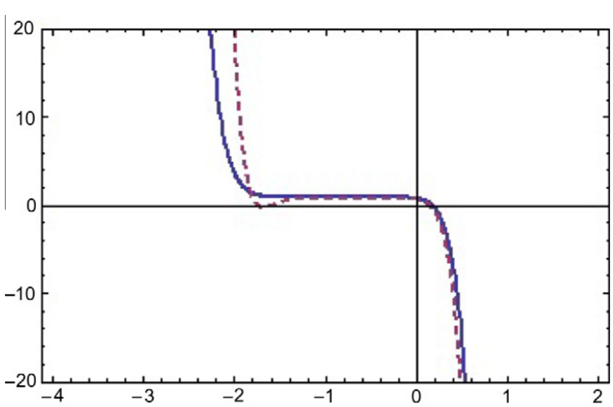

Figure 4 $\hbar$ curves of $u(x, t)$ for $c = 0.8, t = 1, x = 0.5$ $\alpha = 1$ (thick), $\alpha = 0.5$ (dashed).

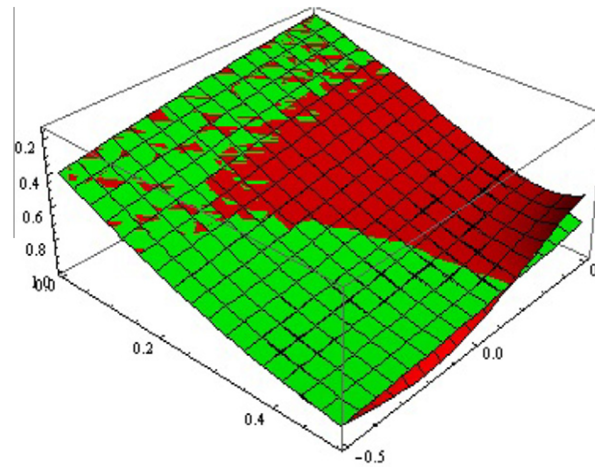

Figure 5 Exact, approx. (red, green) solution surfaces of $u(x, t)$ for $\hbar = -1, c = 0.4, \alpha = 1$.

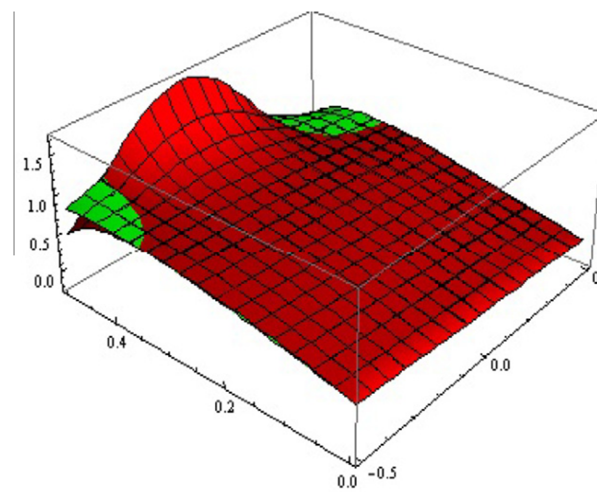

Figure 6 Approx. (red $\alpha = 0.9$, green $\alpha = 0.95$) solutions surfaces of $u(x, t)$ for $\hbar = -1$,.

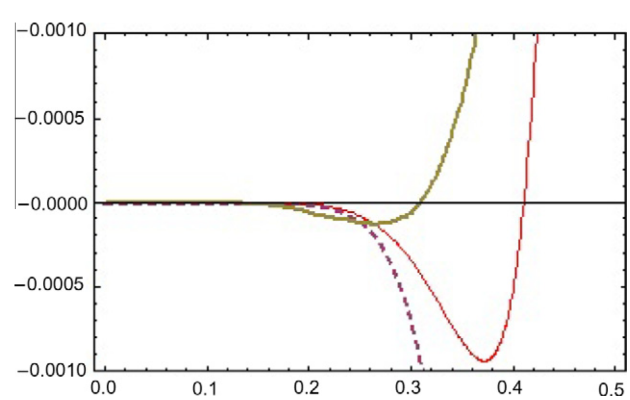

Figure 7 Error curves of $u(x, t)$ for $\alpha = 1, c = 0.4$ for different $\hbar = -1, -0.9, -0.7$ (red, dashed and thick).

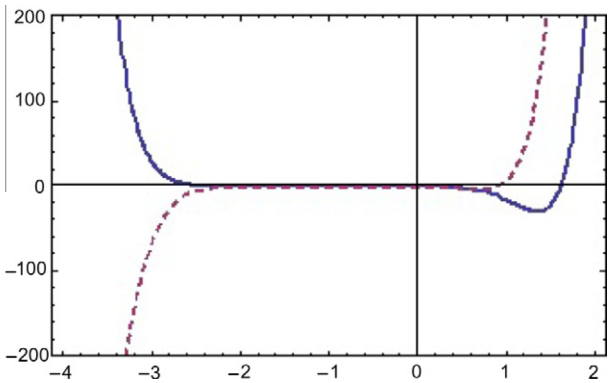

Figure 8 $\hbar$ curves of $u(x, t)$ for $t = 0.1, x = 0.5$ $\alpha = 1$ (thick), $\alpha = 0.9$ (dashed).

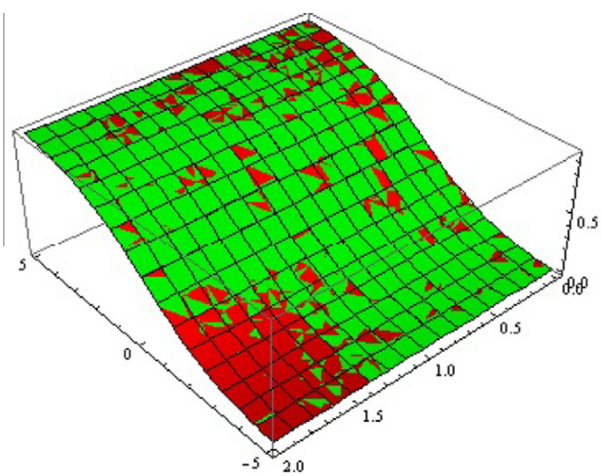

Figure 9 Exact, approx. (green, red) solution surfaces of $u(x, t)$ for $\hbar = -1, \theta = 0.5, \alpha = 1$.

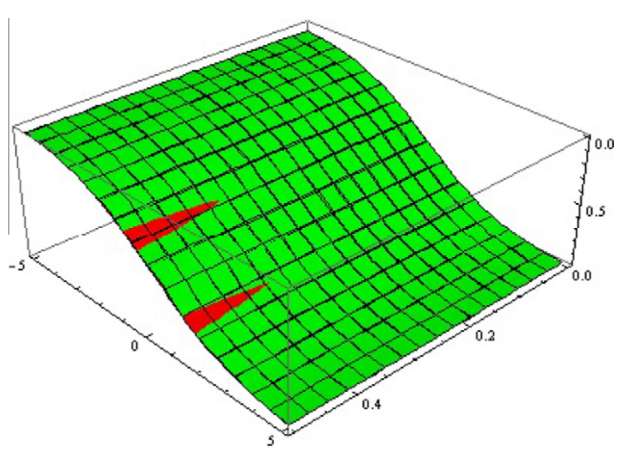

Figure 10 Approx. (red $\alpha = 0.9$, green $\alpha = 0.2$) solutions surfaces of $u(x, t)$ for $\hbar = -1, \theta = 0.6$.

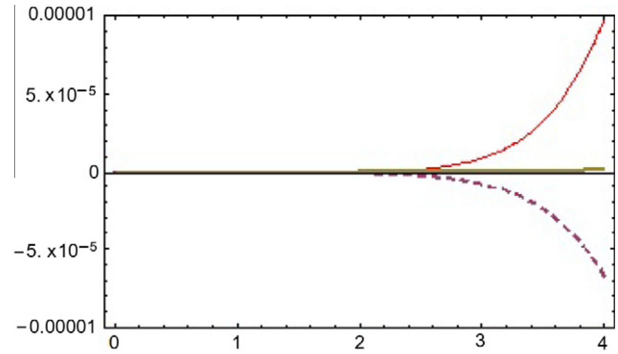

Figure 11 Error curves of $u(x, t)$ for $\alpha = 1, \theta = 0.5, x = 0.6$ for different $\hbar = -1, -0.9, -0.7$ (red, dashed, thick).

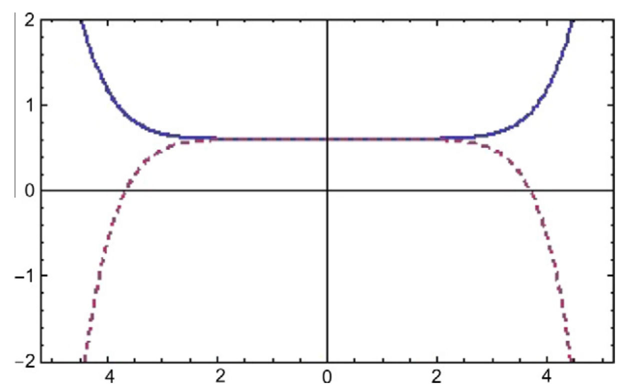

Figure 12 $\hbar$ curves of $u(x, t)$ for $t = 3, \theta = 0.5, x = 0.6$ $\alpha = 0.75$ (thick), $\alpha = 0.5$ (dashed).

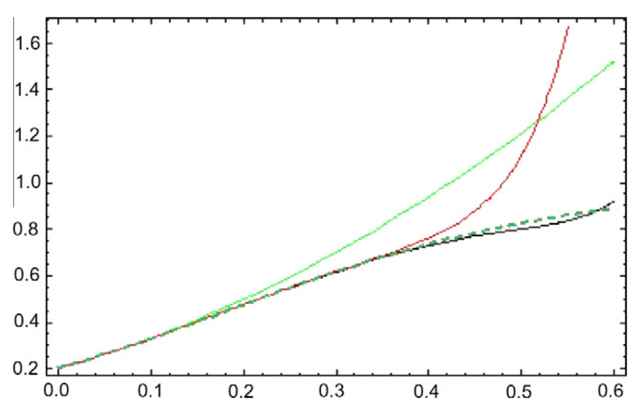

Figure 13 Approximation of $u(x, t)$ for $\alpha = 1, \theta = 0.5, x = 0.2$ DTM, HAM ($\hbar = -1, \hbar = -1.1$), exact (green, black, red, dashed).

is the one whose results did not give the minimum error. The solution surfaces for application 1 are displayed in Figs. 1 and 2. The solution surfaces for application 2 are depicted for different values of constants in Figs. 5 and 6. Finally, the approximate solution of Fitzhugh–Nagumo equation are depicted in Figs. 9 and 10. Figs. 3, 7 and 11 show the values of error function. The valid region of auxillary parameters are depicted in Figs. 4, 8 and 12 for first, second and third applications. The comparison of solution curves of application 2 and 3 is displayed in Figs. 13 and 14. The present solutions show the better performance than DTM solution. Recently Rida et al. (2010) have solved the problems we presented in

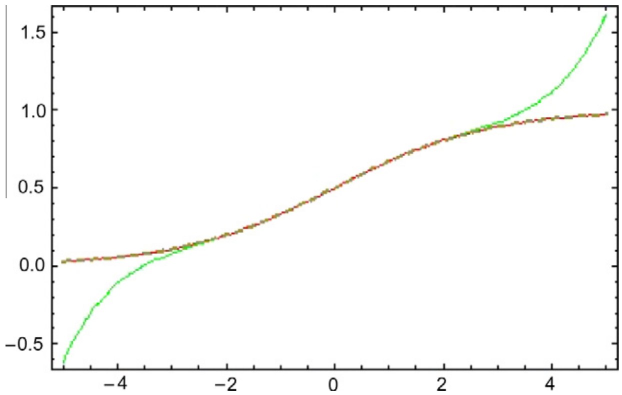

Figure 14 Approximation of $u(x,t)$ for $\alpha = 1, t = 2, \theta = 0.5$ DTM, HAM ($\hbar = -1$), exact (green, red, dashed).

application 2 and 3 using generalized differential transform method (GDTM), a method which provides approximate analytical solutions in the form of an infinite power series for nonlinear equations. The major lacks of GDTM are that it requires transformation and the given differential equation and related initial conditions are transformed into a recurrence equation that finally leads to the solution of a system of algebraic equations as coefficients of a power series solution.

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