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On L_p-estimates for a class of non-local elliptic equations

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Abstract

We consider non-local elliptic operators with kernel $K(y) = a(y)/|y|^{d+\sigma}$, where $0 < \sigma < 2$ is a constant and a is a bounded measurable function. By using a purely analytic method, we prove the continuity of the non-local operator L from the Bessel potential space H_p^σ to L_p, and the unique strong solvability of the corresponding non-local elliptic equations in L_p spaces. As a byproduct, we also obtain interior L_p-estimates. The novelty of our results is that the function a is not necessarily to be homogeneous, regular, or symmetric. An application of our result is the uniqueness for the martingale problem associated to the operator L.

Keywords: Non-local elliptic equations; Bessel potential spaces; Lévy processes; The martingale problem

1. Introduction

Non-local equations such as integro-differential equations for jump Lévy processes have attracted the attention of many mathematicians. These equations arise from models in physics,

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engineering, and finance that involve long-range interactions (see, for instance, [9]). An example is the following non-local elliptic equation associated with pure jump process (see, for instance, [23]):

$$Lu - \lambda u = f \quad \text{in } \mathbb{R}^d, \tag{1.1}$$

where

$$Lu = \int_{\mathbb{R}^d} \big(u(x+y) - u(x) - y \cdot \nabla u(x) \chi^{(\sigma)}(y)\big) K(x,y)\, dy,$$

$$\chi^{(\sigma)} \equiv 0 \quad \text{for } \sigma \in (0,1), \qquad \chi^{(1)} = 1_{y \in B_1}, \qquad \chi^{(\sigma)} \equiv 1 \quad \text{for } \sigma \in (1,2). \tag{1.2}$$

In the above, λ is a nonnegative constant and $K(x,y)$ is a positive kernel which has the following lower and upper bounds:

$$(2-\sigma)\frac{\nu}{|y|^{d+\sigma}} \leqslant K(x,y) \leqslant (2-\sigma)\frac{\Lambda}{|y|^{d+\sigma}}, \tag{1.3}$$

where $0 < \nu \leqslant \Lambda < \infty$ are two constants.

As is well known, if $K(x,y) = c^{-1}|y|^{-d-\sigma}$ with $c = c(d,\sigma) > 0$ and $\sigma \in (0,2)$, we get the fractional Laplace operator $-(-\Delta)^{\sigma/2}$, which has the symbol $-|\xi|^{\sigma}$. In this case, the classical theory for pseudo-differential operators shows that, for any $\lambda > 0$ and $f \in L_p(\mathbb{R}^d)$, $1 < p < \infty$, there exists a unique solution $u \in H_p^{\sigma}(\mathbb{R}^d)$ to Eq. (1.1) satisfying

$$\|u\|_{H_p^{\sigma}(\mathbb{R}^d)} \leqslant N(d,\sigma,\lambda,p)\|f\|_{L_p(\mathbb{R}^d)};$$

see Section 2.1 for the definition of the Bessel potential space $H_p^{\sigma}(\mathbb{R}^d)$. In general, if the symbol of the operator is sufficiently smooth and its derivatives satisfy appropriate decays, the aforementioned L_p-solvability is classical following from the Fourier multiplier theorems (see, for instance [28,15,13]). It should be pointed out the L_p-solvability is also available if the kernel $K(y)$ is of the form $a(y)/|y|^{d+\sigma}$, and $a(y)$ is homogeneous of order zero and sufficiently smooth; see [18,23].

In this paper, as a first step of our project, we extend this type of L_p-solvability to Eq. (1.1)[3] when the kernel K is translation invariant with respect to x, i.e., $K(x,y) = K(y)$, merely measurable in y, and satisfies only the ellipticity condition (1.3). Moreover, if $\sigma = 1$ we make a natural cancellation assumption on K; see (2.1). Note that the operator L has the symbol

$$m(\xi) = \int_{\mathbb{R}^d} \big(e^{iy\cdot\xi} - 1 - iy \cdot \xi \chi^{(\sigma)}(y)\big) K(y)\, dy,$$

which generally lacks sufficient differentiability to apply the classical multiplier theorems.

There has been considerable work concerning regularity issues of solutions to non-local equations, such as the Harnack inequality, Hölder estimates, and non-local versions of the

[3] One can also consider Eq. (1.1) with $\chi^{(\sigma)} = 1_{y \in B_1}$ for all $\sigma \in (0,2)$. For a discussion about this case, see Remark 2.5.

Aleksandrov–Bakelman–Pucci (ABP) estimate. Firstly appeared approaches were probabilistic; see, for example, [7,5,6]. Recently, analytic and PDE techniques have been used to study non-local equations with symmetric kernels in [8,16], and with non-symmetric kernels in [27,17,4, 3].[4] See also [12] for another ABP type estimate for a certain class of fully nonlinear non-local elliptic equations.

On the other hand, to the best of our knowledge, little is known in the literature about the L_p-estimates of non-local operators if K is only measurable and non-symmetric. Our approach in this paper is purely analytic and uses techniques only from PDE, and does not use any multipliers or probabilistic representations of solutions. We obtain a fully equipped L_p-estimate which enables us to get the desired L_p-solvability of Eq. (1.1) in the space $H_p^\sigma(\mathbb{R}^d)$, $\sigma \in (0,2)$; see Theorem 2.1. We note that, in the symmetric case, a related L_p-estimate can be deduced from the main result in a fairly recent paper [2], where a probabilistic approach is used to study Fourier multipliers. To be precise, thanks to the symmetry of $K(y)$, applying Theorem 1 in [2] to the symbol

$$M(\xi) = \frac{\int_{\mathbb{R}^d}(\cos(\xi\cdot y)-1)a^{-1}(y)V(dy)}{\int_{\mathbb{R}^d}(\cos(\xi\cdot y)-1)V(dy)}, \qquad V(dy) = K(y)\,dy,$$

gives

$$\|u\|_{\dot H_p^\sigma(\mathbb{R}^d)} \leqslant N\|Lu\|_{L_p(\mathbb{R}^d)};$$

see Section 2.1 for the definition of the homogeneous space $\dot H_p^\sigma(\mathbb{R}^d)$.

Our proof of L_p-estimates for non-local operators is founded on so-called *mean oscillation estimates* along with the Hardy–Littlewood maximal function theorem and the Fefferman–Stein theorem. This method was used by N.V. Krylov in [20] to treat second-order elliptic and parabolic equations with VMO_x coefficients (see also [14,10] for earlier work), and further developed in a series of papers including [21] and [11] for second-order and higher-order equations with rough coefficients. In this paper, we adapt this method to study non-local operators. One feature of the method is that it does not require a representation formula of solutions via fundamental solutions, which makes it possible to deal with non-local operators with inhomogeneous and merely measurable kernels. The key step in establishing the mean oscillation estimates of solutions is based on the following C^α-estimate for the non-local equation $Lu - \lambda u = f$:

$$[u]_{C^\alpha(B_{1/2})} \leqslant N\int_{\mathbb{R}^d}\frac{|u|}{1+|x|^{d+\sigma}}\,dx + N\,\mathrm{osc}_{B_1}|f|,$$

with a constant N which is independent of the size of $\lambda \geqslant 0$; cf. Corollary 4.3. This estimate is non-local in the sense that the local Hölder norm of the solution u depends on u itself in the whole space. For the proof, we use some ideas from [4]. To proceed from this Hölder estimate

[4] The kernel K is said to be symmetric if $K(y) = K(-y)$. In this case, Lu can be written as

$$Lu(x) = \frac{1}{2}\int_{\mathbb{R}^d}\big(u(x+y)+u(x-y)-2u(x)\big)K(y)\,dy.$$

to the mean oscillation estimate of u, we make a crucial observation that the first term on the right-hand side above can be bounded by the maximal function of u at the origin. We then use this idea to further estimate the mean oscillation of the fractional derivative $(-\Delta)^{\sigma/2}u$.

We remark that during the preparation of this paper we learned that Mikulevicius and Pragarauskas established L_p-estimates for non-local parabolic equations in [25], where they considered both stochastic local and non-local equations using probabilistic methods. The ellipticity condition in [25] is slightly more general than ours replacing ν in (1.3) by a sufficiently smooth, positive, and homogeneous of order zero function, which can be degenerate on the whole space except on an arbitrarily narrow cone with vertex at zero (also see [23]). However, for the strong solvability, the authors of [25] appealed to the continuity estimate of L proved in [2] and [23], which requires either the symmetry of K or the homogeneity and sufficient smoothness of K. A direct consequence of our main result is the strong solvability of the stochastic non-local equations under considerably relaxed conditions; see Remark 2.4.

We state the main result, Theorem 2.1, and its applications in the next section after we introduce a few necessary notation. The proof of Theorem 2.1 will be given in Section 6 after we prove an L_2-estimate in Section 3, a Hölder estimate in Section 4, and finally mean oscillation estimates in Section 5. Section 7 is devoted to several interior local estimates, which are deduced from the global estimate in Theorem 2.1.

2. Main result

2.1. Function spaces and notation

For $p \in (1, \infty)$ and $\sigma > 0$, we use $H_p^\sigma(\mathbb{R}^d)$ to denote the Bessel potential space

$$H_p^\sigma(\mathbb{R}^d) = \{u \in L_p(\mathbb{R}^d):\ (1-\Delta)^{\sigma/2}u \in L_p(\mathbb{R}^d)\},$$

which is equipped with the norm

$$\|u\|_{H_p^\sigma(\mathbb{R}^d)} = \|(1-\Delta)^{\sigma/2}u\|_{L_p(\mathbb{R}^d)}.$$

The homogeneous space is denoted by

$$\dot{H}_p^\sigma(\mathbb{R}^d) = \{u \in S'(\mathbb{R}^d):\ (-\Delta)^{\sigma/2}u \in L_p(\mathbb{R}^d)\},$$

where $S'(\mathbb{R}^d)$ is the space of tempered distributions. We use the semi-norm

$$\|u\|_{\dot{H}_p^\sigma(\mathbb{R}^d)} = \|(-\Delta)^{\sigma/2}u\|_{L_p(\mathbb{R}^d)}.$$

Note that by the inequalities

$$N_1(1+|\xi|^\sigma) \leqslant (1+|\xi|^2)^{\sigma/2} \leqslant N_2(1+|\xi|^\sigma),$$

we have

$$\|u\|_{H_p^\sigma(\mathbb{R}^d)} \approx \|u\|_{L_p(\mathbb{R}^d)} + \|u\|_{\dot{H}_p^\sigma(\mathbb{R}^d)}.$$

Throughout the paper we omit $\mathbb{R}^d$ in $C_0^\infty(\mathbb{R}^d)$, $L_p(\mathbb{R}^d)$, or $H_p^\sigma(\mathbb{R}^d)$ whenever the omission is clear from the context. We write $N(d, \nu, \dots)$ in the estimates to express that the constant N is determined only by the parameters $d, \nu, \dots$.

2.2. *Main theorem*

In addition to the ellipticity condition (1.3), in the case $\sigma = 1$ we assume

$$\int_{\partial B_r} yK(y)\, dS_r(y) = 0, \quad \forall r \in (0, \infty), \tag{2.1}$$

where dS_r is the surface measure on ∂B_r. We remark that (2.1) is needed even for the continuity of L from H_2^σ to L_2; cf. Lemma 3.1. In particular, (2.1) is always satisfied for any symmetric kernels. It is worth noting that due to (2.1) the indicator function $\chi^{(1)}$ can be replaced by 1_{B_r} for any $r > 0$.

Here is the main result of this paper.

Theorem 2.1 *(L_p-solvability). Let $1 < p < \infty$, $\lambda \geqslant 0$, and $0 < \sigma < 2$. Assume that $K = K(y)$ satisfies* (1.3) *and, if $\sigma = 1$, K also satisfies the condition* (2.1). *Then L defined in* (1.2) *is a continuous operator from H_p^σ to L_p. For $u \in H_p^\sigma$ and $f \in L_p$ satisfying*

$$Lu - \lambda u = f \quad \text{in } \mathbb{R}^d, \tag{2.2}$$

we have

$$\|u\|_{\dot{H}_p^\sigma} + \sqrt{\lambda}\|u\|_{\dot{H}_p^{\sigma/2}} + \lambda\|u\|_{L_p} \leqslant N\|f\|_{L_p}, \tag{2.3}$$

where $N = N(d, \nu, \Lambda, \sigma, p)$. Moreover, for any $\lambda > 0$ and $f \in L_p$, there exists a unique strong solution $u \in H_p^\sigma$ of (2.2).

Remark 2.2. Upon using the embedding $C^0 \subset H_p^\sigma$ for $p > d/\sigma$, Theorem 2.1 implies a new uniqueness result for the martingale problem associated with the Lévy type operator L; see, for instance, [18]. For other results about the martingale problem for pure jump processes, we refer the reader to [19,26,24,1] and the references therein.

Remark 2.3. For the sake of brevity, in this paper we do not present the precise dependence of the constant N in (2.3) on the regularity parameter σ. Nevertheless, by keeping track of the constants we find that, if $\sigma \in [\sigma_0, 2)$, where $\sigma_0 \in (0, 2)$, in the symmetric case the constant N in the estimate (2.3) depends on σ_0, not σ. In the non-symmetric case, if $0 < \sigma_0 \leqslant \sigma \leqslant \sigma_1 < 1$ or $1 < \sigma_0 \leqslant \sigma < 2$, then the constant N depends on σ_0 (and σ_1), not σ. In particular, N does not blow up as σ approaches 2. A similar fact is observed in the study of local regularities of non-local equations in [8].

Remark 2.4. One noteworthy result in Theorem 2.1 is the continuity of the operator L from H_p^σ to L_p. One can see from the proofs below that for this continuity the lower bound in the ellipticity condition (1.3) is not needed. This implies that the operators in [25] are continuous from H_p^σ to

L_p under Assumption A [25] and the cancellation condition (2.1) in the case $\sigma = 1$. On the other hand, in [25] it is shown that weak solutions are strong solutions if the operators are continuous. Therefore, the weak solutions obtained in [25] are indeed strong solutions (under the additional cancellation condition (2.1) when $\sigma = 1$).

A natural question is whether the result in Theorem 2.1 can be extended to equations with translation-variant kernels of the form $K(x, y) = a(x, y)|y|^{-d-\sigma}$, under natural conditions on K, say K satisfies the assumptions above and a is uniformly continuous (or smooth) with respect to x. Recall that the classical L_p-theory for second-order equations with uniformly continuous coefficients is built upon the estimates for equations with constant coefficients by using a standard perturbation argument and a partition of unity technique. However, for the non-local operator (1.2), such a perturbation method seems to be out of reach. We note that estimates of this type were obtained in [23] by using the Calderón–Zygmund approach when the function $a(x, y)$ is homogeneous in y of order zero and (some higher order) derivatives of $a(x, y)$ in y are uniformly continuous in x. The L_p-estimate in the translation-variant case remains to be a challenging problem if $a(x, y)$ is inhomogeneous and merely measurable with respect to y.

Remark 2.5. In our main theorem (Theorem 2.1), we consider the operator L in (1.2) with three different $\chi^{(\sigma)}$ depending on the range of σ. In this remark, we discuss the solvability in the unified case $\chi^{(\sigma)} = 1_{y\in B_1}$ for all $\sigma \in (0, 2)$, which is also of interest from the probabilistic point of view. Upon setting

$$\tilde{L}u = \int_{\mathbb{R}^d} \big(u(x + y) - u(x) - y \cdot \nabla u(x) 1_{y\in B_1}\big) K(y)\, dy, \tag{2.4}$$

we observe that

$$\tilde{L}u = Lu + b \cdot \nabla u, \tag{2.5}$$

where

$$b = -\int_{B_1} yK(y)\, dy \quad \text{if } \sigma \in (0, 1), \qquad b = \int_{\mathbb{R}^d\setminus B_1} yK(y)\, dy \quad \text{if } \sigma \in (1, 2).$$

Then the unique solvability in H_p^σ of $\tilde{L}u - \lambda u = f$ follows from that of the equation

$$Lu + b \cdot \nabla u - \lambda u = f \quad \text{in } \mathbb{R}^d, \tag{2.6}$$

where $b = (b_1, \dots, b_d)$ is a constant vector. For Eq. (2.6), as in the proof of Theorem 2.1, it suffices to prove the following estimate for $u \in C_0^\infty$ satisfying (2.6):

$$\|u\|_{\dot{H}_p^\sigma} + \sqrt{\lambda}\|u\|_{\dot{H}_p^{\sigma/2}} + \lambda\|u\|_{L_p} + \|b \cdot \nabla u\|_{L_p} \leqslant N\|f\|_{L_p}, \tag{2.7}$$

where $N = N(d, \nu, \Lambda, \sigma, p)$. This estimate is proved using the results in [25] combined with the continuity of the operator L from H_p^σ to L_p proved in Theorem 2.1. For the reader's convenience, we present a proof at the end of Section 6. We note that, because of (2.5), in general $\tilde{L}$ defined in (2.4) is not a continuous operator from H_p^σ to L_p when $\sigma \in (0,1)$.

3. L_2-estimate

To investigate the L_p-solvability of Eq. (2.2), we first study an L_2-estimate. Recall that

$$-(-\Delta)^{\sigma/2}u(x) = \frac{1}{c}\ \text{P.V.} \int_{\mathbb{R}^d} \big(u(x+y) - u(x)\big)\frac{dy}{|y|^{d+\sigma}},$$

where

$$c = c(d,\sigma) = \frac{\pi^{d/2}2^{2-\sigma}}{\sigma(2-\sigma)}\frac{\Gamma(2-\frac{\sigma}{2})}{\Gamma(\frac{d+\sigma}{2})}. \tag{3.1}$$

Here Γ is the Gamma function. Throughout the paper we always assume that $K = K(y)$ satisfies (1.3) and, if $\sigma = 1$, K also satisfies the condition (2.1).

Lemma 3.1. *The operator L defined in* (1.2) *is continuous from H_2^σ to L_2. Let $\lambda \geqslant 0$ be a constant and $u \in H_2^\sigma$ satisfy*

$$Lu - \lambda u = f \quad \text{in } \mathbb{R}^d,$$

where $f \in L_2(\mathbb{R}^d)$. Then we have

$$\|u\|_{\dot{H}_2^\sigma} + \sqrt{\lambda}\|u\|_{\dot{H}_2^{\sigma/2}} + \lambda\|u\|_{L_2} \leqslant N(d,\nu)\|f\|_{L_2}. \tag{3.2}$$

Proof. We first consider the case $u \in C_0^\infty$. By taking the Fourier transform of (1.2), we have

$$\widehat{Lu}(\xi) = \hat{u}(\xi)\int_{\mathbb{R}^d}\big(e^{i\xi\cdot y} - 1 - iy\cdot\xi\chi^{(\sigma)}(y)\big)K(y)\,dy.$$

Then

$$\begin{aligned}
\int_{\mathbb{R}^d}|Lu|^2\,dx &= \int_{\mathbb{R}^d}\big|\widehat{Lu}(\xi)\big|^2\,d\xi \\
&= \int_{\mathbb{R}^d}\big|\hat{u}(\xi)\big|^2\bigg|\int_{\mathbb{R}^d}\big(e^{i\xi\cdot y} - 1 - iy\cdot\xi\chi^{(\sigma)}(y)\big)K(y)\,dy\bigg|^2\,d\xi \\
&\geqslant \int_{\mathbb{R}^d}\big|\hat{u}(\xi)\big|^2\bigg|\Re\int_{\mathbb{R}^d}\big(e^{i\xi\cdot y} - 1 - iy\cdot\xi\chi^{(\sigma)}(y)\big)K(y)\,dy\bigg|^2\,d\xi \\
&= \int_{\mathbb{R}^d}\big|\hat{u}(\xi)\big|^2\bigg(\int_{\mathbb{R}^d}\big(1-\cos(\xi\cdot y)\big)K(y)\,dy\bigg)^2\,d\xi
\end{aligned}$$

$$\geqslant (2-\sigma)^2\nu^2 \int_{\mathbb{R}^d} \big|\hat{u}(\xi)\big|^2 \bigg(\int_{\mathbb{R}^d} \big(1-\cos(\xi\cdot y)\big)|y|^{-d-\sigma}\,dy \bigg)^2 d\xi$$

$$= \nu^2 c^2 (2-\sigma)^2 \int_{\mathbb{R}^d} \big|(-\Delta)^{\sigma/2} u\big|^2\,dx,$$

where c is from (3.1). Here we used the lower bound in (1.3) and the fact that $1-\cos(\xi\cdot y)$ is non-negative. Note that, for $\sigma \in (0,2)$, there exists $N = N(d)$ such that

$$c(2-\sigma) = \pi^{d/2}\frac{2^{2-\sigma}}{\sigma}\frac{\Gamma(2-\frac{\sigma}{2})}{\Gamma(\frac{d+\sigma}{2})} \geqslant N(d).$$

Hence it follows that

$$\int \big|Lu(x)\big|^2\,dx \geqslant N(d,\nu)\|u\|^2_{\dot{H}^\sigma_2}. \tag{3.3}$$

Similarly,

$$\begin{aligned}
-\int_{\mathbb{R}^d} uLu\,dx &= -\int_{\mathbb{R}^d} \widehat{Lu}(\xi)\overline{\hat{u}(\xi)}\,d\xi \\
&= -\int_{\mathbb{R}^d} \big|\hat{u}(\xi)\big|^2 \int_{\mathbb{R}^d} \big(e^{i\xi\cdot y} - 1 - iy\cdot\xi\chi^{(\sigma)}(y)\big)K(y)\,dy\,d\xi \\
&= -\int_{\mathbb{R}^d} \big|\hat{u}(\xi)\big|^2 \bigg(\Re \int_{\mathbb{R}^d} \big(e^{i\xi\cdot y} - 1 - iy\cdot\xi\chi^{(\sigma)}(y)\big)K(y)\,dy\bigg)\,d\xi \\
&= \int_{\mathbb{R}^d} \big|\hat{u}(\xi)\big|^2 \int_{\mathbb{R}^d} \big(1-\cos(\xi\cdot y)\big)K(y)\,dy\,d\xi \\
&\geqslant (2-\sigma)\nu \int_{\mathbb{R}^d} \big|\hat{u}(\xi)\big|^2 \int_{\mathbb{R}^d} \big(1-\cos(\xi\cdot y)\big)|y|^{-d-\sigma}\,dy\,d\xi \\
&= \nu c(2-\sigma) \int_{\mathbb{R}^d} \big|(-\Delta)^{\sigma/4}u\big|^2\,dx.
\end{aligned} \tag{3.4}$$

From the equality

$$\int |Lu-\lambda u|^2\,dx = \int |f|^2\,dx,$$

we finally obtain the estimate (3.2) for $u \in C_0^\infty$ by collecting (3.3) and (3.4).

For the general case, we need to show the continuity of L. The symbol of L is given by

$$m(\xi)=\int_{\mathbb{R}^d}\big(e^{iy\cdot\xi}-1-iy\cdot\xi(1_{\sigma\in(1,2)}+1_{y\in B_1}1_{\sigma=1})\big)K(y)\,dy.$$

Clearly, $m(0)=0$. In the sequel, we assume $\xi\neq 0$. By using the upper bound of K in (1.3) and the change of variable $y\to y/|\xi|$, it is easily seen that for $\sigma\in(0,1)$ or $\sigma\in(1,2)$, we have $|m(\xi)|\leqslant N(d,\sigma,\Lambda)|\xi|^\sigma$. If $\sigma=1$, from (2.1) we get

$$m(\xi)=\int_{\mathbb{R}^d}\big(e^{iy\cdot\xi}-1-iy\cdot\xi 1_{|\xi|y\in B_1}\big)K(y)\,dy,$$

which gives $|m(\xi)|\leqslant N|\xi|$ by using the same argument. Therefore, in any case we have

$$\|Lu\|_{L_2}=\big\|\hat{u}(\xi)m(\xi)\big\|_{L_2}\leqslant N\big\|\hat{u}(\xi)|\xi|^\sigma\big\|_{L_2}\leqslant N\|u\|_{\dot{H}_2^\sigma},\tag{3.5}$$

which implies that L is a continuous operator from H_2^σ to L_2. To prove the estimate (3.2) for general $u\in H_p^\sigma$, we use the fact that C_0^∞ is dense in H_2^σ and the continuity of the operator $L-\lambda$ from H_2^σ to L_2. This completes the proof of the lemma. □

Remark 3.2. We note that the proofs of (3.3) and (3.4) do not use the cancellation condition when $\sigma=1$. These inequalities can also be verified without using the Fourier transform. Indeed, (3.4) follows from the identity

$$-2\int uLu\,dx=\iint\big(u(x+y)-u(x)\big)^2K(y)\,dy\,dx$$

and the ellipticity condition (1.3). For (3.3), we decompose K into its symmetric and skew-symmetric parts $K=K_e+K_o$, where

$$K_e(y)=\frac{1}{2}\big(K(y)+K(-y)\big),\qquad K_o(y)=\frac{1}{2}\big(K(y)-K(-y)\big).$$

Clearly, K_e satisfies (1.3). Let L_e and L_o be the corresponding operators with kernels K_e and K_o, respectively. It is easily seen that

$$\int L_euL_ou\,dx=0.$$

Therefore, we have

$$\int|Lu|^2\,dx\geqslant\int|L_eu|^2\,dx:=I.$$

Since K_e is symmetric,

$$I=\iiint\big(u(x+y)-u(x)\big)\big(u(x+z)-u(x)\big)K_e(y)K_e(z)\,dy\,dz\,dx.$$

Using the change of variables $y \to -y$ and $x \to x+y$, we get

$$I = \iiint \big(u(x) - u(x+y)\big)\big(u(x+y+z) - u(x+y)\big) K_e(y) K_e(z)\, dy\, dz\, dx.$$

Adding the above two expressions of I gives

$$2I = \iiint \big(u(x) - u(x+y)\big)\big(u(x+y+z) - u(x+y) - u(x+z) + u(x)\big) \\ \times K_e(y) K_e(z)\, dy\, dz\, dx.$$

Now we use the change of variables $z \to -z$ and $x \to x+z$ to obtain another expression of $2I$, which is the same as above with $u(x+y+z) - u(x+z)$ in place of $u(x) - u(x+y)$. By adding these two expressions of $2I$, we finally reach

$$4 \int \big|Lu(x)\big|^2 dx = \iiint \big(u(x+y+z) - u(x+y) - u(x+z) + u(x)\big)^2 \\ \times K_e(y) K_e(z)\, dy\, dz\, dx,$$

which along with (1.3) gives (3.3).

For the solvability result, we present the following two lemmas, which are versions of those in [22, Chapter 1] for non-local operators. For later references, the operator in these lemmas is a bit more general than that in Theorem 2.1 having a drift term $b \cdot \nabla u$. The first lemma is a maximum principle.

Lemma 3.3 *(A maximum principle). Let $\lambda > 0$ be a constant, $b = (b_1, \dots, b_d)$ be a bounded measurable function in $\mathbb{R}^d$, and u be a smooth function in $\mathbb{R}^d$ satisfying $u(x) \to 0$ as $x \to \infty$. Assume that $Lu + b \cdot \nabla u - \lambda u = 0$ in $\mathbb{R}^d$. Then $u \equiv 0$ in $\mathbb{R}^d$.*

Proof. We prove the lemma by contradiction. Suppose that $\sup_{\mathbb{R}^d} u > 0$. Since u tends to 0 as $x \to \infty$, we can find $x_0 \in \mathbb{R}^d$ such that $u(x_0) = \sup_{\mathbb{R}^d} u$. Then from (1.2), it is easily seen that $Lu(x_0) \leqslant 0$. This together with $u(x_0) > 0$ and $\nabla u(x_0) = 0$ gives $Lu - \lambda u < 0$ at x_0, which contradicts the assumption in the lemma. Therefore, we must have $\sup_{\mathbb{R}^d} u \leqslant 0$. Similarly, $\inf_{\mathbb{R}^d} u \geqslant 0$. This completes the proof of the lemma. □

Lemma 3.4. *Let $\lambda > 0$ be a constant and $b = (b_1, \dots, b_d)$ be a constant vector in $\mathbb{R}^d$. Then the set $(L + b \cdot \nabla - \lambda) C_0^\infty$ is dense in L_p for any $p \in (1, \infty)$.*

Proof. Assume the assertion is not true. Then by the Hahn–Banach theorem and Riesz's representation theorem, there is a nonzero function $g \in L_{p/(p-1)}$ such that

$$\int_{\mathbb{R}^d} \big(Lu(x) + b \cdot \nabla u(x) - \lambda u(x)\big) g(x)\, dx = 0 \tag{3.6}$$

for any $u \in C_0^\infty$. Let L^* be the non-local operator L with $K(y)$ replaced by $K(-y)$. Then we see that, for each $y \in \mathbb{R}^d$,

$$L^*(u*g)(y) - b\cdot\nabla(u*g)(y) - \lambda u*g(y) = \int_{\mathbb{R}^d}\big(Lv(x) + b\cdot\nabla v(x) - \lambda v(x)\big)g(x)\,dx = 0,$$

where $v(x) = u(y-x) \in C_0^\infty$ and the last equality is due to (3.6) with v in place of u. Because $u \in C_0^\infty$ and $g \in L_{p/(p-1)}$, the function $u*g(y)$ is smooth and tends to zero as $y \to \infty$. By Lemma 3.3 applied to the operator $L^* - b\cdot\nabla - \lambda$, we get that $u*g \equiv 0$ in $\mathbb{R}^d$. Bearing in mind that $u \in C_0^\infty$ is arbitrary, we conclude $g \equiv 0$ in $\mathbb{R}^d$, which contradicts our assumption that g is a nonzero function. The lemma is proved. □

Now we are ready to prove the following solvability result.

Proposition 3.5 *(L_2-solvability). For any $\lambda > 0$ and $f \in L_2$, there exists a unique strong solution $u \in H_2^\sigma$ to $Lu - \lambda u = f$ in $\mathbb{R}^d$ satisfying* (3.2).

Proof. Due to Lemma 3.4, we can find a sequence $u_n \in C_0^\infty$ such that $Lu_n - \lambda u_n$ converges to f in L_2. By Lemma 3.1, we have

$$\|u_n\|_{\dot H_2^\sigma} + \sqrt{\lambda}\|u_n\|_{\dot H_2^{\sigma/2}} + \lambda\|u_n\|_{L_2} \leqslant N(d,\nu)\|Lu_n - \lambda u_n\|_{L_2} \tag{3.7}$$

and

$$\|u_n - u_m\|_{\dot H_2^\sigma} + \sqrt{\lambda}\|u_n - u_m\|_{\dot H_2^{\sigma/2}} + \lambda\|u_n - u_m\|_{L_2} \leqslant N(d,\nu)\big\|L(u_n - u_m) - \lambda(u_n - u_m)\big\|_{L_2}.$$

Therefore, $\{u_n\}$ is a Cauchy sequence in H_2^σ and there is a limiting function $u \in H_2^\sigma$. By the continuity estimate (3.5) and (3.7), u is a strong solution to $Lu - \lambda u = f$ and satisfies (3.2). Finally, the uniqueness follows from the estimate (3.2). The proposition is proved. □

Remark 3.6. In the proof of Proposition 3.5, instead of relying on Lemmas 3.3 and 3.4, one may also use the method of continuity and the solvability of $-(-\Delta)^{\sigma/2} - \lambda u = f$ in H_2^2. The same remark applies to the proof of Theorem 2.1.

4. Hölder estimate

In this section we prove a Hölder estimate of solutions to the equation $Lu - \lambda u = f$. The novelty of the result here is that the constant in the estimate is independent of $\lambda \geqslant 0$. Our proof is based on the arguments developed in [4]. In the case $\lambda = 0$, similar Hölder estimates with very different proofs can be found in [8] for symmetric kernels and very recently in [17] for non-symmetric kernels. We note that more general nonlinear Pucci type operators are treated in [8,17].

Theorem 4.1 *(C^α-estimate). Let $\lambda \geqslant 0$, $0 < \sigma < 2$, $1/2 \leqslant r < R < 1$, and $f \in L_\infty(B_1)$. Let $u \in C^2_{\mathrm{loc}}(B_1) \cap L_1(\mathbb{R}^d, \omega)$ with $\omega(x) = 1/(1+|x|^{d+\sigma})$ such that*

$$Lu - \lambda u = f$$

in B_R. *Then for any* $\alpha \in (0, \min\{1, \sigma\})$, *we have*

$$[u]_{C^\alpha(B_r)} \leqslant N\Big((R-r)^{-\alpha} \sup_{B_R} |u| + (R-r)^{-d-\alpha} \|u\|_{L_1(\mathbb{R}^d,\omega)} + (R-r)^{\sigma-\alpha} \operatorname{osc}_{B_R} f\Big),$$

where $N = N(d, \nu, \Lambda, \sigma, \alpha)$.

Proof. Denote $r_1 = (R-r)/2$, and $\bar{r} = (R+r)/2$. Set $w(x) = I_{B_R}(x)u(x)$. For $x \in B_{\bar{r}}$, we have $\nabla u(x) = \nabla w(x)$ and thus

$$\begin{aligned} Lu(x) &= \int_{\mathbb{R}^d} \big(u(x+z) - u(x) - z \cdot \nabla u(x)\chi^{(\sigma)}(z)\big) K(z)\, dz \\ &= \int_{\mathbb{R}^d} \big(w(x+z) - w(x) - z \cdot \nabla w(x)\chi^{(\sigma)}(z)\big) K(z)\, dz \\ &\quad + \int_{\mathbb{R}^d} \big(u(x+z) - w(x+z)\big) K(z)\, dz \\ &= Lw(x) + \int_{|z| \geqslant r_1} \big(u(x+z) - w(x+z)\big) K(z)\, dz. \end{aligned}$$

Hence in $B_{\bar{r}}$

$$\lambda w(x) - Lw(x) = g(x) - f(x),$$

where

$$g(x) = \int_{|z| \geqslant r_1} \big(u(x+z) - w(x+z)\big) K(z)\, dz.$$

Note that

$$\|g\|_{L_\infty(B_R)} \leqslant N r_1^{-d-\sigma} \|u\|_{L_1(\mathbb{R}^d,\omega)}, \tag{4.1}$$

where $N = N(d, \Lambda, \sigma)$.

For $x_0 \in B_r$, we set

$$M(x, y) := w(x) - w(y) - \phi(x-y) - \Gamma(x),$$

where $\phi(z) = C_1|z|^\alpha$, $\alpha \in (0, \min\{1, \sigma\})$, and $\Gamma(x) = C_2|x - x_0|^2$. We will find $C_1, C_2 \in (0, \infty)$ depending only on d, ν, Λ, σ, $\|u\|_{L_\infty(B_R)}$, $\|u\|_{L_1(\mathbb{R}^d,\omega)}$, $\operatorname{osc}_{B_R} f$, r_1, but independent of the choice of $x_0 \in B_r$, such that

$$\sup_{x,y \in \mathbb{R}^d} M(x, y) \leqslant 0. \tag{4.2}$$

This proves the assertion in the theorem. More specifically, using the fact the C_1 and C_2 are independent of the choice of $x_0 \in B_r$, we obtain

$$\bigl|u(x) - u(y)\bigr| \leqslant C_1|x-y|^\alpha, \quad x, y \in B_r,$$

where C_1 will be taken below to be the right-hand side of the Hölder estimate in the theorem.

To prove (4.2), we first take

$$C_2 := 8r_1^{-2}\|u\|_{L_\infty(B_R)}.$$

Then, for $x \in \mathbb{R}^d \setminus B_{r_1/2}(x_0)$,

$$w(x) - w(y) \leqslant 2\|u\|_{L_\infty(B_R)} \leqslant C_2|x-x_0|^2.$$

This shows that

$$M(x,y) \leqslant 0, \quad x \in \mathbb{R}^d \setminus B_{r_1/2}(x_0). \tag{4.3}$$

To get a contradiction, let us assume that there exist $x, y \in \mathbb{R}^d$ such that $M(x,y) > 0$. By (4.3) we know that $x \in B_{r_1/2}(x_0) \subset B_{(\bar{r}+r)/2}$. Moreover, if $M(x,y) > 0$, then

$$w(x) - w(y) > C_1|x-y|^\alpha, \quad \text{i.e.,} \quad |x-y|^\alpha < \frac{2\|u\|_{L_\infty(B_R)}}{C_1}. \tag{4.4}$$

If we take a sufficiently large C_1 so that $C_1 \geqslant 2^{1+\alpha}r_1^{-\alpha}\|u\|_{L_\infty(B_R)}$, the above inequalities show that $y \in B_{\bar{r}}$. Therefore, the assumption that $M(x,y) > 0$ for some $x, y \in \mathbb{R}^d$ (and the continuity of u on B_R) enables us to assume that there exist $\bar{x}, \bar{y} \in B_{\bar{r}}$ satisfying $\sup_{x,y\in\mathbb{R}^d} M(x,y) = M(\bar{x},\bar{y}) > 0$.

Note that at $\bar{x}, \bar{y} \in B_{\bar{r}}$ we have

$$g(\bar{y}) - f(\bar{y}) = \lambda w(\bar{y}) - Lw(\bar{y}),$$
$$-g(\bar{x}) + f(\bar{x}) = -\lambda w(\bar{x}) + Lw(\bar{x}).$$

Thus, upon observing $w(\bar{y}) - w(\bar{x}) < 0$, it follows that

$$\begin{aligned} -2\|g\|_{L_\infty(B_R)} - \operatorname{osc}_{B_R} f &\leqslant \lambda\bigl(w(\bar{y}) - w(\bar{x})\bigr) + Lw(\bar{x}) - Lw(\bar{y}) \\ &\leqslant Lw(\bar{x}) - Lw(\bar{y}) := I. \end{aligned} \tag{4.5}$$

We decompose K into a symmetric part K_1 and non-symmetric part K_2, where

$$K_1(z) = \min\bigl\{K(z), K(-z)\bigr\}, \qquad K_2(z) = K(z) - K_1(z).$$

Clearly, the kernel K_1 also satisfies (1.3), and $K_2 \geqslant 0$ has the upper bound in (1.3). Let L_1 and L_2 be the elliptic operators with kernels K_1 and K_2, respectively. Then I in (4.5) can be written as

$$I = I_1 + I_2,$$

where

$$I_1 := L_1 w(\bar{x}) - L_1 w(\bar{y}), \qquad I_2 := L_2 w(\bar{x}) - L_2 w(\bar{y}). \tag{4.6}$$

Thanks to the symmetry of K_1, we have

$$I_1 = \frac{1}{2} \int_{\mathbb{R}^d} J(\bar{x}, \bar{y}, z) K_1(z)\, dz,$$

where

$$J(\bar{x}, \bar{y}, z) = w(\bar{x} + z) + w(\bar{x} - z) - 2w(\bar{x}) - w(\bar{y} + z) - w(\bar{y} - z) + 2w(\bar{y}).$$

Since $M(x, y)$ attains its maximum at $\bar{x}, \bar{y}$, we have

$$\begin{aligned} &w(\bar{x} + z) - w(\bar{y} + z) - \phi(\bar{x} - \bar{y}) - \Gamma(\bar{x} + z) \\ &\quad \leqslant w(\bar{x}) - w(\bar{y}) - \phi(\bar{x} - \bar{y}) - \Gamma(\bar{x}), \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} &w(\bar{x} - z) - w(\bar{y} - z) - \phi(\bar{x} - \bar{y}) - \Gamma(\bar{x} - z) \\ &\quad \leqslant w(\bar{x}) - w(\bar{y}) - \phi(\bar{x} - \bar{y}) - \Gamma(\bar{x}) \end{aligned}$$

for all $z \in \mathbb{R}^d$. These two inequalities lead us to

$$J(\bar{x}, \bar{y}, z) \leqslant \Gamma(\bar{x} + z) + \Gamma(\bar{x} - z) - 2\Gamma(\bar{x}), \quad z \in \mathbb{R}^d. \tag{4.8}$$

By again the assumption that $M(x, y)$ has the maximum at $\bar{x}, \bar{y}$, we have

$$\begin{aligned} &w(\bar{x} + z) - w(\bar{y} - z) - \phi(\bar{x} - \bar{y} + 2z) - \Gamma(\bar{x} + z) \\ &\quad \leqslant w(\bar{x}) - w(\bar{y}) - \phi(\bar{x} - \bar{y}) - \Gamma(\bar{x}), \end{aligned}$$

and

$$\begin{aligned} &w(\bar{x} - z) - w(\bar{y} + z) - \phi(\bar{x} - \bar{y} - 2z) - \Gamma(\bar{x} - z) \\ &\quad \leqslant w(\bar{x}) - w(\bar{y}) - \phi(\bar{x} - \bar{y}) - \Gamma(\bar{x}) \end{aligned}$$

for all $z \in \mathbb{R}^d$. Hence it follows that, for any $z \in \mathbb{R}^d$,

$$\begin{aligned} J(\bar{x}, \bar{y}, z) &\leqslant \phi(\bar{x} - \bar{y} + 2z) + \phi(\bar{x} - \bar{y} - 2z) - 2\phi(\bar{x} - \bar{y}) \\ &\quad + \Gamma(\bar{x} + z) + \Gamma(\bar{x} - z) - 2\Gamma(\bar{x}). \end{aligned} \tag{4.9}$$

Set $a = \bar{x} - \bar{y}$. Since $\bar{x}, \bar{y}$ satisfy (4.4), we have $|a| < r_1/2$. Also set, for some $\eta_1, \eta_2 \in (0, 1/2)$,

$$\mathcal{C} = \big\{|z| < \eta_1 |a|\colon\ |z \cdot a| \geqslant (1 - \eta_2)|a||z|\big\}.$$

Then $\mathcal{C} \subset B_{r_1}$ and

$$\begin{aligned} 2I_1 = &\int_{|z| \geqslant r_1} J(\bar{x}, \bar{y}, z) K_1(z)\, dz + \int_{B_{r_1} \setminus \mathcal{C}} J(\bar{x}, \bar{y}, z) K_1(z)\, dz \\ &+ \int_{\mathcal{C}} J(\bar{x}, \bar{y}, z) K_1(z)\, dz := T_1 + T_2 + T_3. \end{aligned} \tag{4.10}$$

Note that

$$T_1 \leqslant N(d, \Lambda, \sigma) r_1^{-\sigma} \|u\|_{L_\infty(B_R)}.$$

By (4.8) it follows

$$T_2 \leqslant \int_{B_{r_1} \setminus \mathcal{C}} \big(\Gamma(\bar{x} + z) + \Gamma(\bar{x} - z) - 2\Gamma(\bar{x})\big) K_1(z)\, dz \leqslant N r_1^{2-\sigma} C_2,$$

where $N = N(d, \Lambda)$, but N is independent of η_1, η_2 in the definition of $\mathcal{C}$. Now using (4.9) we obtain

$$\begin{aligned} T_3 \leqslant &\int_{\mathcal{C}} \big(\phi(\bar{x} - \bar{y} + 2z) + \phi(\bar{x} - \bar{y} - 2z) - 2\phi(\bar{x} - \bar{y})\big) K_1(z)\, dz \\ &+ \int_{\mathcal{C}} \big(\Gamma(\bar{x} + z) + \Gamma(\bar{x} - z) - 2\Gamma(\bar{x})\big) K_1(z)\, dz := T_{3,1} + T_{3,2}. \end{aligned}$$

The term $T_{3,2}$ is again bounded by $N r_1^{2-\sigma} C_2$, where $N = N(d, \Lambda)$. Finally, by Lemma 4.2 below,

$$T_{3,1} \leqslant -N(d, \nu, \alpha) C_1 |a|^{\alpha - \sigma}.$$

Thus, we get from (4.10) and the choice of C_2 that

$$I_1 \leqslant N(d, \Lambda, \sigma) r_1^{-\sigma} \|u\|_{L_\infty(B_R)} - N(d, \nu, \alpha) C_1 |a|^{\alpha - \sigma}. \tag{4.11}$$

Next we estimate $I_2 = L_2 w(\bar{x}) - L_2 w(\bar{y})$ in (4.6). We consider separately three cases: $\sigma < 1$, $\sigma = 1$, and $\sigma > 1$.

Case 1: $\sigma \in (0, 1)$. In this case,

$$I_2 = \Bigg(\int\limits_{|z|\geqslant r_1} + \int\limits_{B_{r_1}} \Bigg) \big(w(\bar{x}+z) - w(\bar{x}) - w(\bar{y}+z) + w(\bar{y}) \big) K_2(z)\,dz$$
$$:= T_4 + T_5. \tag{4.12}$$

Similar to T_1, we bound T_4 by $N(d,\Lambda,\sigma) r_1^{-\sigma} \|u\|_{L_\infty(B_R)}$. Since $\sigma \in (0,1)$ and $|\bar{x} - x_0| < r_1/2$ by (4.3), from (4.7) we have

$$\begin{aligned} T_5 &\leqslant \int\limits_{B_{r_1}} \big(\Gamma(\bar{x}+z) - \Gamma(\bar{x}) \big) K_2(z)\,dz \\ &\leqslant 2(2-\sigma)\Lambda C_2 \int\limits_{B_{r_1}} \big(|z|^2 + 2|z||\bar{x}-x_0| \big) \frac{1}{|z|^{d+\sigma}}\,dz \\ &\leqslant N(d,\Lambda,\sigma) r_1^{2-\sigma} C_2. \end{aligned}$$

Therefore, we get from (4.12) and the choice of C_2 that

$$I_2 \leqslant N(d,\Lambda,\sigma) r_1^{-\sigma} \|u\|_{L_\infty(B_R)}. \tag{4.13}$$

Combining (4.5), (4.11), (4.13), and (4.1) we finally have

$$\begin{aligned} 0 \leqslant\ & N(d,\Lambda,\sigma)\big(\operatorname{osc}_{B_R} f + r_1^{-\sigma} \|u\|_{L_\infty(B_R)} + r_1^{-d-\sigma} \|u\|_{L_1(\mathbb{R}^d,\omega)} \big) \\ & - N(d,\nu,\alpha) C_1 |a|^{\alpha-\sigma} := J. \end{aligned}$$

Choose C_1 so that $C_1 \geqslant 2^{1+\alpha} r_1^{-\alpha} \|u\|_{L_\infty(B_R)}$ as well as

$$C_1 \geqslant N(d,\Lambda,\sigma) r_1^{\sigma-\alpha} \big(\operatorname{osc}_{B_R} f + r_1^{-\sigma} \|u\|_{L_\infty(B_R)} + r_1^{-d-\sigma} \|u\|_{L_1(\mathbb{R}^d,\omega)} \big) / N(d,\nu,\alpha).$$

Then, for $\alpha \in (0, \min\{1,\sigma\})$, by (4.4) $|a|^{\alpha-\sigma} r_1^{\sigma-\alpha} > 1$ and

$$J \leqslant N(d,\Lambda,\sigma)\big(\operatorname{osc}_{B_R} f + r_1^{-\sigma} \|u\|_{L_\infty(B_R)} + r_1^{-d-\sigma} \|u\|_{L_1(\mathbb{R}^d,\omega)} \big) \big(1 - |a|^{\alpha-\sigma} r_1^{\sigma-\alpha} \big) < 0.$$

This contradicts the fact that $J \geqslant 0$.

Case 2: $\sigma = 1$. Note that, because K_1 is symmetric, both K_1 and K_2 satisfy (2.1). Therefore, 1_{B_1} can be replaced by $1_{B_{r_1}}$ in the definition of L_2, and we have $I_2 = T_4 + T_5$, where

$$T_4 = \int\limits_{|z|\geqslant r_1} \big(w(\bar{x}+z) - w(\bar{x}) - w(\bar{y}+z) + w(\bar{y}) \big) K_2(z)\,dz,$$

$$T_5 = \int\limits_{B_{r_1}} \big(w(\bar{x}+z) - w(\bar{x}) - w(\bar{y}+z) + w(\bar{y}) - z\cdot\big(\nabla w(\bar{x}) - \nabla w(\bar{y})\big) \big) K_2(z)\,dz.$$

Then we bound T_4 as in Case 1.

Since $M(x, y)$ attains its maximum at the interior point $(\bar{x}, \bar{y})$, we easily get

$$\nabla w(\bar{x}) = \nabla \phi(\bar{x} - \bar{y}) + \nabla \Gamma(\bar{x}), \qquad \nabla w(\bar{y}) = \nabla \phi(\bar{x} - \bar{y}). \tag{4.14}$$

For T_5, using (4.7) and (4.14), we have

$$\begin{aligned} T_5 &\leqslant \int_{B_{r_1}} \big(\Gamma(\bar{x} + z) - \Gamma(\bar{x}) - z \cdot \big(\nabla w(\bar{x}) - \nabla w(\bar{y})\big) 1_{B_1}\big) K_2(z)\, dz \\ &= \int_{B_{r_1}} \big(\Gamma(\bar{x} + z) - \Gamma(\bar{x}) - z \cdot \nabla \Gamma(\bar{x})\big) K_2(z)\, dz \\ &= \int_{B_{r_1}} C_2 |z|^2 K_2(z)\, dz \\ &\leqslant N(d, \Lambda) r_1^{2-\sigma} C_2. \end{aligned}$$

Then we argue as in Case 1 to get the contradiction.

Case 3: $\sigma \in (1, 2)$. Now $I_2 = T_4 + T_5$, where

$$T_4 = \int_{|z| \geqslant r_1} \big(w(\bar{x} + z) - w(\bar{x}) - w(\bar{y} + z) + w(\bar{y}) - z \cdot \big(\nabla w(\bar{x}) - \nabla w(\bar{y})\big)\big) K_2(z)\, dz,$$

$$T_5 = \int_{B_{r_1}} \big(w(\bar{x} + z) - w(\bar{x}) - w(\bar{y} + z) + w(\bar{y}) - z \cdot \big(\nabla w(\bar{x}) - \nabla w(\bar{y})\big)\big) K_2(z)\, dz.$$

Because $\sigma \in (1, 2)$, $|\bar{x} - x_0| < r_1/2$, and $C_2 = 8 r_1^{-2} \|u\|_{L_\infty(B_R)}$, by (4.14) we have

$$\begin{aligned} T_4 &\leqslant \int_{|z| \geqslant r_1} \big(4\|u\|_{L_\infty(B_R)} + |z| \big|\nabla \Gamma(\bar{x})\big|\big) K_2(z)\, dz \\ &\leqslant N(d, \Lambda, \sigma) r_1^{-\sigma} \|u\|_{L_\infty(B_R)}. \end{aligned}$$

It follows from (4.7) and (4.14) that

$$\begin{aligned} T_5 &\leqslant \int_{B_{r_1}} \big(\Gamma(\bar{x} + z) - \Gamma(\bar{x}) - z \cdot \nabla \Gamma(\bar{x})\big) K_2(z)\, dz \\ &= \int_{B_{r_1}} C_2 |z|^2 K_2(z)\, dz \\ &\leqslant N(d, \Lambda) r_1^{2-\sigma} C_2. \end{aligned}$$

So we again argue as in Case 1 to arrive at the contradiction.

Therefore, we conclude that (4.2) holds true in all three cases. The theorem is proved. □

Recall that $a = \bar{x} - \bar{y}$ and

$$T_{3,1} = \int_{\mathcal{C}} \big(\phi(a+2z) + \phi(a-2z) - 2\phi(a)\big) K_1(z)\,dz,$$

where

$$\mathcal{C} = \big\{|z| < \eta_1|a|:\ |z\cdot a| \geqslant (1-\eta_2)|a||z|\big\}.$$

Lemma 4.2. *There exist $\eta_1, \eta_2 \in (0, 1/2)$, depending only on α, such that*

$$T_{3,1} \leqslant -NC_1|a|^{\alpha-\sigma}, \tag{4.15}$$

where $N = N(d, \nu, \alpha) > 0$.

Proof. The idea of the proof is to use the local concavity of the function $|x|^\alpha$ in the radial direction. Set $\eta(t) = a + 2tz$, where $a = \bar{x} - \bar{y}$. Then

$$\varphi(t) := \phi(a+2tz) = \phi\big(\eta(t)\big).$$

Since $\phi(x) = C_1|x|^\alpha$, we have

$$\frac{\partial\phi}{\partial x_i}(x) = C_1\frac{\partial}{\partial x_i}\big(|x|^\alpha\big) = C_1\alpha x_i|x|^{\alpha-2},$$
$$\frac{\partial^2\phi}{\partial x_i\partial x_j}(x) = C_1\alpha(\alpha-2)x_ix_j|x|^{\alpha-4} + C_1\alpha|x|^{\alpha-2}I_{i=j}.$$

Hence

$$\varphi'(t) = \sum_{i=1}^d \frac{\partial\phi}{\partial x_i}\big(\eta(t)\big)\frac{d\eta_i(t)}{dt} = \sum_{i=1}^d \frac{\partial\phi}{\partial x_i}\big(\eta(t)\big)2z_i$$

and

$$\begin{aligned}\varphi''(t) &= \sum_{i,j=1}^d \frac{\partial^2\phi}{\partial x_i\partial x_j}\big(\eta(t)\big)4z_iz_j \\ &= 4C_1\alpha(\alpha-2)\big|\eta(t)\big|^{\alpha-4}\big|\eta(t)\cdot z\big|^2 + 4C_1\alpha\big|\eta(t)\big|^{\alpha-2}|z|^2 \\ &= 4C_1\alpha|a+2tz|^{\alpha-4}\big[(\alpha-2)\big|(a+2tz)\cdot z\big|^2 + |a+2tz|^2|z|^2\big].\end{aligned}$$

Observe that, on $\mathcal{C}$,

$$|a+2tz|^2 \leqslant (1+2\eta_1)^2|a|^2,$$

$$\begin{aligned}\big|(a+2tz)\cdot z\big| &= \big|a\cdot z+2t|z|^2\big| \geqslant |a\cdot z|-2|z|^2\\ &\geqslant (1-\eta_2)|a||z|-2|z|^2 \geqslant (1-2\eta_1-\eta_2)|z||a|\end{aligned}$$

for all $t\in[-1,1]$. Thus upon noting $\alpha-2<0$ we get

$$\varphi''(t)\leqslant 4C_1\alpha|a+2tz|^{\alpha-4}\big[(\alpha-2)(1-2\eta_1-\eta_2)^2+(1+2\eta_1)^2\big]|a|^2|z|^2. \tag{4.16}$$

Since $(1-2\eta_1-\eta_2)^2\to 1$ and $(1+2\eta_1)^2\to 1$ as $\eta_1,\eta_2\searrow 0$, there exist sufficiently small $\eta_1,\eta_2\in(0,1/2)$, depending only on $\alpha\in(0,1)$, such that

$$(\alpha-2)(1-2\eta_1-\eta_2)^2+(1+2\eta_1)^2\leqslant(\alpha-1)/2.$$

This together with (4.16) implies that

$$\varphi''(t)\leqslant -2C_1\alpha(1-\alpha)|a+2tz|^{\alpha-4}|a|^2|z|^2.$$

From this and the fact that

$$|a+2tz|^{\alpha-4}\geqslant(1+2\eta_1)^{\alpha-4}|a|^{\alpha-4}\geqslant 2^{\alpha-4}|a|^{\alpha-4},$$

we arrive at

$$\varphi''(t)\leqslant -2^{\alpha-3}C_1\alpha(1-\alpha)|a|^{\alpha-2}|z|^2,\quad t\in[-1,1],\ z\in\mathcal{C}. \tag{4.17}$$

On the other hand, by the mean value theorem for difference quotients, there exists $t_0\in(-1,1)$ satisfying

$$\varphi(1)+\varphi(-1)-2\varphi(0)=\varphi''(t_0).$$

Using this equality and (4.17), we have

$$T_{3,1}\leqslant -\int_{\mathcal{C}} 2^{\alpha-3}C_1\alpha(1-\alpha)|a|^{\alpha-2}|z|^2K_1(z)\,dz. \tag{4.18}$$

From the definition of $\mathcal{C}$ it follows that

$$\int_{\mathcal{C}}|z|^2K_1(z)\,dz\geqslant\nu(2-\sigma)\int_{\mathcal{C}}|z|^{2-d-\sigma}\,dz=N(d,\nu,\eta_2)\eta_1^{2-\sigma}|a|^{2-\sigma}.$$

Combining this with (4.18) and recalling the fact that η_1, η_2 depend only on α, we finally obtain the inequality (4.15). □

In the next section we will need a bound of the C^α norm of u only in terms of f and the weighted L_1 norm of u. To this end, in the corollary below we use an iteration argument to drop the term $\sup_{B_R}|u|$ on the right-hand side of the estimate in Theorem 4.1.

Corollary 4.3. *Let $\lambda \geqslant 0$, $0 < \sigma < 2$, and $f \in L_\infty(B_1)$. Let $u \in C^2_{\mathrm{loc}}(B_1) \cap L_1(\mathbb{R}^d, \omega)$ with $\omega(x) = 1/(1 + |x|^{d+\sigma})$ such that*

$$Lu - \lambda u = f$$

in B_1. Then for any $\alpha \in (0, \min\{1, \sigma\})$, we have

$$[u]_{C^\alpha(B_{1/2})} \leqslant N\|u\|_{L_1(\mathbb{R}^d,\omega)} + N \operatorname{osc}_{B_1} f, \tag{4.19}$$

where $N = N(d, \nu, \Lambda, \sigma, \alpha)$.

Proof. Set

$$r_n = 1 - 2^{-n-1}, \qquad B_{(n)} = B_{r_n}, \quad n = 0, 1, 2, \ldots.$$

Theorem 4.1 gives, for $n = 0, 1, 2, \ldots,$

$$[u]_{C^\alpha(B_{(n)})} \leqslant N_1\Big(2^{2n} \sup_{B_{(n+1)}} |u| + 2^{(d+\alpha)n}\|u\|_{L_1(\mathbb{R}^d,\omega)} + \operatorname{osc}_{B_{(n+1)}} f\Big), \tag{4.20}$$

where $N_1 = N_1(d, \nu, \Lambda, \sigma, \alpha)$ is a constant independent of n. To estimate the first term on the right-hand side of (4.20), by the well-known interpolation inequality, we have

$$\sup_{B_{(n+1)}} |u| \leqslant \varepsilon[u]_{C^\alpha(B_{(n+1)})} + N\varepsilon^{-d/\alpha}\|u\|_{L_1(B_{(n+1)})}, \quad \forall\varepsilon \in (0, 1). \tag{4.21}$$

Upon taking $\varepsilon = (N_1 2^{2n+3d/\alpha})^{-1}$ and combining (4.20) and (4.21), we get

$$\begin{aligned} [u]_{C^\alpha(B_{(n)})} &\leqslant 2^{-3d/\alpha}[u]_{C^\alpha(B_{(n+1)})} + N2^{2nd/\alpha}\|u\|_{L_1(B_1)} \\ &\quad + N2^{(d+\alpha)n}\|u\|_{L_1(\mathbb{R}^d,\omega)} + N \operatorname{osc}_{B_1} f. \end{aligned} \tag{4.22}$$

We multiply both sides of (4.22) by $2^{-3dn/\alpha}$ and sum over n to obtain

$$\begin{aligned} \sum_{n=0}^{\infty} 2^{-3dn/\alpha}[u]_{C^\alpha(B_{(n)})} &\leqslant \sum_{n=0}^{\infty} 2^{-3d(n+1)/\alpha}[u]_{C^\alpha(B_{(n+1)})} \\ &\quad + N\sum_{n=0}^{\infty} 2^{-dn/\alpha}\|u\|_{L_1(B_1)} + N\sum_{n=0}^{\infty} 2^{-3dn/\alpha+(d+\alpha)n}\|u\|_{L_1(\mathbb{R}^d,\omega)} \\ &\quad + N\sum_{n=0}^{\infty} 2^{-3dn/\alpha} \operatorname{osc}_{B_1} f, \end{aligned}$$

which immediately yields (4.19). The corollary is proved. □

5. Mean oscillation estimates

This section is devoted to several mean oscillation estimates for u and its fractional derivative $(-\Delta)^{\sigma/2}u$ by using the L_2 estimate in Section 3 and the Hölder estimate established in Section 4.

We recall the maximal function theorem and the Fefferman–Stein theorem. Let the maximal and sharp functions of g defined on $\mathbb{R}^d$ be given by

$$\mathcal{M}g(x) = \sup_{r>0} \fint_{B_r(x)} \left|g(y)\right| dy,$$

$$g^{\#}(x) = \sup_{r>0} \fint_{B_r(x)} \left|g(y) - (g)_{B_r(x)}\right| dy.$$

Then

$$\|g\|_{L_p} \leqslant N \left\|g^{\#}\right\|_{L_p}, \qquad \|\mathcal{M}g\|_{L_p} \leqslant N\|g\|_{L_p}, \tag{5.1}$$

if $g \in L_p$, where $1 < p < \infty$ and $N = N(d, p)$. As is well known, the first inequality above is due to the Fefferman–Stein theorem on sharp functions and the second one to the Hardy–Littlewood maximal function theorem (this inequality also holds trivially when $p = \infty$). Throughout the paper we denote

$$(f)_\Omega = \frac{1}{|\Omega|} \int_\Omega f(x)\,dx = \fint_\Omega f(x)\,dx,$$

where $|\Omega|$ is the d-dimensional Lebesgue measure of Ω.

Lemma 5.1. *Let $\lambda \geqslant 0$, $0 < \sigma < 2$, and $f \in C^\infty_{\mathrm{loc}} \cap L_\infty(\mathbb{R}^d)$ satisfying $f = 0$ in B_2. Let $u \in H^\sigma_2 \cap C^\infty_b(\mathbb{R}^d)$ satisfy*

$$Lu - \lambda u = f \quad \text{in } \mathbb{R}^d. \tag{5.2}$$

Then for all $\alpha \in (0, \min\{1, \sigma\})$,

$$[u]_{C^\alpha(B_{1/2})} \leqslant N \sum_{k=0}^{\infty} 2^{-k\sigma} \big(|u|\big)_{B_{2^k}}, \tag{5.3}$$

$$\big[(-\Delta)^{\sigma/2}u\big]_{C^\alpha(B_{1/2})} \leqslant N \left(\sum_{k=0}^{\infty} 2^{-k\sigma} \big(\big|(-\Delta)^{\sigma/2}u\big|\big)_{B_{2^k}} + \mathcal{M}f(0) \right), \tag{5.4}$$

where $N = N(d, \nu, \Lambda, \sigma, \alpha)$.

Note that the right-hand side of (5.3) and the first term on the right-hand side of (5.4) are bounded by $\mathcal{M}u(0)$ and $\mathcal{M}((-\Delta)^{\sigma/2}u)(0)$, respectively. Therefore, Lemma 5.1 implies that the local Hölder norms of u and its fractional derivative $(-\Delta)^{\sigma/2}u$ can be controlled by the maximal functions of u, $(-\Delta)^{\sigma/2}u$, and f. This enables us to adapt the approach in [20].

Proof of Lemma 5.1. First note that we have $u, (-\Delta)^{\sigma/2}u \in C^2_{\mathrm{loc}}(B_1) \cap L_1(\mathbb{R}^d, \omega)$ with $\omega(x) = 1/(1+|x|^{d+\sigma})$. Since $f = 0$ in B_2, by Corollary 4.3,

$$[u]_{C^\alpha(B_{1/2})} \leqslant N\|u\|_{L_1(\mathbb{R}^d,\omega)}. \tag{5.5}$$

Set

$$B_{(0)} = B_1, \qquad B_{(k)} = B_{2^k} \setminus B_{2^{k-1}}, \quad k \geqslant 1.$$

Note that

$$\begin{aligned}
\|u\|_{L_1(\mathbb{R}^d,\omega)} &= \int_{\mathbb{R}^d} |u(x)| \frac{1}{1+|x|^{d+\sigma}}\,dx \\
&= \sum_{k=0}^{\infty} \int_{B_{(k)}} |u(x)| \frac{1}{1+|x|^{d+\sigma}}\,dx \\
&\leqslant N \sum_{k=0}^{\infty} 2^{-k\sigma} \big(|u|\big)_{B_{2^k}}.
\end{aligned}$$

This together with (5.5) gives (5.3).

To prove (5.4), we apply $(-\Delta)^{\sigma/2}$ to the both sides of (5.2) and obtain

$$(L-\lambda)(-\Delta)^{\sigma/2}u = (-\Delta)^{\sigma/2} f.$$

Again by Corollary 4.3,

$$\big[(-\Delta)^{\sigma/2}u\big]_{C^\alpha(B_{1/2})} \leqslant N\big\|(-\Delta)^{\sigma/2}u\big\|_{L_1(\mathbb{R}^d,\omega)} + N\sup_{B_1}\big|(-\Delta)^{\sigma/2} f\big|. \tag{5.6}$$

In exactly the same way above, we bound the first term on the right-hand side of (5.6) by

$$N\sum_{k=0}^{\infty} 2^{-k\sigma}\big(\big|(-\Delta)^{\sigma/2}u\big|\big)_{B_{2^k}}.$$

Next we estimate the second term on the right-hand side of (5.6). For $|x| < 1$, we have

$$\begin{aligned}
\big|-(-\Delta)^{\sigma/2} f(x)\big| &= \frac{1}{c}\left|\mathrm{P.V.}\int_{\mathbb{R}^d} \big(f(x+y) - f(x)\big)\frac{1}{|y|^{d+\sigma}}\,dy\right| \\
&\leqslant N \int_{|y|>1/2} \big|f(x+y)\big| \frac{1}{|y|^{d+\sigma}+1}\,dy,
\end{aligned} \tag{5.7}$$

where the inequality above is due to the fact that

$$f(x) = 0 \quad \text{if } |x| < 2, \qquad f(x+y) = 0 \quad \text{if } |x| < 1,\ |y| < 1/2.$$

Similar to the estimate of $\|u\|_{L_1(\mathbb{R}^d,\omega)}$ above, we bound the right-hand side of (5.7) by

$$N \sum_{k=0}^{\infty} 2^{-k\sigma} \big(|f|\big)_{B_{2^k}} \leqslant N(d,\sigma)\mathcal{M}f(0).$$

The lemma is proved. □

By using a simple scaling argument, we obtain the following corollary.

Corollary 5.2. *Let $\lambda \geqslant 0$, $0 < \sigma < 2$, $r > 0$, $\kappa \geqslant 2$, and $f \in C_{\mathrm{loc}}^{\infty} \cap L_\infty(\mathbb{R}^d)$ satisfying $f = 0$ in $B_{2\kappa r}$. Let $u \in H_2^\sigma \cap C_b^\infty(\mathbb{R}^d)$ satisfy*

$$Lu - \lambda u = f \quad \text{in } \mathbb{R}^d.$$

Then for all $\alpha \in (0, \min\{1,\sigma\})$,

$$[u]_{C^\alpha(B_{\kappa r/2})} \leqslant N(\kappa r)^{-\alpha} \sum_{k=0}^{\infty} 2^{-k\sigma} \big(|u|\big)_{B_{2^k \kappa r}},$$

$$\big[(-\Delta)^{\sigma/2} u\big]_{C^\alpha(B_{\kappa r/2})} \leqslant N(\kappa r)^{-\alpha} \left(\sum_{k=0}^{\infty} 2^{-k\sigma} \big(\big|(-\Delta)^{\sigma/2} u\big|\big)_{B_{2^k \kappa r}} + \mathcal{M}f(0) \right),$$

where $N = N(d, \nu, \Lambda, \sigma, \alpha)$.

Proof. Let $R = \kappa r$, $w(x) = u(Rx)$, and $g(x) = R^\sigma f(Rx)$. Set L_1 to be a non-local operator with the kernel $K_1(z) = R^{d+\sigma} K(Rz)$. Then we see that K_1 satisfies (1.3) and $w \in L_1(\mathbb{R}^d, \omega)$. Moreover,

$$L_1 w - R^\sigma \lambda w = g \quad \text{in } \mathbb{R}^d,$$

where $g = 0$ in B_2. Applying Lemma 5.1 to w, we obtain (5.3) and (5.4) with w in place of u. Turning w back to u gives the desired inequalities. □

Note that, for example,

$$\big(\big|u - (u)_{B_r}\big|\big)_{B_r} \leqslant 2^\alpha r^\alpha [u]_{C^\alpha(B_{\kappa r/2})}$$

for $\kappa \geqslant 2$. This combined with the inequalities in the above corollary leads us to

Corollary 5.3. *Let $\lambda \geqslant 0$, $0 < \sigma < 2$, $r > 0$, $\kappa \geqslant 2$, and $f \in C_{\mathrm{loc}}^{\infty} \cap L_\infty(\mathbb{R}^d)$ satisfying $f = 0$ in $B_{2\kappa r}$. Let $u \in H_2^\sigma \cap C_b^\infty(\mathbb{R}^d)$ satisfy*

$$Lu - \lambda u = f \quad \text{in } \mathbb{R}^d.$$

Then for all $\alpha \in (0, \min\{1, \sigma\})$,

$$\left(\left|u-(u)_{B_r}\right|\right)_{B_r} \leqslant N\kappa^{-\alpha}\sum_{k=0}^{\infty} 2^{-k\sigma}\left(|u|\right)_{B_{2^k\kappa r}},$$

$$\left(\left|(-\Delta)^{\sigma/2}u-\left((-\Delta)^{\sigma/2}u\right)_{B_r}\right|\right)_{B_r} \leqslant N\kappa^{-\alpha}\left(\sum_{k=0}^{\infty} 2^{-k\sigma}\left(\left|(-\Delta)^{\sigma/2}u\right|\right)_{B_{2^k\kappa r}}+\mathcal{M}f(0)\right),$$

where $N = N(d, \nu, \Lambda, \sigma, \alpha)$.

The proposition below is the main result of this section. It reads that the mean oscillations of u and $(-\Delta)^{\sigma/2}u$ can be controlled by their maximal functions together with the maximal function of f^2.

Proposition 5.4 *(Mean oscillation estimate). Let* $\lambda > 0$, $0 < \sigma < 2$, $r > 0$, $\kappa \geqslant 2$, *and* $f \in C^{\infty}_{\mathrm{loc}} \cap L_{\infty}$. *Let* $u \in H^{\sigma}_2 \cap C^{\infty}_b(\mathbb{R}^d)$ *satisfy*

$$Lu - \lambda u = f \quad \text{in } \mathbb{R}^d.$$

Then for all $\alpha \in (0, \min\{1, \sigma\})$,

$$\lambda\left(\left|u-(u)_{B_r}\right|\right)_{B_r} + \left(\left|(-\Delta)^{\sigma/2}u-\left((-\Delta)^{\sigma/2}u\right)_{B_r}\right|\right)_{B_r} \leqslant N\kappa^{-\alpha}\left(\lambda\mathcal{M}u(0)+\mathcal{M}\left((-\Delta)^{\sigma/2}u\right)(0)\right)+N\kappa^{d/2}\left(\mathcal{M}\left(f^2\right)(0)\right)^{1/2}, \tag{5.8}$$

where $N = N(d, \nu, \Lambda, \sigma, \alpha)$.

Proof. Take a cut-off function $\eta \in C^{\infty}_0(B_{4\kappa r})$ such that $\eta = 1$ in $B_{2\kappa r}$. Due to Proposition 3.5, there is a unique H^{σ}_2-solution to

$$Lw - \lambda w = \eta f.$$

Since $\eta f \in C^{\infty}_0$, by the classical theory, we know that $w \in H^{\sigma}_2 \cap C^{\infty}_b$. It follows from Lemma 3.1 that

$$\lambda\|w\|_{L_2} + \left\|(-\Delta)^{\sigma/2}w\right\|_{L_2} \leqslant N(d,\nu)\|\eta f\|_{L_2},$$

which yields, for any $R > 0$,

$$\left(\lambda|w| + \left|(-\Delta)^{\sigma/2}w\right|\right)_{B_R} \leqslant N\left(R^{-1}\kappa r\right)^{d/2}\left(f^2\right)^{1/2}_{B_{4\kappa r}} \leqslant N\left(R^{-1}\kappa r\right)^{d/2}\left(\mathcal{M}\left(f^2\right)(0)\right)^{1/2}. \tag{5.9}$$

Now $v := u - w \in H^{\sigma}_2 \cap C^{\infty}_b$ satisfies

$$Lv - \lambda v = (1-\eta)f.$$

Notice that $(1-\eta)f=0$ in $B_{2\kappa r}$. By Corollary 5.3, we have

$$\begin{aligned}&\lambda\big(\big|v-(v)_{B_r}\big|\big)_{B_r}+\big(\big|(-\Delta)^{\sigma/2}v-\big((-\Delta)^{\sigma/2}v\big)_{B_r}\big|\big)_{B_r}\\&\quad\leqslant N\lambda\kappa^{-\alpha}\sum_{k=0}^{\infty}2^{-k\sigma}\big(|v|\big)_{B_{2^k\kappa r}}+N\kappa^{-\alpha}\Bigg(\sum_{k=0}^{\infty}2^{-k\sigma}\big(\big|(-\Delta)^{\sigma/2}v\big|\big)_{B_{2^k\kappa r}}+\mathcal{M}f(0)\Bigg).\end{aligned}$$

This together with the triangle inequality, (5.9), and the inequality $\mathcal{M}f(0)\leqslant(\mathcal{M}(f^2)(0))^{1/2}$ gives

$$\begin{aligned}&\lambda\big(\big|u-(u)_{B_r}\big|\big)_{B_r}+\big(\big|(-\Delta)^{\sigma/2}u-\big((-\Delta)^{\sigma/2}u\big)_{B_r}\big|\big)_{B_r}\\&\quad\leqslant\lambda\big(\big|v-(v)_{B_r}\big|\big)_{B_r}+\big(\big|(-\Delta)^{\sigma/2}v-\big((-\Delta)^{\sigma/2}v\big)_{B_r}\big|\big)_{B_r}\\&\qquad+N\lambda\big(|w|\big)_{B_r}+N\big(\big|(-\Delta)^{\sigma/2}w\big|\big)_{B_r}\\&\quad\leqslant N\kappa^{-\alpha}\sum_{k=0}^{\infty}2^{-k\sigma}\big(\lambda|v|+\big|(-\Delta)^{\sigma/2}v\big|\big)_{B_{2^k\kappa r}}+N\kappa^{d/2}\big(\mathcal{M}\big(f^2\big)(0)\big)^{1/2}\\&\quad\leqslant N\kappa^{-\alpha}\sum_{k=0}^{\infty}2^{-k\sigma}\big(\lambda|u|+\big|(-\Delta)^{\sigma/2}u\big|\big)_{B_{2^k\kappa r}}+N\kappa^{d/2}\big(\mathcal{M}\big(f^2\big)(0)\big)^{1/2},\end{aligned}$$

which is clearly less than the right-hand side of (5.8). In the last inequality above, we used (5.9) with $R=2^k\kappa r$, $k=0,1,\ldots$. The proposition is proved. □

Next, we show that the inequality (5.8) holds true if we interchange the roles of $-(-\Delta)^{\sigma/2}$ and L.

Lemma 5.5. *Let $\lambda>0$, $0<\sigma<2$, $r>0$, $\kappa\geqslant 2$, and $f\in C^\infty_{\mathrm{loc}}\cap L_\infty$. Let $u\in H^\sigma_2\cap C^\infty_b(\mathbb{R}^d)$ satisfy*

$$-(-\Delta)^{\sigma/2}u-\lambda u=f\quad\text{in }\mathbb{R}^d.\tag{5.10}$$

Then for all $\alpha\in(0,\min\{1,\sigma\})$,

$$\begin{aligned}&\lambda\big(\big|u-(u)_{B_r}\big|\big)_{B_r}+\big(\big|Lu-(Lu)_{B_r}\big|\big)_{B_r}\\&\quad\leqslant N\kappa^{-\alpha}\big(\lambda\mathcal{M}u(0)+\mathcal{M}(Lu)(0)\big)+N\kappa^{d/2}\big(\mathcal{M}\big(f^2\big)(0)\big)^{1/2},\end{aligned}$$

where $N=N(d,\nu,\Lambda,\sigma,\alpha)$.

Proof. We follow the proof of Proposition 5.4 with necessary changes outlined below. As before, we decompose u as a sum of w and v. For the estimate of w corresponding to (5.9), by using (3.5) and (3.2) we have

$$\big(\lambda|w|+|Lw|\big)_{B_R}\leqslant N\big(R^{-1}\kappa r\big)^{d/2}\big(\mathcal{M}\big(f^2\big)(0)\big)^{1/2}.$$

Since the operator L in Lemma 5.1 can be set to be $(-\Delta)^{\sigma/2}$, one can still use (5.3) for the Hölder estimate of v. Now for the Hölder estimate of Lv, we need an estimate similar to (5.4):

$$[Lu]_{C^\alpha(B_{1/2})} \leqslant N\left(\sum_{k=0}^{\infty} 2^{-k\sigma}\big(|Lu|\big)_{B_{2^k}} + \mathcal{M}f(0)\right)$$

provided that $f = 0$ in B_2. We apply L to the both sides of (5.10) and obtain

$$\big((-\Delta)^{\sigma/2} - \lambda\big)Lu = Lf.$$

By Corollary 4.3,

$$[Lu]_{C^\alpha(B_{1/2})} \leqslant N\|Lu\|_{L_1(\mathbb{R}^d,\omega)} + N\sup_{B_1}|Lf|. \tag{5.11}$$

We bound the first term on the right-hand side of (5.11) as in the proof of Lemma 5.1. To estimate the second term, we notice that since $f = 0$ in B_2, for any $|x| < 1$ we have $\nabla f(x) = 0$, and thus

$$\begin{aligned}
\big|Lf(x)\big| &= \left|\int_{\mathbb{R}^d}\big(f(x+y) - f(x) - y\cdot\nabla f(x)\chi^{(\sigma)}(y)\big)K(y)\,dy\right| \\
&= \left|\int_{\mathbb{R}^d}\big(f(x+y) - f(x)\big)K(y)\,dy\right| \\
&\leqslant N\int_{|y|>1/2}\big|f(x+y)\big|\frac{1}{|y|^{d+\sigma}+1}\,dy,
\end{aligned}$$

which is bounded by $N\mathcal{M}f(0)$ as desired. The remaining proof is the same as that of Proposition 5.4. □

6. L_p-estimate

We finally complete the proof of the L_p solvability of $Lu - \lambda u = f$ by providing the proof of Theorem 2.1.

Proof of Theorem 2.1. First we prove the estimate (2.3) for $u \in C_0^\infty$ and $\lambda > 0$. In this case, clearly we have $u \in H_2^\sigma \cap C_b^\infty(\mathbb{R}^d)$ and $f \in C_{\mathrm{loc}}^\infty \cap L_\infty$. When $p = 2$, the estimate is proved in Lemma 3.1.

Next we consider the case when $p \in (2,\infty)$. Set $\alpha = \min\{1,\sigma\}/2$. Then by Proposition 5.4 combined with translations we have, for all $x \in \mathbb{R}_d$, $r > 0$ and $\kappa \geqslant 2$,

$$\begin{aligned}
&\lambda\big(\big|u - (u)_{B_r(x)}\big|\big)_{B_r(x)} + \big(\big|(-\Delta)^{\sigma/2}u - \big((-\Delta)^{\sigma/2}u\big)_{B_r(x)}\big|\big)_{B_r(x)} \\
&\quad\leqslant N\kappa^{-\alpha}\big(\lambda\mathcal{M}u(x) + \mathcal{M}\big((-\Delta)^{\sigma/2}u\big)(x)\big) + N\kappa^{d/2}\big(\mathcal{M}\big(f^2\big)(x)\big)^{1/2},
\end{aligned}$$

where $N = N(d,\nu,\Lambda,\sigma)$. Take the supremum of the left-hand side of the inequality with respect to $r > 0$ to get

$$\lambda u^{\#}(x) + \big((-\Delta)^{\sigma/2}u\big)^{\#}(x) \leqslant N\kappa^{-\alpha}\big(\lambda \mathcal{M}u(x) + \mathcal{M}\big((-\Delta)^{\sigma/2}u\big)(x)\big) + N\kappa^{d/2}\big(\mathcal{M}\big(f^2\big)(x)\big)^{1/2}.$$

By applying the Fefferman–Stein theorem on sharp functions and the Hardy–Littlewood maximal function theorem to the above inequality (see the inequalities in (5.1)), we obtain

$$\begin{aligned}\lambda\|u\|_{L_p} + \big\|(-\Delta)^{\sigma/2}u\big\|_{L_p} &\leqslant N\lambda\big\|u^{\#}\big\|_{L_p} + \big\|\big((-\Delta)^{\sigma/2}u\big)^{\#}\big\|_{L_p} \\ &\leqslant N\kappa^{-\alpha}\big(\lambda\|\mathcal{M}u\|_{L_p} + \big\|\mathcal{M}\big((-\Delta)^{\sigma/2}u\big)\big\|_{L_p}\big) + N\kappa^{d/2}\big\|\mathcal{M}\big(f^2\big)\big\|_{L_{p/2}}^{1/2} \\ &\leqslant N\kappa^{-\alpha}\big(\lambda\|u\|_{L_p} + \big\|(-\Delta)^{\sigma/2}u\big\|_{L_p}\big) + N\kappa^{d/2}\|f\|_{L_p},\end{aligned}$$

where $N = N(d, \nu, \Lambda, \sigma, p)$. It then only remains to take a sufficiently large κ so that $N\kappa^{-\alpha} \leqslant 1/2$. For the case $\lambda = 0$ and $u \in C_0^\infty$, since the estimate (2.3) holds for any $\lambda > 0$, we take the limit as $\lambda \searrow 0$.

To prove (2.3) for general $u \in H_p^\sigma$, we need a continuity estimate of L as in Lemma 3.1. Thanks to Lemma 5.5, the argument using sharp and maximal functions as above yields, for any $\lambda > 0$,

$$\lambda\|u\|_{L_p} + \|Lu\|_{L_p} \leqslant N\big\|-(-\Delta)^{\sigma/2}u - \lambda u\big\|_{L_p},$$

with a constant N independent of λ. Letting $\lambda \to 0$, we get for any $u \in C_0^\infty$,

$$\|Lu\|_{L_p} \leqslant N\|u\|_{\dot{H}_p^\sigma}, \tag{6.1}$$

which implies that L is a continuous operator from H_p^σ to L_p. Since C_0^∞ is dense in H_p^σ, we obtain (2.3) in its full generality.

Now the unique solvability of the equation in the case $p \in (2, \infty)$ follows from the same argument as in Proposition 3.5 with p in place of 2 along with Lemma 3.4 as well as the estimates (6.1) and (2.3).

For $p \in (1, 2)$, we use a duality argument. Let L^* be the non-local operator with kernel $K(-y)$. Denote $q = p/(p-1) \in (2, \infty)$. For any $g \in L_q$, by the H_q^σ-solvability there is a unique solution $v \in H_q^\sigma$ to the equation

$$L^*v - \lambda v = g \quad \text{in } \mathbb{R}^d.$$

It is easily seen that L^* is the adjoint operator of L. Therefore, for any $u \in C_0^\infty$,

$$\begin{aligned}\int_{\mathbb{R}^d} g(-\Delta)^{\sigma/2}u\,dx &= \int_{\mathbb{R}^d}\big(L^*v - \lambda v\big)(-\Delta)^{\sigma/2}u\,dx \\ &= \int_{\mathbb{R}^d}(-\Delta)^{\sigma/2}v(Lu - \lambda u)\,dx.\end{aligned} \tag{6.2}$$

By using (2.3) with q in place of p, from (6.2) we have

$$\left|\int_{\mathbb{R}^d} g(-\Delta)^{\sigma/2}u\,dx\right| \leqslant \left\|(-\Delta)^{\sigma/2}v\right\|_{L_q}\|Lu-\lambda u\|_{L_p}$$
$$\leqslant N\|g\|_{L_q}\|Lu-\lambda u\|_{L_p}.$$

Since $g \in L_q$ is arbitrary, we then get

$$\left\|(-\Delta)^{\sigma/2}u\right\|_{L_p} \leqslant N\|Lu-\lambda u\|_{L_p},$$

which along with a similar estimate of $\lambda\|u\|_{L_p}$ yields (2.3) for any $u \in C_0^\infty$. For general $u \in H_p^\sigma$, as before we need a continuity estimate of L. For any $g \in L_q$, let $v \in H_q^\sigma$ be the equation

$$-(-\Delta)^{\sigma/2}v - \lambda v = g \quad \text{in } \mathbb{R}^d.$$

For any $u \in C_0^\infty$, we have

$$\int_{\mathbb{R}^d} gLu\,dx = \int_{\mathbb{R}^d} \bigl(-(-\Delta)^{\sigma/2}v - \lambda v\bigr)Lu\,dx$$
$$= \int_{\mathbb{R}^d} L^*v\bigl(-(-\Delta)^{\sigma/2}u - \lambda u\bigr)\,dx. \tag{6.3}$$

By the continuity of L^*, from (6.3) we have

$$\left|\int_{\mathbb{R}^d} gLu\,dx\right| \leqslant \left\|L^*v\right\|_{L_q}\left\|-(-\Delta)^{\sigma/2}u-\lambda u\right\|_{L_p}$$
$$\leqslant N\|g\|_{L_q}\left\|(-\Delta)^{\sigma/2}u-\lambda u\right\|_{L_p}.$$

Since $g \in L_q$ is arbitrary, we then get

$$\|Lu\|_{L_p} \leqslant N\left\|(-\Delta)^{\sigma/2}u-\lambda u\right\|_{L_p}.$$

Letting $\lambda \to 0$ gives the continuity of L from H_p^σ to L_p. The rest of the proof is the same as in the case $p \in (2,\infty)$. The theorem is proved. □

Proof of the estimate (2.7). We take a smooth function $\eta \in C_0^\infty((-2,2))$ satisfying $\eta(t)=1$ for $t \in [-1,1]$. Fix a $T>0$. It is easily seen that $U(t,x) := \eta(t/T)u(x) \in C_0^\infty((-2T,2T)\times\mathbb{R}^d)$ satisfies

$$-D_tU(t,x) + LU(t,x) + b\cdot\nabla U(t,x) - \lambda U(t,x) = \eta(t/T)f(x) - u(x)\eta'(t/T)/T.$$

Define $V(t,x) = U(t,x-bt)$. Then $V \in C_0^\infty((-2T,2T)\times\mathbb{R}^d)$ and satisfies

$$-D_tV(t,x) + LV(t,x) - \lambda V(t,x) = \eta(t/T)f(x-bt) - u(x-bt)\eta'(t/T)/T.$$

It follows from the results in [25] combined with the continuity L from H_p^σ to L_p proved in Theorem 2.1 (see Remark 2.4) that

$$\begin{aligned}&\big\|(-\Delta)^{\sigma/2}V\big\|_{L_p(((-2T,2T)\times\mathbb{R}^d)}+\lambda\|V\|_{L_p(-2T,2T)\times\mathbb{R}^d)}\\&\quad\leqslant N\big\|\eta(t/T)f(x-bt)-u(x-bt)\eta'(t/T)/T\big\|_{L_p((-2T,2T)\times\mathbb{R}^d)},\end{aligned}$$

which implies

$$\big\|(-\Delta)^{\sigma/2}u\big\|_{L_p}+\lambda\|u\|_{L_p}\leqslant N\|f\|_{L_p}+NT^{-1}\lambda\|u\|_{L_p}$$

with a constant $N=N(d,\nu,\Lambda,\sigma,p)$. Letting $T\to\infty$, we get

$$\|u\|_{\dot H_p^\sigma}+\sqrt{\lambda}\|u\|_{\dot H_p^{\sigma/2}}+\lambda\|u\|_{L_p}\leqslant N\|f\|_{L_p}.$$

To complete the proof, we use Eqs. (2.6) and (6.1) to bound the L_p norm of $b\cdot\nabla u$ by

$$\|Lu\|_{L_p}+\lambda\|u\|_{L_p}+\|f\|_{L_p}\leqslant N\|f\|_{L_p}. \qquad \Box$$

7. Local estimates

From the global estimate in Theorem 2.1, by using a more or less standard localization argument one can obtain the following interior estimates.

$$\big\|(-\Delta)^{\sigma/2}u\big\|_{L_p(B_1)}\leqslant N\|f\|_{L_p(B_2)}+N\|u\|_{L_p(\mathbb{R}^d,\omega)} \tag{7.1}$$

for $\sigma\in(0,1)$,

$$\big\|(-\Delta)^{\sigma/2}u\big\|_{L_p(B_1)}\leqslant N\|f\|_{L_p(B_2)}+N\|u\|_{L_p(\mathbb{R}^d,\omega)}+N\|Du\|_{L_p(B_4)} \tag{7.2}$$

for $\sigma\in(1,2)$, and

$$\big\|(-\Delta)^{\sigma/2}u\big\|_{L_p(B_1)}\leqslant N\|f\|_{L_p(B_2)}+N(\varepsilon)\|u\|_{L_p(\mathbb{R}^d,\omega)}+\varepsilon\|Du\|_{L_p(B_4)} \tag{7.3}$$

for $\sigma=1$ and any $\varepsilon\in(0,1)$. Here the weight function ω is defined in Theorem 4.1.

For the proof of this claim, we take a cut-off function $\eta\in C_0^\infty(B_2)$ satisfying $\eta\equiv1$ on B_1. Then it is easily seen that

$$L(\eta u)-\lambda\eta u=\eta f+L(\eta u)-\eta Lu.$$

Applying the global estimate in Theorem 2.1 to the equation above gives

$$\begin{aligned}\big\|(-\Delta)^{\sigma/2}(\eta u)\big\|_{L_p(\mathbb{R}^d)}&\leqslant N\big\|\eta f+L(\eta u)-\eta Lu\big\|_{L_p(\mathbb{R}^d)}\\&\leqslant N\|f\|_{L_p(B_2)}+N\big\|L(\eta u)-\eta Lu\big\|_{L_p(\mathbb{R}^d)}.\end{aligned}$$

Thus, by the triangle inequality,

$$\begin{aligned}\big\|(-\Delta)^{\sigma/2}u\big\|_{L_p(B_1)} &\leqslant \big\|\eta(-\Delta)^{\sigma/2}u\big\|_{L_p(\mathbb{R}^d)}\\ &\leqslant N\|f\|_{L_p(B_2)}+N\big\|L(\eta u)-\eta Lu\big\|_{L_p(\mathbb{R}^d)}\\ &\quad+\big\|(-\Delta)^{\sigma/2}(\eta u)-\eta(-\Delta)^{\sigma/2}u\big\|_{L_p(\mathbb{R}^d)}. \end{aligned} \tag{7.4}$$

It suffices to estimate the second term on the right-hand side above since the estimate of the third term is similar. We compute

$$L(\eta u)-\eta Lu=\int_{\mathbb{R}^d}\big(\big(\eta(x+y)-\eta(x)\big)u(x+y)-y\cdot\nabla\eta(x)u(x)\chi^{(\sigma)}(y)\big)K(y)\,dy.$$

(i) For $\sigma\in(0,1)$, we have

$$\begin{aligned}\big|L(\eta u)-\eta Lu\big| &\leqslant \int_{\mathbb{R}^d}\big|\big(\eta(x+y)-\eta(x)\big)u(x+y)\big|K(y)\,dy\\ &\leqslant N\bigg(\int_{B_1}+\int_{B_1^c}\bigg)\big|\big(\eta(x+y)-\eta(x)\big)\big|u(x+y)\big|\big||y|^{-d-\sigma}\,dy. \end{aligned}$$

By using the obvious bound

$$\big|\eta(x+y)-\eta(x)\big|\leqslant N|y|1_{|x|<3}\quad \text{for } y\in B_1, \tag{7.5}$$

we get

$$\begin{aligned}\big|L(\eta u)-\eta Lu\big| &\leqslant N\int_{B_1}1_{|x|<3}\big|u(x+y)\big||y|^{1-d-\sigma}\,dy\\ &\quad+\int_{B_1^c}\big|u(x+y)\big|\big(1_{|x+y|<2}+1_{|x|<2}\big)|y|^{-d-\sigma}\,dy. \end{aligned} \tag{7.6}$$

By Minkowski's inequality and Hölder's inequality,

$$\big\|L(\eta u)-\eta Lu\big\|_{L_p(\mathbb{R}^d)}\leqslant N\|u\|_{L_p(\mathbb{R}^d,\omega)}, \tag{7.7}$$

which together with (7.4) yields (7.1). Indeed, to obtain the above estimate the last term in (7.6) is calculated as follows.

$$\begin{aligned}&\bigg\|\int_{|y|>1}1_{|x|<2}\big|u(x+y)\big||y|^{-d-\sigma}\,dy\bigg\|_{L_p(\mathbb{R}^d)}\\ &\quad\leqslant 2\int_{\mathbb{R}^d}\big\|u(\cdot+y)\big\|_{L_p(B_2)}\omega(y)\,dy \end{aligned}$$

$$\leqslant 2\Big(\int_{\mathbb{R}^d} \|u\|^p_{L_p(B_2(y))}\omega(y)\,dy\Big)^{1/p}\Big(\int_{\mathbb{R}^d}\omega(y)\,dy\Big)^{1/q}$$

$$\leqslant N\Big(\int_{\mathbb{R}^d}\big|u(x)\big|^p\int_{B_2(x)}\omega(y)\,dy\,dx\Big)^{1/p} \leqslant N\|u\|_{L_p(\mathbb{R}^d,\omega)},$$

where $q = p/(p-1)$.

(ii) For $\sigma \in (1,2)$, we have

$$\big|L(\eta u) - \eta L u\big| \leqslant \int_{\mathbb{R}^d}\big|\big(\eta(x+y)-\eta(x)\big)u(x+y) - y\cdot\nabla\eta(x)u(x)\big|K(y)\,dy$$
$$\leqslant I_1 + I_2, \tag{7.8}$$

where

$$I_1 := \int_{\mathbb{R}^d}\big|\big(\eta(x+y)-\eta(x)\big)\big(u(x+y)-u(x)\big)\big|K(y)\,dy,$$

$$I_2 := \int_{\mathbb{R}^d}\big|\big(\eta(x+y)-\eta(x)-y\cdot\nabla\eta(x)\big)u(x)\big|K(y)\,dy.$$

Note that

$$\big|u(x+y)-u(x)\big| \leqslant |y|\int_0^1\big|\nabla u(x+ty)\big|\,dt.$$

We use (7.5) and the bound above to estimate I_1 by

$$I_1 := \Big(\int_{B_1} + \int_{B_1^c}\Big)\big|\big(\eta(x+y)-\eta(x)\big)\big(u(x+y)-u(x)\big)\big|K(y)\,dy$$

$$\leqslant N\int_{B_1}\int_0^1 1_{|x|<3}\big|\nabla u(x+ty)\big||y|^{2-d-\sigma}\,dt\,dy$$

$$+ N\int_{B_1^c}\big(\big|u(x+y)\big|+\big|u(x)\big|\big)\big(1_{|x+y|<2}+1_{|x|<2}\big)|y|^{-d-\sigma}\,dy.$$

By Minkowski's inequality and Hölder's inequality as used for (7.7),

$$\|I_1\|_{L_p(\mathbb{R}^d)} \leqslant N\|Du\|_{L_p(B_4)} + N\|u\|_{L_p(\mathbb{R}^d,\omega)}. \tag{7.9}$$

Note that by the mean value theorem,

$$\left|\eta(x+y)-\eta(x)-y\cdot\nabla\eta(x)\right|\leqslant N|y|^2 1_{|x|<3}\quad\text{for } y\in B_1.$$

Thus we have

$$I_2\leqslant N\left|u(x)\right|1_{|x|<3}\int_{B_1}|y|^{2-d-\sigma}\,dy$$
$$+N\left|u(x)\right|\int_{B_1^c}\left(1_{|x+y|<2}+1_{|x|<2}\left(1+|y|\right)\right)|y|^{-d-\sigma}\,dy.$$

Again, by Minkowski's inequality and Hölder's inequality,

$$\|I_2\|_{L_p(\mathbb{R}^d)}\leqslant N\|u\|_{L_p(\mathbb{R}^d,\omega)},$$

which together with (7.8) and (7.9) gives

$$\left\|L(\eta u)-\eta Lu\right\|_{L_p(\mathbb{R}^d)}\leqslant N\|u\|_{L_p(\mathbb{R}^d,\omega)}+N\|Du\|_{L_p(B_4)},$$

and thus (7.2).

(iii) In the last case $\sigma=1$, by using (2.1), for any $\delta\in(0,1)$ we have

$$\left|L(\eta u)-\eta Lu\right|\leqslant I_3+I_4+I_5,$$

where

$$I_3:=\int_{B_\delta}\left|\left(\eta(x+y)-\eta(x)\right)\left(u(x+y)-u(x)\right)\right|K(y)\,dy,$$
$$I_4:=\int_{B_\delta}\left|\left(\eta(x+y)-\eta(x)-y\cdot\nabla\eta(x)\right)u(x)\right|K(y)\,dy,$$
$$I_5:=\int_{B_\delta^c}\left|\left(\eta(x+y)-\eta(x)\right)u(x+y)\right|K(y)\,dy.$$

We bound I_3 and I_4 in the same way as I_1 and I_2 to get

$$I_3\leqslant N\int_{B_\delta}\int_0^1 1_{|x|<3}\left|\nabla u(x+ty)\right||y|^{1-d}\,dt\,dy,$$
$$I_4\leqslant N\int_{B_\delta}1_{|x|<3}\left|u(x)\right||y|^{1-d}\,dy,$$

and bound I_5 as in the first case to get

$$I_5 \leqslant N \int_{B_1 \setminus B_\delta} 1_{|x|<3} \big|u(x+y)\big| |y|^{-d} \, dy + N \int_{B_1^c} \big|u(x+y)\big| \big(1_{|x+y|<2} + 1_{|x|<2}\big) |y|^{-d-1} \, dy.$$

Thus, by Minkowski's inequality and Hölder's inequality,

$$\big\| L(\eta u) - \eta L u \big\|_{L_p(\mathbb{R}^d)} \leqslant N\delta \|Du\|_{L_p(B_4)} + N\big(1 - \log(\delta)\big) \|u\|_{L_p(\mathbb{R}^d, \omega)}.$$

By choosing a suitable δ, we obtain (7.3). The claim is proved.

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