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HOMOLOGICAL LOCALIZATION OF π-MODULES*

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1. Introduction

A.K. Bousfield recently characterized the topological spaces which are determined up to homotopy by their integral homology type [6]. As part of this characterization he introduced and studied, for any group π, a *homology localization functor* E on the category of π-modules. In theory, for any π-module M the localization $E(M)$ can be constructed either as a transfinite direct limit [6: 8.4, 11.5] or as a transfinite inverse limit [7, § 8]; there are also some concrete formulas known for $E(M)$ in special cases ([5: 2.7, 2.10, 2.11], [10, Theorem 4]). In practice, however, the exact nature of the functor E and of its relationship to more familiar homological constructions has remained unclear. The present paper is an attempt to remedy this, with a view toward topological applications in [13] and [14].

Recall that a map $f: M \to M'$ of π-modules is said to be an $H\mathbb{Z}$*-map* if the induced homomorphism $H_i(\pi; M) \to H_i(\pi; M')$ is an isomorphism for $i = 0$ and an epimorphism for $i = 1$. A π-module N is said to be $H\mathbb{Z}$*-local* or *Bousfield* if every $H\mathbb{Z}$-map $M \to M'$ induces a one-one correspondence $\mathrm{Hom}_\pi(M', N) \to \mathrm{Hom}_\pi(M, N)$. Bousfield has shown [6, 5.4] that for any group π there exists a functor E on the category of π-modules such that

(i) for all M, $E(M)$ is Bousfield, and

(ii) there is a natural $H\mathbb{Z}$-map $M \to E(M)$.

This E is called the $H\mathbb{Z}$*-localization* functor; it is additive, right exact, and has many other properties ([6, § 8], [7, §§ 7–9]).

The plan of this paper is as follows. Section 2 gives a fairly complete description of E under the single assumption that π is a finitely presented group. Section 3 studies the question of when E is naturally equivalent to the *zero'th left derived functor* of the familiar *lower central series completion functor* (2.2); this is the case for many interesting groups (3.1, 3.8), but *not* for all finitely presented groups (3.6). Finally, Section 4 uses the results of Section 3 and a duality construction to produce exceptionally simple formulas for E in some instructive special cases (4.9).

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Throughout the paper, π denotes a given group, $R = \mathbf{Z}[\pi]$ its integral group ring, and $I \subseteq R$ the augmentation ideal. The terms *π-module* and *R-module* are used synonymously. With the exception of the first argument of $\mathrm{Tor}^R_*(-,-)$, all unspecified modules are *left* modules. The additive group of integers $\mathbf{Z}$ is always considered to be a trivial left or right π-module, in the sense that each element of π acts on $\mathbf{Z}$ as the identity map; thus $H_i(\pi; -)$ is another name for the functor $\mathrm{Tor}^R_i(\mathbf{Z}, -)$.

A *tower* $\{M_s\}_s$ of abelian groups, π-modules, etc., is a family of such objects, indexed by the non-negative integer s, together with maps $M_{s+1} \to M_s$. In most cases the maps are obvious and are not made explicit. The elementary algebraic properties of towers [8; III, §2] play a large role in this paper, as do the related properties of the inverse limit functor $\varprojlim$ and its right derived functor $\varprojlim^1$ ([19], [8: IX, §2]).

I would like to thank A.K. Bousfield, K. Brown and E. Dror for valuable ideas; to some extent Section 3 of this paper overlaps their work.

2. Finitely presented groups

The purpose of this section is to compute the $H\mathbf{Z}$ localization functor E on the category of π-modules if π is a *finitely presented* group, that is, if π admits a presentation with a finite number of generators and a finite number of relations. The main result is

2.1. Theorem. *Suppose that π is a finitely presented group. Then for any π-module M there are natural exact sequences*

$$0 \to \varprojlim{}^1\{\mathrm{Tor}^R_1(R/I^s, I^s \cdot M\}_s \to E(M) \to C(M) \to 0$$

and

$$0 \to \varprojlim\{\mathrm{Tor}^R_1(R/I^s, I^s \cdot M)\}_s \to \varprojlim\{I^s \otimes_R (I^s \cdot M)\}_s$$
$$\to M \to E(M) \to \varprojlim{}^1\{I^s \otimes_R (I^s \cdot M)\}_s \to 0.$$

2.2. Remark. The *lower central series completion* $\varprojlim\{M/I^s \cdot M\}_s$ of M is denoted by $C(M)$. The first exact sequence of 2.1 shows that $E(M)$ is isomorphic to $C(M)$ if and only if $\varprojlim^1\{\mathrm{Tor}^R_1(R/I^s, I^s \cdot M)\}_s$ vanishes. The second is interesting insofar as it sheds light on the kernel of the $H\mathbf{Z}$-localization map $M \to E(M)$.

2.3. Remark. The proof of 2.1 contains an explicit formula for $E(M)$.

There is one lemma.

2.4. Lemma. *For any group π and π-module M there is a natural pro-exact sequence*

$$0 \to \{\mathrm{Tor}_1^R(R/I^s, I^s \cdot M)\}_s \to \{I^s \otimes_R (I^s \cdot M)\}_s \to \{I^s \cdot M\}_s \to 0.$$

Moreover, the middle tower in this sequence is superperfect, in the sense that the towers $\{H_i(\pi; I^s \otimes_R (I^s \cdot M))\}_s$ $(i = 0, 1)$ *are both pro-trivial.*

2.5. Remark. The proof of 2.4 appears below. It is well-known that $I \otimes_R (I \cdot M)$ is a superperfect π-module if π is a *perfect* group; in fact, $I \otimes_R (I \cdot M)$ is the *universal central extension* of the perfect π-module $I \cdot M$ [12, 6.2]. The point of 2.4 is that even if π is not perfect (so that $I^2 \neq I$) a similar construction can be made at the expense of passing to towers.

Proof of 2.1. Let W_s denote $I^s \otimes_R (I^s \cdot M)$, and let $p_s: W_s \to W_{s-1}$ be the structure maps of the tower $\{W_s\}_s$. Multiplication gives maps $q_s: W_s \to M$ which are compatible with the maps $W_s \to W_{s-1}$ and fit into exact sequences

$$(2.6) \qquad W_s \xrightarrow{q_s} M \to M/I^{2s} \cdot M \to 0.$$

Let W denote the infinite product $\prod_{s \geq 0} W_s$. Define a map $\partial': W \to M$ by

$$\partial'(w_0, w_1, \ldots, w_s, \ldots) = q_0(w_0)$$

and a map $\partial'': W \to W$ by

$$\partial''(w_0, w_1, \ldots, w_s, \ldots) = (w_0 - p_1 w_1, w_1 - p_2 w_2, \ldots, w_s - p_{s+1} w_{s+1}, \ldots).$$

(Clearly $\ker \partial'' = \varprojlim \{W_s\}_s$ and $\mathrm{coker}\, \partial'' = \varprojlim^1 \{W_s\}_s$.) Let $\partial: W \to M \oplus W$ be the direct sum of ∂' and ∂''.

Let X denote $\mathrm{coker}(\partial)$. We claim that

(a) X is a Bousfield π-module, and

(b) the composite of the inclusion $M \to M \oplus W$ and the projection

$M \oplus W \to X$ is an $H\mathbb{Z}$-map.

Statements (a) and (b) together imply that X is naturally isomorphic to the $H\mathbb{Z}$-localization $E(M)$ of M.

To see (a), note that the "chain complex" $\mathbf{C}$ given by

$$\partial: W \to M \oplus W$$

is the inverse limit of a tower $\{\mathbf{C}_s\}$ of chain complex epimorphisms. Here $\mathbf{C}_s$ is

$$\partial_s: \prod_{t \leq s} W_t \to M \oplus \Big(\prod_{t \leq s-1} W_t\Big)$$

where $\partial_s = \partial'_s + \partial''_s$, with

$$\partial'_s(w_0, w_1, \ldots, w_s) = q_0(w_0)$$

and

$$\partial''_s(w_0, w_1, \ldots, w_s) = (w_0 - p_1 w_1, w_1 - p_2 w_2, \ldots, w_{s-1} - p_s w_s).$$

Thus by [19] there is a short exact sequence

$$0 \to \varprojlim{}^1 \{h_1 C_s\}_s \to X \to \varprojlim \{h_0 C_s\}_s \to 0$$

where h_i denotes the i^{th} homology group functor. By Lemma 2.4 and sequence (2.6), this exact sequence reads

$$0 \to \varprojlim{}^1 \{\mathrm{Tor}_1^R(R/I^s, I^s \cdot M)\}_s \to X \to C(M) \to 0.$$

Since $M/I^s \cdot M$ and $\mathrm{Tor}_1^R(R/I^s, I^s \cdot M)$ are nilpotent π-modules, it follows easily from [6: 8.5, 8.7, 8.9] that X is Bousfield.

Let Y denote image $(\partial: W \to M \oplus W)$. The long exact homology sequence of

$$0 \to Y \to M \oplus W \to X \to 0$$

shows that in order to prove (b) it is enough to show that the composite $Y \to M \oplus W \to W$ (where the second map is projection) is an $H\mathbb{Z}$-map.

Consider the commutative triangle

$$\begin{array}{ccc} H_0(\pi; W) & \longrightarrow & H_0(\pi; W) \\ & \searrow \quad \nearrow & \\ & H_0(\pi; Y) & \end{array}$$

where the top map is induced by ∂''. The hypothesis on π implies that $H_0(\pi; -)$ commutes with arbitrary direct products [9] so that the kernel and cokernel of the top map are isomorphic to $\varprojlim{}^i \{H_0(\pi; W_s)\}_s$, $i = 0, 1$, respectively. By 2.4 both of these groups vanish, so the top map is an isomorphism. Since the map $H_0(\pi; W) \to H_0(\pi; Y)$ is clearly epimorphic, this implies that the map $H_0(\pi; Y) \to H_0(\pi; W)$ is also an isomorphism. A similar argument, using the fact that $H_1(\pi; -)$ commutes with arbitrary direct products and the vanishing of $\varprojlim{}^1 \{H_1(\pi; W_s)\}_s$, shows that the map $H_1(\pi; Y) \to H_1(\pi; W)$ must be epimorphic.

The second exact sequence of 2.1 arises as the long exact homology sequence of the chain complex short exact sequence

$$\begin{array}{ccccccc} 0 \to & 0 & \to & W & \to & W & \to 0 \\ & \downarrow & & \downarrow \partial & & \downarrow \partial'' & \\ 0 \to & M & \to & M \oplus W & \to & W & \to 0. \end{array}$$

Proof of 2.4. It is clear that the tower $\{R/I^s \otimes_R (I^s \cdot M)\}_s$ is pro-trivial, so that if the short exact sequence

$$0 \to \{I^s\}_s \to R \to \{R/I^s\}_s \to 0$$

is tensored on the right with $\{I^s \cdot M\}_s$, what results is the pro-exact sequence of the lemma. Tensoring the short exact sequence

$$0 \to I \to R \to \mathbb{Z} \to 0$$

on the right with $\{I^s \otimes_R (I^s \cdot M)\}_s$ shows that the second half of the lemma is equivalent to the statement that the multiplication map $\{I\otimes_R I^s\otimes_R (I^s\cdot M)\}_s \to \{I^s\otimes_R (I^s\cdot M)\}_s$ is a pro-isomorphism.

The pro-exact sequence

$$0\to \{\mathrm{Tor}_1^R(\mathbb{Z}, I^s)\}_s \to \{I\otimes_R I^s\}_s \to \{I^s\}_s \to 0$$

results from tensoring the second exact sequence above on the right with $\{I^s\}_s$ and noting that $\{\mathbb{Z}\otimes_R I^s\}_s$ is pro-trivial. Tensoring further on the right with $\{I^s\cdot M\}_s$ gives

$$\{\mathrm{Tor}_1^R(R, I^s)\otimes_R (I^s\cdot M)\}_s \to \{I\otimes_R I^s\otimes_R I^s\cdot M)\}_s \to \{I^s\otimes_R (I^s\cdot M)\}_s \to 0.$$

The tower $\{\mathrm{Tor}_1^R(\mathbb{Z}, I^s)\}_s$ is a tower of nilpotent right π-modules, since $\mathrm{Tor}_1^R(\mathbb{Z}, I^s)$ is isomorphic as a right π-module to $\mathrm{Tor}_2^R(\mathbb{Z}, R/I^s)$. Since $\{I^s\cdot M\}_s$ is a *perfect* tower of π-modules, that is, $\{\mathbb{Z}\otimes_R (I^s\cdot M)\}_s$ is pro-trivial, it follows from direct sum and long exact sequence arguments together with induction on the nilpotency class of N that $\{N\otimes_R (I^s\cdot M)\}_s$ is pro-trivial for any nilpotent right π-module N. Thus $\{\mathrm{Tor}_1^R(\mathbb{Z}, I^s)\otimes_R (I^s\cdot M)\}_s$ is pro-trivial, since this tower is the diagonal of a double tower $\{\mathrm{Tor}_1^R(\mathbb{Z}, I^s)\otimes_R (I^t\cdot M)\}_{s,t}$ with pro-trivial columns. This completes the proof.

3. Lower central series completion

Recall that the lower central series completion $C(M)$ of a π-module M (2.2) is defined as the inverse limit $\varprojlim\{M/I^s\cdot M\}_s$. This section looks at the problem of deciding when the $H\mathbb{Z}$-localization functor E for π-modules is naturally equivalent to the *zero'th left derived functor* C_0 of C [11: V, §5]. The goal is to understand when the functor E admits a classical description and to determine when the results of Section 4 can be brought to bear on the problem of computing E in a simple way.

If M is any π-module, the natural $H\mathbb{Z}$-map $M\to E(M)$ induces an isomorphism $C(M)\approx C(E(M)$ [7: 8.7, 9.1] whose inverse, composed with the obvious map $E(M)\to C(E(M))$, gives a natural map $E(M)\to C(M)$. Since there is a natural transformation $C_0\to C$ which is universal with respect to natural transformations of right exact functors into C, this natural transformation $E\to C$ lifts to a unique natural transformation $E\to C_0$.

3.1. Theorem (cf. [10: Theorem 4]). *If π is a finitely generated pre-nilpotent group, the natural transformation $E\to C_0$ is a natural equivalence.*

3.2. Remark. The *lower central series* subgroups $\Gamma_s(\pi)$ of π are defined inductively by

$$\Gamma_1(\pi)=\pi$$

$$\Gamma_{s+1}(\pi)=[\pi,\Gamma_s(\pi)] \qquad s\geq 1.$$

The group π is said to be *pre-nilpotent* [10: 2.3] if there is some integer N such that $\Gamma_{N+1}(\pi)=\Gamma_N(\pi)$. This is equivalent to requiring that there be a normal subgroup $\Gamma(\pi)$ of π such that

(i) $[\pi,\Gamma(\pi)]=\Gamma(\pi)$, and

(ii) $\pi/\Gamma(\pi)$ is a nilpotent group.

All abelian groups and more generally nilpotent groups are pre-nilpotent, as are all perfect groups and all finite groups.

3.3. Remark. The finite generation condition in 3.1 can be replaced by the assumption that

(i) $H_1(\pi;\mathbf{Z})$ is a finitely generated abelian group, and

(ii) $H_1(\Gamma(\pi);\mathbf{Z})$ is a finitely generated $\pi/\Gamma(\pi)$-module.

Example 10.6 of [7] shows that some such assumption is necessary.

For finitely presented groups there is a generalization of 3.1 which admits a converse. Let $\Phi_s(\pi)$ $(s\geq 2)$ denote the kernel of the natural map $H_2(\pi;\mathbf{Z})\to H_2(\pi/\Gamma_{s-1}(\pi);\mathbf{Z})$. The natural inclusions $\Phi_{s+1}(\pi)\to\Phi_s(\pi)$ give rise to a tower $\{\Phi_s(\pi)\}_s$ of abelian groups. In the same way the natural surjections $\pi/\Gamma_{s+1}(\pi)\to\pi/\Gamma_s(\pi)$ give rise to towers $\{H_i(\pi/\Gamma_s(\pi);\mathbf{Z})\}_s$ $(i\geq 0)$.

3.4. Theorem. *If π is a finitely presented group, the natural transformation $E\to C_0$ is a natural equivalence if and only if*

(i) $\varprojlim^1\{\Phi_s(\pi)\}_s=0$, *and*

(ii) $\varprojlim^1\{H_3(\pi/\Gamma_s(\pi);\mathbf{Z})\}_s=0$.

There is an interesting topological variant of 3.4. In the statement, $\pi^\wedge$ stands for the *lower central series completion* $\varprojlim\{\pi/\Gamma_s(\pi)\}_s$ of the *group* π, and $\mathbf{Z}_\infty$ denotes the Bousfield–Kan *integral nilpotent completion functor* [8].

3.5. Theorem. *If π is a finitely presented group, the natural transformation $E\to C_0$ is a natural equivalence if and only if the canonical epimorphism $\pi_1(\mathbf{Z}_\infty(K(\pi,1)))\to\pi^\wedge$ is an isomorphism.*

3.6. Example. Suppose that σ is an infinite cyclic group generated by α, and that M is a free abelian group on two generators x_1,x_2. Let σ act on M by $\alpha\cdot x_i=-x_i$ $(i=1,2)$ and let π be the semi-direct product of σ with M. It is clear that π is a finitely presented group. An explicit calculation shows that $\varprojlim^1\{\Phi_s(\pi)\}_s$ does not vanish, so by 3.4 the natural transformation $E\to C_0$ of functors on the category of π-modules is not a natural equivalence. In line with 3.5, it is not hard to show that there is an exact sequence

$$1\to\mathbf{Z}\to\mathbf{Z}_2^\wedge\to\pi_1\mathbf{Z}_\infty K(\pi,1)\to\pi^\wedge\to 1$$

where $\mathbf{Z}_2^\wedge$ denotes the 2-adic integers and the map $\mathbf{Z}\to\mathbf{Z}_2^\wedge$ is 2-adic completion.

The basis for 3.1, 3.4 and 3.5 is

3.7. Proposition.

(a) *For any group* π, *the natural transformation* $E \to C_0$ *is a natural equivalence if and only if the obvious map* $F \to C(F)$ *is an* $H\mathbb{Z}$*-map for all free* π*-modules* F.

(b) *If* π *is finitely generated the conditions of* (a) *hold if and only if* $H_1(\pi; C(F))$ *vanishes for all free* π*-modules* F

(c) *If* π *is finitely presented, the conditions of* (a) *hold if and only if* $\varprojlim^1\{H_2(\pi; R/I^s)\}_s = 0$.

3.8. Example. Bousfield has pointed out [7, 10.2] that $\varprojlim^1\{H_2(\pi; R/I^s)\}_s$ vanishes if π is a finitely presented group such that $H_2(\pi; \mathbb{Z})$ is finite. Thus for such a group, in particular, for a finitely generated free group, E is naturally equivalent to C_0.

The remainder of this section is taken up with proofs.

Recall that a tower $\{M_s\}_s$ is said to be *stable* or *Mittag–Leffler* if for each $s \geq 0$ there is a $k \geq 0$ such that image $(M_{s+j} \to M_s)$ equals image $(M_{s+k} \to M_s)$ for all $j \geq k$. It is easy to see that $\varprojlim^1\{M_s\}_s$ vanishes if $\{M_s\}_s$ is Mittag–Leffler; the converse, in general, does not hold. However

3.9. Lemma. *If* $\{M_s\}_s$ *is a tower of finitely generated abelian groups, then either*

(i) $\{M_s\}_s$ *is Mittag–Leffler and* $\varprojlim^1\{M_s\}_s = 0$, *or*

(ii) $\varprojlim^1\{M_s\}_s$ *is uncountable.*

This appears in [17].

Proof of 3.7. (a) Since both E and C_0 are right exact functors, the natural transformation $E \to C_0$ is a natural equivalence if and only if it induces an isomorphism $E(F) \to C_0(F)(= C(F))$ for any free π-module F. Choose some free F. The module $C(F)$ is Bousfield, since it is an inverse limit of nilpotent π-modules [6: 8.5, 8.9]; thus the map $E(F) \to C(F)$ is an isomorphism if and only if it is an $H\mathbb{Z}$-map. Since the natural map $F \to E(F)$ is an $H\mathbb{Z}$-map, the map $E(F) \to C(F)$ is an $H\mathbb{Z}$-map if and only if the map $F \to C(F)$ is.

(b) If π is finitely generated then I is finitely generated as a (right) R-module, so that techniques of [19] give a short exact sequence

$$0 \to \varprojlim^1\{H_1(\pi; F/I^s \cdot F)\}_s \to H_0(\pi; C(F)) \to \varprojlim\{H_0(\pi; F/I^s \cdot F)\}_s \to 0$$

for any π-module F. If F is free, the tower $\{H_1(\pi; F/I^s \cdot F)\}_s = \{I^s \cdot F/I^{s+1} \cdot F\}_s$ is pro-trivial, and the tower $\{H_0(\pi; F/I^s \cdot F)\}_s$ is isomorphic to the constant tower $H_0(\pi; F)$. This proves (b).

(c) If π is finitely presented then I is finitely presented as a (right) R-module, so that techniques of [19] give another short exact sequence

$$0 \to \varprojlim^1\{H_2(\pi; F/I^s \cdot F)\}_s \to H_1(\pi; C(F)) \to \varprojlim\{H_1(\pi; F/I^s \cdot F)\}_s \to 0$$

for any π-module F. If F is free, the tower $\{H_1(\pi; F/I^s \cdot F)\}_s = \{I^s \cdot F/I^{s+1} \cdot F\}_s$ is pro-trivial as before; this implies that the map $F \to C(F)$ is an $H\mathbb{Z}$-map if and only if $\varprojlim^1\{H_2(\pi; F/I^s \cdot F)\}_s$ vanishes.

Suppose that $\varprojlim^1\{H_2(\pi; R/I^s)\}_s$ vanishes. An induction on s shows that $H_2(\pi; R/I^s)$ ($s \geq 0$) is a finitely generated abelian group, so that 3.9 implies that the tower $\{H_2(\pi; R/I^s)\}_s$ is Mittag–Leffler. Since direct sums of a Mittag–Leffler tower with itself remain Mittag–Leffler, it follows that the tower $\{H_2(\pi; F/I^s \cdot F)\}_s$ is Mittag–Leffler for any free π-module F. This proves (c).

3.10. Lemma. *Let ν be a finitely generated nilpotent group and let $J \subseteq \mathbb{Z}[\nu]$ be the augmentation ideal. Then if $i > 0$ the tower $\{H_i(\nu; \mathbb{Z}[\nu]/J^s)\}_s$ is pro-trivial.*

Proof. If $i > 0$, the tower $\{H_i(\nu; \mathbb{Z}[\nu]/J^s)\}_s$ is isomorphic to the tower $\{H_{i-1}(\nu; J^s)\}_s$. Since $\{H_0(\nu; J^s)\}_s = \{J^s/J^{s+1}\}_s$ is pro-trivial, the lemma follows from [15, Theorem 2] and the fact that $\mathbb{Z}[\nu]$ is Noetherian [15; (5)].

Proof of 3.1. Let F be a free π-module. According to 3.7, it is enough to show that $H_1(\pi; C(F)) = 0$. Let Γ be $\Gamma(\pi)$, and let ν be the finite generated nilpotent group π/Γ. The proof will consist in showing that $E^2_{0,1} = E^2_{1,0} = 0$ in the Lyndon–Hochschild–Serre spectral sequence

$$E^2_{i,j} = H_i(\nu; H_j(\Gamma; C(F))) \Longrightarrow H_{i+j}(\pi; C(F)).$$

Let $J \subseteq \mathbb{Z}[\nu]$ be the augmentation ideal. It is well-known [10: Lemma 2] that Γ acts trivially on each of the nilpotent π-modules $F/I^s \cdot F$, so that the modules $F/I^s \cdot F$ and $C(F)$ are in fact given as modules over ν. Let F' be the free ν-module $H_0(\Gamma; F)$. It follows easily that for each s there is a canonical ν-module isomorphism

$$F/I^s \cdot F \approx F'/J^s \cdot F'$$

and thus, as a ν-module, $C(F)$ is isomorphic to $\varprojlim(F'/J^s \cdot F'\}_s$. Since $\mathbb{Z}[\nu]$ is Noetherian, techniques of [19] give a short exact sequence

$$0 \to \varprojlim^1\{H_2(\nu; F'/J^s \cdot F')\}_s \to H_1(\nu; C(F))$$
$$\to \varprojlim\{H_1(\nu; F'/J^s \cdot F')\}_s \to 0.$$

Since direct sums of a pro-trivial tower with itself remain pro-trivial, Lemma 3.10 shows that $H_1(\nu; C(F)) = E^2_{1,0}$ vanishes.

It remains to show that $E^2_{0,1} = H_0(\nu; H_1(\Gamma; C(F)))$ vanishes. Since Γ acts trivially on $C(F)$, $E^2_{0,1}$ is isomorphic to $M \otimes_{\mathbb{Z}[\nu]} C(F)$, where for brevity M denotes $H_1(\Gamma; \mathbb{Z})$ considered as a right ν-module via the canonical anti-automorphism of $\mathbb{Z}[\nu]$. Since $H_1(\Gamma; \mathbb{Z})$ is finitely generated over $\mathbb{Z}[\nu]$ [10: Lemma 3] and $\mathbb{Z}[\nu]$ is Noetherian, there is a short exact sequence [19]

$$0 \to \varprojlim{}^1\{\mathrm{Tor}_1^{\mathbb{Z}[\nu]}(M, F'/J^s \cdot F')\}_s$$
$$\to M \otimes_{\mathbb{Z}[\nu]} C(F) \to \varprojlim\{M \otimes_{\mathbb{Z}[\nu]} F'/J^s \cdot F'\}_s \to 0.$$

Thus it is certainly enough to show that $\mathrm{Tor}_i^{\mathbb{Z}[\nu]}(M, N) = 0$ for any $i \geq 0$ and any nilpotent ν-module N. Using induction on the nilpotent class of N together with simple co-limit and long exact sequence arguments, it is possible to reduce this to the case in which N is the trivial ν-module $\mathbb{Z}$. However, $\mathrm{Tor}_i^{\mathbb{Z}[\nu]}(M, \mathbb{Z})$ is isomorphic to $H_i(\nu; H_1(\Gamma, \mathbb{Z}))$. Since $[\pi, \Gamma] = \Gamma$, the zero'th homology group $H_0(\nu; H_1(\Gamma; \mathbb{Z}))$ vanishes, so the proof can be completed by using [15: Theorem 1].

3.11. Lemma. *Suppose that π is a finitely generated group and that $\{M_s\}_{s\geq 0}$ is a tower of Bousfield π-modules. Then $\varprojlim{}^1\{M_s\}_s$ vanishes if and only if $\varprojlim{}^1\{H_0(\pi; M_s)\}_s$ does.*

Proof. Let W denote the infinite product $\Pi_{s\geq 0} M_s$, and let $\partial: W \to W$ be the map given by

$$\partial(m_0, m_1, \ldots, m_s, \ldots) = (m_0 - p_1(m_1), \ldots, m_s - p_{s+1}(m_{s+1}), \ldots)$$

where the maps $p_{s+1}: M_{s+1} \to M_s$ are the tower maps. Then W is a Bousfield π-module [6: 8.5] and coker (∂) is isomorphic to $\varprojlim{}^1\{M_s\}_s$. Since π is finitely generated the functor $H_0(\pi; -)$ commutes with arbitrary products (cf. [9], [19]), so that coker$(H_0(\pi; \partial))$ is isomorphic to $\varprojlim{}^1\{H_0(\pi; M_s)\}_s$. The lemma follows from the fact that in general a map $M \to N$ of Bousfield π-modules is an epimorphism if and only if the induced map $H_0(\pi; M) \to H_0(\pi; N)$ is an epimorphism [7: 7.8].

Proof of 3.4. The proof will show that if π is finitely presented conditions (i) and (ii) of 3.4 are equivalent to the vanishing of $\varprojlim{}^1\{H_0(\pi; \mathrm{Tor}_2^R(R/I^s, \mathbb{Z}))\}_s$. The theorem then follows from 3.11, 3.7(c), and the observation that tower $\{\mathrm{Tor}_2^R(R/I^s; \mathbb{Z})\}_s$ is isomorphic to $\{\mathrm{Tor}_2^R(\mathbb{Z}, R/I^s)\}_s = \{H_2(\pi, R/I^s)\}_s$ via the canonical anti-automorphism of $R = \mathbb{Z}[\pi]$.

For each $s \geq 1$ let ν_s be the finitely generated nilpotent group $\pi/\Gamma_s(\pi)$. It is well-known that the π-module structure on R/I^s factors through a natural ν_s-module structure [10, Lemma 2]; in fact, as a ν_s-module R/I^s is naturally isomorphic to $\mathbb{Z}[\nu_s]/(J_s)^s$, where $J_s \subseteq \mathbb{Z}[\nu_s]$ is the augmentation ideal. The action of ν_s on R/I^s induces an action of ν_s on $\mathrm{Tor}_j^R(R/I^s, M)$ for any π-module M and $j \geq 0$.

The main ingredient in the proof of 3.4 is a certain first quadrant spectral sequence tower of homological type:

$$\{E_s^2(i, j) = H_i(\nu_s; \mathrm{Tor}_j^R(R/I^s, M))\}_s \Longrightarrow H_{i+j}(\pi; M).$$

The symbolism means that for a given π-module M the spectral sequence tower converges to a limit which is pro-isomorphic, in each dimension $n \geq 0$, to the constant tower $H_n(\pi; M)$. This is a standard composition-of-functors spectral sequence tower corresponding to the natural pro-isomorphism between the constant tower $H_0(\pi; M)$ and $\{H_0(\nu_s; R/I^s \otimes_R M)\}_s$. To set up the spectral sequence tower in the usual way it is necessary to check that $\{H_i(\nu_s; R/I^s \otimes_R F)\}_s$ is pro-trivial whenever $i > 0$ and F is a free π-module. This follows from the fact that $\{H_i(\nu_s; R/I^s \otimes_R F)\}_s$ is the direct sum of a number of copies of the diagonal of a double tower $\{H_i(\nu_s; \mathbf{Z}[\nu_s]/(J_s)^t)\}_{s,t}$ whose columns, by 3.10 are all pro-trivial.

In the above spectral sequence, let M be the trivial π-module $\mathbf{Z}$. The tower $\{\mathrm{Tor}_1^R(R/I^s, \mathbf{Z})\}_s = \{I^s/I^{s+1}\}_s$ is pro-trivial, so the towers $\{E_s^2(i,1)\}_s$ $(i \geq 0)$ are also pro-trivial. In addition, $\{E_s^2(i,0)\}_s$ $(i \geq 0)$ is isomorphic to $\{H_p(\nu_s; \mathbf{Z})\}_s$ and $\{E_s^2(2,0)\}_s$ is isomorphic to $\{H_0(\pi; \mathrm{Tor}_2^R(R/I^s, \mathbf{Z}))\}_s$, which shows that in low dimensions the spectral sequence tower degenerates into the long pro-exact sequence

$$H_3(\pi; \mathbf{Z}) \to \{H_3(\nu_s; \mathbf{Z})\}_s \to \{H_0(\pi; \mathrm{Tor}_2^R(R/I^s, \mathbf{Z}))\}_s$$
$$\to H_2(\pi; \mathbf{Z}) \to \{H_2(\nu_s; \mathbf{Z})\}_s \to 0.$$

Let $\Psi_s(\pi)$ denote coker $(H_3(\pi; \mathbf{Z}) \to H_3(\nu_s; \mathbf{Z}))$. The short pro-exact sequence

$$0 \to \{\Psi_s(\pi)\}_s \to \{H_0(\pi; \mathrm{Tor}_2^R(R/I^s, \mathbf{Z}))\}_s \to \{\Phi_s(\pi)\}_s \to 0$$

gives rise to a long exact sequence

$$0 \to \varprojlim\{\Psi_s(\pi)\}_s \to \varprojlim\{H_0(\pi; \mathrm{Tor}_2^R(R/I^s, \mathbf{Z}))\}_s$$
$$\to \varprojlim\{\Phi_s(\pi)\}_s \to \varprojlim{}^1\{\Psi_s(\pi)\}_s$$
$$\to \varprojlim{}^1\{H_0(\pi; \mathrm{Tor}_2^R(R/I^s, \mathbf{Z}))\}_s \to \varprojlim{}^1\{\Phi_s(\pi)\}_s \to 0.$$

Thus $\varprojlim{}^1\{H_0(\pi; \mathrm{Tor}_2^R(R/I^s, \mathbf{Z}))\}_s$ vanishes if and only if $\varprojlim{}^1\{\Phi_s(\pi)\}_s$ vanishes and $\varprojlim\{\Phi_s(\pi)\}_s$ maps onto $\varprojlim{}^1\{\Psi_s(\pi)\}_s$. However, since ν_s is a finitely generated nilpotent group $\Psi_s(\pi)$ is a finitely generated abelian group, so, by 3.9, if the subgroup $\varprojlim\{\Phi_s(\pi)\}_s$ of $H_2(\pi, \mathbf{Z})$ maps onto $\varprojlim{}^1\{\Psi_s(\pi)\}_s$, this latter group must vanish. The proof is finished by the observation that the pro-exact sequence

$$H_3(\pi, \mathbf{Z}) \to \{H_3(\nu_s; \mathbf{Z})\}_s \to \{\Psi_s(\pi)\}_s \to 0$$

gives use to an exact sequence

$$0 = \varprojlim{}^1 H_3(\pi; \mathbf{Z}) \to \varprojlim{}^1\{H_3(\nu_s; \mathbf{Z})\}_s \to \varprojlim{}^1\{\Psi_s(\pi)\}_s \to 0.$$

Proof of 3.5. This follows from 3.7(c), [16, 3.1] and [8: IX, §3]. The existence of a canonical epimorphism $\pi_1\mathbf{Z}_\infty(K(\pi,1)) \to \pi^\wedge$ comes from the fact that the tower $\{\pi_1\mathbf{Z}_s(K(\pi,1))\}_s$ is naturally pro-isomorphic to $\{\pi/\Gamma_s(\pi)\}_s$ [8: pp. 30, 125, 251]. Note that the spectral sequence which figured in the proof of 3.4 is essentially the Serre spectral sequence of the fibration tower

$$\{\widetilde{\mathbf{Z}_s K(\pi,1)}\}_s \to \{\mathbf{Z}_s K(\pi,1)\} \to \{K(\pi_1\mathbf{Z}_s K(\pi,1),1)\}_s,$$

where tilde denotes universal cover.

4. Groups of type (FP)

A group π is said to be of type (FP) if the trivial π-module $\mathbb{Z}$ has a resolution of finite length made up of finitely generated projective R-modules. This section provides a way to compute the left derived functors C_i $(i \geq 0)$ of the lower central series completion functor C (2.2) on the category of modules over a group of type (FP). In favorable cases (4.8) this method gives a simple homological formula for the functors C_i. If C_0 happens to coincide with the $H\mathbb{Z}$-localization functor E (3.1, 3.4, 3.8), the zero-dimensional part of this formula is a simple expression for E itself.

A brief sketch of the method may help to clarify the peculiarities of the modules involved. The starting point is to write the modules $M/I^s \cdot M$ which appear in the definition of $C(M)$ as $\mathrm{Tor}_0^R(R/I^s, M)$. This notation makes clear that the formation of $M/I^s \cdot M$ depends on the left R-module structure of M and the right R-module structure of R/I^s. This right R-module structure of R/I^s can be dualized in a more or less standard way to get *left* R-modules, in terms of which $\mathrm{Tor}_*^R(R/I^s, M)$ can be expressed using $\mathrm{Ext}_R^*(-, M)$. The dualization process transforms the inverse system $\{R/I^s\}_s$ into a direct system of dual modules; it turns out that the derived functors of C can be computed by first taking a direct limit of these dual modules and then applying $\mathrm{Ext}_R^*(-, M)$. The extra left R-module structure on each R/I^s is reflected in a right R-module structure on the dual modules; this passes to the direct limit and induces the usual left R-module structure on $C_*(M)$ when $\mathrm{Ext}_R^*(-, M)$ is applied.

Given the group π, let $K_j(\pi)$ $(j \geq 0)$ be the direct limit $\varinjlim_s \mathrm{Ext}_R^j(R/I^s, R)$, where Ext is taken in the sense of right R-modules, and the maps in the direct system are induced by the usual epimorphisms $R/I^s \to R/I^{s-1}$. Each $K_j(\pi)$ has commuting left and right π-module structures: the left action of π is induced by the usual left action of π on R, and the right action of π by the usual left action of π on each R/I^s. It is not hard to see that if the trivial left or right π-module $\mathbb{Z}$ possess a projective resolution of finite length over R

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0,$$

for instance, if π is of type (FP), then $K_j(\pi)$ vanishes for $j > n+1$.

The basic result of this section is

4.1. Theorem. *If π is of type* (FP), *then for any π-module M there is a strongly convergent second quadrant spectral sequence of homological type*:

$$E^2_{i,j} = \mathrm{Ext}_R^{-i}(K_j(\pi), M) \Longrightarrow C_{i+j}(M).$$

4.2. Remark. The Ext's which form the E^2-term of this spectral sequence are of course computed in the sense of left R-modules. However, the extra right action of R on $K_j(\pi)$ furnishes the groups $\mathrm{Ext}_R^{-i}(K_j(\pi), M)$ with a left R-module structure which commutes with the differentials and, on passage to E^∞, corresponds to the usual left action of R on $C_*(M)$.

4.3. Remark. The above spectral sequence exists as long as π is of type $(\overline{\mathrm{FP}})$, that is, as long as the trivial π-module $\mathbb{Z}$ admits a (possibly infinite) resolution by finitely generated projective R-modules. However, the spectral sequence does not necessarily converge under this weaker hypothesis, even if only a finite number of the modules $K_j(\pi)$ are non-zero. An example of such failure to converge can be obtained by letting π be a non-trivial finite perfect group.

The group π is said to be a duality group [3, 0.3] of *dimension n* if

(i) π is of type (FP) (cf. [9]),

(4.4) (ii) $H^k(\pi; R) = 0$, $k \neq n$,

(iii) $H^n(\pi; R)$ is torsion free.

Condition (ii) and (iii) are equivalent to

(ii)′ $\mathrm{Ext}^k_R(\mathbb{Z}, R) = 0$, $k \neq n$,

(iii)′ $\mathrm{Ext}^n_R(\mathbb{Z}, R) = D$ is torsion-free,

where $\mathbb{Z}$ is the trivial R-module and the Ext's are taken in the sense of *right* R-modules. The abelian group D with left π-action induced by the extra left action of π on R is called the *dualizing module* for π. (This differs slightly but inessentially from the usual definition of dualizing module [3, 1.2].) Many examples of duality groups are given in [3] and [4].

If π is a duality group the modules $K_j(\pi)$ are especially accessible, in view of

4.5. Proposition. *Suppose that π is a duality group of dimension n, with dualizing module D. Then $K_j(\pi)$ vanishes for $j \neq n, n+1$, and there are natural isomorphisms*

$$K_n(\pi) \approx D \otimes_{\mathbb{Z}} \varinjlim_s \mathrm{Hom}_{\mathbb{Z}}(R/I^s, \mathbb{Z})$$

$$K_{n+1}(\pi) \approx D \otimes_{\mathbb{Z}} \varinjlim_s \mathrm{Ext}_{\mathbb{Z}}(R/I^s, \mathbb{Z}).$$

4.6. Remarks. Under the indicated isomorphism, the right action of π on $K_n(\pi)$ is induced by the left action of π on each R/I^s, while the left action of π on $K_n(\pi)$ is a diagonal action induced by the left action of π on D and the right action of π on each R/I^s. A corresponding statement holds for $K_{n+1}(\pi)$.

4.7. Remark. A result similar to 4.5 holds without the restriction 4.4(iii)′ that D be torsion-free. In this more general setting the formula for $K_n(\pi)$ is replaced by a short exact sequence

$$0 \to \varinjlim_s D \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(R/I^s, \mathbb{Z}) \to K_n(\pi) \to \varinjlim_s \mathrm{Tor}^{\mathbb{Z}}(D, \mathrm{Ext}_{\mathbb{Z}}(R/I^s, \mathbb{Z})) \to 0.$$

The formula for $K_{n+1}(\pi)$ remains unchanged.

It follows from 4.5 that for a duality group π the spectral sequence of 4.1 collapses into a long exact sequence. There is even further collapse if the lower central series quotient $\pi/\Gamma_s(\pi)$ (3.2) of π are torsion-free. In this case the abelian groups R/I^s are torsion-free [1, 1.3], so (4.13) $K_{n+1}(\pi)$ vanishes.

4.8. Corollary. *Suppose that π is a duality group of dimension n with torsion-free lower central series quotients. Let D be the dualizing module for π. Then for any π-module M there are natural isomorphisms,*

$$C_j(M) \approx \operatorname{Ext}_R^{n-j}\left(D \otimes_{\mathbf{Z}} \varinjlim_s \operatorname{Hom}_{\mathbf{Z}}(R/I^s, \mathbf{Z}), M\right), \qquad j \geq 0.$$

The formulas of 4.5 and 4.8 simplify if the dualizing module D is additively isomorphic to an infinite cyclic group, that is, if π is a *Poincaré duality group* [2].

4.9. Example. Suppose that π is a finitely generated torsion-free nilpotent group. By [2, 3.1.2] π is an (oriented) Poincaré duality group (that is, a Poincaré duality group which acts trivially on its dualizing module). Thus if π also has torsion-free lower central series quotients, there are natural isomorphisms (3.1, 4.8)

$$E(M) \approx C_0(M) \approx \operatorname{Ext}_R^n\left(\varinjlim_s \operatorname{Hom}_{\mathbf{Z}}(R/I^s, \mathbf{Z}), M\right)$$

for any π-module M, where n is the duality dimension of π. For instance, if π is an infinite cyclic group generated by α there is a natural isomorphism

$$E(M) \approx \operatorname{Ext}_R^1(K, M)$$

where K is additively isomorphic to the free abelian group on a countable number of generators $x_0, x_1, x_2, \ldots$ and has identical left and right π-actions given by

$$\alpha \cdot x_0 = x_0 \cdot \alpha = x_0$$

$$\alpha \cdot x_i = x_i \cdot \alpha = x_i - x_{i-1} \qquad (i > 0),$$

that is

$$(1-\alpha) \cdot x_i = x_{i-1} \qquad (i > 0).$$

It will be convenient to have some simple lemmas to refer to in the proofs of 4.1 and 4.5.

Let M be a right R-module. A projective resolution

$$\to P_{n+1} \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

of M is said to have *length* $\leq n$ if P_i vanishes for $i > n$. The resolution is said to be *finite* if it has length $\leq n$ for some n and each P_i is a finitely generated projective right R-module.

Let

$$0 \to M' \to M \to M'' \to 0$$

be a short exact sequence of right R-module.

4.10. Lemma. *If M and M' have finite projective resolutions of length $\leq n$, then M'' has a finite projective resolution of length $\leq n+1$.*

In fact, the resolution of M'' can be chosen to be the *mapping cone* [18, p. 46] of a suitable map between resolutions of M' and M.

4.11. Lemma [11: V, 2.2]. *Let $\mathbf{F}'$ and $\mathbf{F}''$ be projective resolutions of M' and M'' respectively. Then there is a resolution $\mathbf{F}$ of M which fits into a short exact sequence*

$$0 \to \mathbf{F}' \to \mathbf{F} \to \mathbf{F}'' \to 0$$

covering the original short exact sequence of modules.

Note that if the differentials are discounted $\mathbf{F}$ is isomorphic to the direct sum of $\mathbf{F}''$ and $\mathbf{F}'$.

4.12. Lemma. *Let M be a left π-module. Define a right action of π on $R \otimes M = R \otimes_{\mathbf{Z}} M$ by*

$$(r \otimes m) \cdot \alpha = (r \cdot \alpha) \otimes (\alpha^{-1} \cdot m) \qquad r \in R, m \in M, \alpha \in \pi.$$

Then $\mathrm{Tor}_i^R(R \otimes M, \mathbf{Z})$ vanishes unless $i = 0$, and the left action of π on R makes $\mathrm{Tor}_0^R(R \otimes M, \mathbf{Z})$ into a left π-module which is naturally isomorphic to M.

4.13. Lemma. *If π is a group of type (FP) the π-modules I^s/I^{s+1} and R/I^s ($s \leq 1$) are finitely generated as abelian groups.*

In fact, the conclusion of 4.13 holds if and only if $H_1(\pi; \mathbf{Z}) = I/I^2$ is finitely generated. This is seen by induction on s, using the multiplication surjections $(I/I^2) \otimes (I^{s-1}/I^s) \to I^s/I^{s+1}$.

Proof of 4.1. Since π is of type (FP), the trivial left or right π-module $\mathbf{Z}$ possesses a finite projective resolution over R of length, say, $\leq n$. It follows from 4.13 and 4.10 that the trivial right π-modules I^s/I^{s+1} ($s \geq 1$) possess finite projective resolutions over R of length $\leq n+1$.

By 4.11, it is possible to construct inductively resolutions $\mathbf{F}_s$ of the right R-modules R/I^s ($s \geq 1$) such that

(4.14) (i) $\mathbf{F}_s$ is a finite projective resolution of length $\leq n+1$ and
(ii) there are surjective maps $\mathbf{F}_s \to \mathbf{F}_{s-1}$ which cover the usual surjections $R/I^s \to R/I^{s-1}$.

Let $\mathbf{F}$ denote the cochain complex $\varinjlim_s \mathrm{Hom}_R(\mathbf{F}_s, R)$, where Hom is of necessity taken in the sense of right R-modules and the maps in the direct systems are induced by those of 4.14(ii). The left action of R on itself makes $\mathbf{F}$ into a cochain complex of left R-modules; since direct limits are exact, the i^{th} cohomology group of $\mathbf{F}$ ($i \geq 0$) is naturally isomorphic to $K_i(\pi)$.

For any *left* R-module M, let $W_i(M)$ $(i \leq 0)$ be the i^{th} homology group of the chain complex $\text{Hom}_R(\mathbf{F}, M)$. It is not hard to see that the abelian groups $W_i(M)$ are actually left R-modules in a natural way. In fact, left multiplication by any element r of R induces right R-module endomorphisms $\mu(r)_s: R/I^s \to R/I^s$. Using 4.14(ii) these can be lifted to resolution maps $\lambda(r)^s: \mathbf{F}_s \to \mathbf{F}_s$ which form a *coherent* family, in the sense that the diagrams

$$\begin{array}{ccc} \mathbf{F}_s & \longrightarrow & \mathbf{F}_{s-1} \\ \lambda(r)_s \downarrow & & \downarrow \lambda(r)_{s-1} \\ \mathbf{F}_s & \longrightarrow & \mathbf{F}_{s-1} \end{array}$$

all commute. Moreover, any two such coherent families of resolution endomorphisms corresponding to the same element of R are coherently chain homotopic. This coherent left action of R, up to chain homotopy, on the resolutions $\mathbf{F}_s$ induces a right action of R, up to cochain homotopy, on the cochain complex $\mathbf{F}$, and therefore an actual left action of R on the homology groups $W_i(M)$.

The dual $\text{Hom}_R(P, R)$ of a finitely generated projective right R-module P is finitely generated projective left R-module. Together with 4.24(ii) this implies that $\mathbf{F}$ is a cochain complex of projective left R-modules. Thus a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of left R-modules gives rise to a short exact sequence

$$0 \to \text{Hom}_R(\mathbf{F}, M') \to \text{Hom}_R(\mathbf{F}, M) \to \text{Hom}_R(\mathbf{F}, M'') \to 0$$

of chain complexes, and hence to a long exact homology sequence

$$(4.15) \qquad \to W_i(M') \to W_i(M) \to W_i(M'') \to W_{i-1}(M') \to \cdots \to W_0(M'') \to 0.$$

If P is a finitely generated projective right R-module and M is a left R-module the natural map

$$P \otimes_R M \to \text{Hom}_R(\text{Hom}_R(P, R), M)$$

is an isomorphism. Consequently, for any left R-module M there are natural isomorphisms

$$\begin{aligned} \text{Hom}_R(\mathbf{F}, M) &= \text{Hom}_R\left(\varinjlim_s \text{Hom}_R(\mathbf{F}_s, R), M\right) \\ &\approx \varprojlim \{\text{Hom}_R(\text{Hom}_R(\mathbf{F}_s, R), M)\}_s \\ &\approx \varprojlim \mathbf{F}_s \otimes_R M. \end{aligned}$$

The techniques of [19] thus imply that there are natural short exact sequences

$$0 \to \varprojlim{}^1 \{\text{Tor}_{i+1}^R(R/I^s, M)\}_s \to W_i(M) \to \varprojlim \{\text{Tor}_i^R(R/I^s, M)\}_s \to 0.$$

In particular, there is a natural map $W_0(M) \to C(M)$; if M is free, this map is an isomorphism and the modules $W_i(M)$ vanish for $i>0$. In view of 4.15, this shows that W_i is naturally equivalent to the i^{th} left derived functor C_i of C.

Since $\mathbf{F}$ is a cochain complex of projective modules, the desired spectral sequence is just the hypercohomology spectral sequence of [11: XVII, §2]. This converges strongly for the trivial reason that $\mathbf{F}$ vanishes above dimension $n+1$.

Proof of 4.5. For the purposes of this proof $\mathrm{Ext}_R^*(-,-)$ will always denote Ext in the sense of *right* R-modules.

The proof depends on the fact that for any right π-module M there is a natural first quadrant composition-of-functors spectral sequence of cohomological type

$$E_{i,j}^2 = \mathrm{Ext}_R^i(\mathbb{Z}, \mathrm{Ext}_{\mathbb{Z}}^j(M,R)) \Longrightarrow \mathrm{Ext}_R^{i+j}(M,R).$$

Here the action of π on $\mathrm{Hom}_{\mathbb{Z}}(M,R)$ is given by

$$(f\cdot\alpha)(x) = (f(x\cdot\alpha^{-1}))\cdot\alpha \qquad f\in\mathrm{Hom}_{\mathbb{Z}}(M,R)$$
$$x\in M, \alpha\in\pi$$

and there is a corresponding description of the action of R on $\mathrm{Ext}_{\mathbb{Z}}^1(M,R)$.

If M is R/I^s, the E^2-term of this spectral sequence can be computed in two steps. First of all, R/I^s and R are, respectively, finitely generated (4.13) and free as abelian groups, so that there are natural isomorphisms

$$\mathrm{Ext}_{\mathbb{Z}}^j(R/I^s, R) \approx R\otimes_{\mathbb{Z}}\mathrm{Ext}_{\mathbb{Z}}^j(R/I^s,\mathbb{Z}), \qquad j=0,1$$

which can be made into isomorphisms of right R-modules by putting the proper module structure on the right hand sides.

Secondly, the fact that π is a duality group of dimension n implies that for any right π-module M there are natural isomorphisms

(4.16) $\quad \mathrm{Ext}_R^i(\mathbb{Z}, M) \approx \mathrm{Tor}_{n-i}^R(D\otimes_{\mathbb{Z}} M, \mathbb{Z})$

where $\mathbb{Z}$, as an argument of $\mathrm{Tor}^R(-,-)$, denotes the trivial left R-module and the right action of π on $D\otimes_{\mathbb{Z}} M$ is given by

$$(d\otimes m)\cdot\alpha = (\alpha^{-1}d)\otimes(m\cdot\alpha), \qquad \alpha\in\pi, d\in D, m\in M.$$

Consequently, there are natural isomorphisms

$$E_{i,j}^2 \approx \mathrm{Ext}_R^i(\mathbb{Z}, \mathrm{Ext}_{\mathbb{Z}}^j(R/I^s, R)) \approx \mathrm{Tor}_{n-i}^R(R\otimes_{\mathbb{Z}} D\otimes_{\mathbb{Z}}\mathrm{Ext}_{\mathbb{Z}}^j(R/I^s,\mathbb{Z}),\mathbb{Z}).$$

The proof is finished by applying 4.12 and taking a direct limit over s.

The claim in remark 4.7 can be proved by showing that if D is not torsion-free the isomorphisms 4.16 are replaced by isomorphisms

$$\mathrm{Ext}_R^i(\mathbb{Z}, M) \approx \mathrm{Tor}_{n-i}^R(M, D).$$

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