DISCRETE APPLIED MATHEMATICS

ELSEVIER Discrete Applied Mathematics 82 (1998) 231–246

On aggregating two linear Diophantine equations

Nan Zhu, Kevin Broughan*

Department of Mathematics, University of Waikato, Private Bag 3105, Hamilton, New Zealand

Received 5 January 1995; revised 9 July 1997

Abstract

The problem of aggregating a general system of two linear Diophantine equations with integer coefficients and non-negative integer variables, to form a single linear Diophantine equation with the same solution space, is investigated. New procedures, which generalize and improve upon some results in the literature, are given. Some or all of the variables may be given upper bounds.

Keywords: Linear Diophantine equation; Aggregation method; Knapsack problem

1. Introduction

By aggregating a system of two linear Diophantine equations with integer coefficients we mean reducing the system:

$$\sum_{j\in N} a_{1j}x_j = b_1, \quad \sum_{j\in N} a_{2j}x_j = b_2, \qquad x_j \geqslant 0 \text{ integer}, \quad \text{for all } j\in N, \tag{1.1}$$

into a single linear Diophantine equation having the same solution space in non-negative integer variables:

$$\sum_{j\in N} (t_1a_{1j} + t_2a_{2j})x_j = t_1b_1 + t_2b_2, \qquad x_j \geqslant 0 \text{ integer}, \quad \text{for all } j\in N, \tag{1.2}$$

where $N=\{1,\ldots,n\}$ and the multipliers t_1 $(\neq 0)$ and t_2 $(\neq 0)$ are relatively prime integers. The main focus of this work is on finding suitable values for t_1 and t_2, which give small coefficients and right-hand-side value (in the sense of the absolute value) in the aggregated equation (1.2).

It is obvious that any solution to system (1.1) also solves (1.2) for any multipliers t_1 and t_2. The reverse may be obtained by giving some conditions on the multipliers. The earliest work on aggregating system (1.1) is due to Mathews [17]. Elmaghraby and Wig [7] introduce the aggregation method in number theory into the field of integer

* Corresponding author. E-mail: kab@hoiho.math.waikato.ac.nz.

0166-218X/98/$19.00
PII S0166-218X(97)00123-6

linear programming (ILP). By sequentially reducing the original constraints to a single constraint, the original ILP problem can be reduced to an equivalent general knapsack problem (KP) in $m-1$ steps, where m is the number of original constraints of the ILP problem. Therefore, the ILP problem can be solved by solving the KP. If this KP is easier to be solved than the original ILP problem and its construction can be accomplished in a reasonable amount of time, aggregation is worthwhile.

Many methods have been given for aggregating system (1.1). For example, see the works of (arranged alphabetically): Anthonisse [1], Babayev and Glover [2], Babayev and Mardanov [4], Bradley [5], Elimam and Elmaghraby [6], Garfinkel and Nemhauser [8], Glover [9], Glover and Babayev [10], Glover and Woolsey [11], Greenberg [13], Hammer and Rosenberg [14], Kendall and Zionts [15], Lin [16], Onyekwelu [18], Padberg [19], and Zhu [20].

One approach to the aggregation problem is provided in [9]. To aggregate a non-homogeneous system (1.1) (i.e. a system with at least one $b_i \neq 0$), Glover constructs t_1 and t_2 that must satisfy n inequalities. Similar to this work, some procedures to achieve a single inequality for t_1 and t_2 to aggregate non-homogeneous and homogeneous systems (1.1) are devised in [6]. The same approach for non-homogeneous systems (1.1) is taken in [4, 13, 20].

In this paper we assume that for a general system (1.1) and aggregated equation (1.2), the first p individual variables have been given upper bounds, i.e. $x_j \leqslant u_j$ $(j=1,\dots,p;\ p\leqslant n)$. We first give a basic theorem and then some procedures, resulting from the theorem, for solving the aggregation problem. These aggregation procedures generalize and improve upon some results in the literature.

2. The basic result

We use the following notation:

$$N=\{1,\dots,p,\dots,n\}, \qquad w_j=c_1a_{2j}-c_2a_{1j}, \quad (j\in N),$$

$$w_0=c_2a_{1,0}-c_1a_{2,0}=c_2b_1-c_1b_2, \quad x_0=1,$$

$$i\in I=\{1,2\}, \quad S(X)=S(X;c_1,c_2)=\sum_{j\in N\cup\{0\}} w_jx_j,$$

$$-\infty<L\leqslant \min_{X\in T_2} S(X), \quad \max_{X\in T_2} S(X)\leqslant U<+\infty,$$

$$T_1=\{X\colon X=(x_1,\dots,x_p,\dots,x_n) \text{ satisfying } (1.1)\},$$

$$T_2=\{X\colon X=(x_1,\dots,x_p,\dots,x_n) \text{ satisfying } (1.2)\},$$

where the $\{c_i\}$ are arbitrary given integers with $c_i\neq 0$ for at least one $i\in I$, and L and U are any values satisfying the given inequalities. Obviously, $T_1\subseteq T_2$, and if $L>0$ or $U<0$, then system (1.1) is infeasible.

In the following, we have

Theorem 2.1. *For arbitrary integer coefficients* $\{a_{ij}\}$ *and* $\{b_i\}$, *let* $\{c_i\}$ *be arbitrary given integers with* $c_i \neq 0$ *for at least one* $i \in I$. *Then system* (1.1) *is equivalent to single equation* (1.2) *if*

$$t_1c_1 + t_2c_2 > \max\{-L,\ U\}, \tag{2.1}$$

where $\{t_i\}$ *are relatively prime integers.*

Proof. Let $X = \{x_j\}$ be any member of T_2. Then

$$\sum_{j \in N} a_{1j}x_j = b_1 + t_2q \tag{2.2}$$

for some integer q. It follows from (1.2) that

$$\sum_{j \in N} a_{2j}x_j = b_2 - t_1q, \tag{2.3}$$

because t_1 and t_2 are relatively prime integers.

Since $L \leqslant \min_{X \in T_2} S(X)$, so $-L \geqslant \max_{X \in T_2} -S(X)$. By (2.1), we have

$$\begin{aligned} t_1c_1 + t_2c_2 &> \max\{-L,\ U\} \geqslant \max\{-S(X),\ S(X)\} = |S(X)| \\ &= \left|\sum_{j \in N \cup \{0\}} w_jx_j\right| = \left|c_1\sum_{j \in N} a_{2j}x_j - c_2\sum_{j \in N} a_{1j}x_j + c_2b_1 - c_1b_2\right| \\ &= |t_1c_1 + t_2c_2| \cdot |q| = (t_1c_1 + t_2c_2) \cdot |q|. \end{aligned}$$

The above last equality is true because $|S(X)| \geqslant 0$, i.e. $t_1c_1 + t_2c_2$ has been guaranteed to be positive by (2.1). So $1 > |q|$. Therefore, $q = 0$ and $X \in T_1$. Thus $T_1 \supseteq T_2$, and therefore $T_1 = T_2$. □

In normal circumstances, it is difficult to obtain values for $\min_{X \in T_2} S(X)$ and $\max_{X \in T_2} S(X)$ directly, since each is a single equality constrained (t_1, t_2)-parameter ILP problem with the first p individual variables having been given upper bounds. However, under certain conditions the corresponding lower and upper bound values L and U can be obtained. In Sections 3–5, we shall use Theorem 2.1 to aggregate system (1.1) with some additional conditions.

3. The system with no given upper bounded variables

In this section, let $p = 0$ in system (1.1), i.e. a prior stipulation of individual variable upper bounds on (1.1) is not required. Two cases are discussed.

3.1. Case I

In this case, we assume $\{a_{ij}\}$, $\{b_i\}$, $\{t_i\}$ in system (1.1) are positive integers, and $\{c_i\}$ are arbitrary given integers with $c_i > 0$ for at least one $i \in I$.

It is easy to see that the problems of $\min_{X \in T_2} S(X)$ and $\max_{X \in T_2} S(X)$ are equality constrained (t_1, t_2)-parameter KPs. In the following we obtain values for L and U by two methods. First we derive an inequality:

Lemma 3.1. *Let the numbers d_1 and d_2 be real, a_1 and a_2 be positive, and l_1 and l_2 be non-negative with at least one $l_i > 0$ $(i \in I)$. Then the following inequality holds:*

$$\frac{l_1 d_1 + l_2 d_2}{l_1 a_1 + l_2 a_2} \leqslant \max\left\{\frac{d_1}{a_1}, \frac{d_2}{a_2}\right\}. \tag{3.1}$$

Proof. Assume $d_1/a_1 \geqslant d_2/a_2$. Then $\max\{d_1/a_1, d_2/a_2\} = d_1/a_1$. Let $d_1/a_1 = h$, $d_2/a_2 = k$, where h and k are real. Then

$$\frac{l_1 d_1 + l_2 d_2}{l_1 a_1 + l_2 a_2} = \frac{l_1 h a_1 + l_2 k a_2}{l_1 a_1 + l_2 a_2} \leqslant \frac{h(l_1 a_1 + l_2 a_2)}{l_1 a_1 + l_2 a_2} = \frac{d_1}{a_1}. \qquad \square$$

One method to obtain values for L and U is based on a property of the continuous KP:

$$\begin{aligned} &\max && \sum_{j \in N} c_j x_j \\ &\text{s.t.} && \sum_{j \in N} a_j x_j = b, \\ &x_j \geqslant 0, && \text{for all } j \in N, \end{aligned} \tag{3.2}$$

where the $a_j > 0$, $b \geqslant 0$, and c_j are arbitrary. The optimal value of the objective function is $c_k \cdot b/a_k$ with an optimal solution: $x_k^* = b/a_k$, and $x_j^* = 0$ otherwise, where $\max_{j \in N}\{c_j/a_j\} = c_k/a_k$. The value $\lfloor c_k \cdot b/a_k \rfloor$ is an integer upper bound of the objective function of the corresponding integer KP, where $\lfloor x \rfloor$ is the greatest integer $\leqslant x$.

We use the notation

$$L_1 = w_0 - \max_i\left\{\left\lfloor b_i \cdot \max_j A_{ij} \right\rfloor\right\}, \quad U_1 = w_0 + \max_i\left\{\left\lfloor b_i \cdot \max_j B_{ij} \right\rfloor\right\},$$

where $A_{ij} = -w_j/a_{ij}$, $B_{ij} = w_j/a_{ij}$, for all $i \in I$ and $j \in N$. We have

Theorem 3.1. *For arbitrary positive integers $\{a_{ij}\}$ and $\{b_i\}$, let $\{c_i\}$ be arbitrary given integers with $c_i > 0$ for at least one $i \in I$. Then (1.1) is equivalent to (1.2) if*

$$t_1 c_1 + t_2 c_2 > \max\{-L_1, U_1\}, \tag{3.3}$$

where $\{t_i\}$ are relatively prime positive integers.

Proof. Using the above property of the continuous KP, we have

$$\begin{aligned}\max_{X\in T_2} S(X) &= w_0 + \max_{X\in T_2}\sum_{j\in N} w_j x_j \\ &\leqslant w_0 + \max_{j\in N}\{w_j\cdot(t_1b_1 + t_2b_2)/(t_1a_{1j}+t_2a_{2j})\} \\ &= w_0 + \max_{j\in N}\{(t_1b_1\cdot w_j + t_2b_2\cdot w_j)/(t_1a_{1j}+t_2a_{2j})\} \\ &\leqslant w_0 + \max\left\{b_1\cdot\max_{j\in N}\{w_j/a_{1j}\},\ b_2\cdot\max_{j\in N}\{w_j/a_{2j}\}\right\}.\end{aligned}$$

The final inequality above can be proved using inequality (3.1). Similarly, we have

$$\begin{aligned}-\min_{X\in T_2} S(X) = \max_{X\in T_2} -S(X) &= -w_0 + \max_{X\in T_2}\sum_{j\in N}(-w_j)x_j \\ &\leqslant -w_0 + \max\left\{b_1\cdot\max_{j\in N}\{-w_j/a_{1j}\},\ b_2\cdot\max_{j\in N}\{-w_j/a_{2j}\}\right\}.\end{aligned}$$

Hence,

$$\min_{X\in T_2} S(X) \geqslant w_0 - \max_{i\in I}\left\{\left\lfloor b_i\cdot\max_{j\in N} A_{ij}\right\rfloor\right\}.$$

Thus, L_1 and U_1 can be used as L and U in condition (2.1) of Theorem 2.1. Therefore, system (1.1) is equivalent to Eq. (1.2). □

In the following, we derive another method to obtain values for L and U in the condition (2.1) by using the upper bound on the sum of a subset of the variables, respectively. We use the notation:

$$L_2 = w_0 - \left\{\max_{w_j<0}|w_j|\right\}\cdot\max_i\left\{\left\lfloor b_i\Big/\min_{w_j<0} a_{ij}\right\rfloor\right\},$$

$$U_2 = w_0 + \left\{\max_{w_j>0} w_j\right\}\cdot\max_i\left\{\left\lfloor b_i\Big/\min_{w_j>0} a_{ij}\right\rfloor\right\},$$

where all $i\in I$ and $j\in N$. We have

Theorem 3.2. *For arbitrary positive integers $\{a_{ij}\}$ and $\{b_i\}$, let $\{c_i\}$ be arbitrary given integers with $c_i>0$ for at least one $i\in I$. Then (1.1) is equivalent to (1.2) if*

$$t_1c_1 + t_2c_2 > \max\{-L_2,\ U_2\}, \tag{3.4}$$

where $\{t_i\}$ are relatively prime positive integers.

Proof. From Eq. (1.2), we have $\sum_{w_j>0}(t_1a_{1j}+t_2a_{2j})x_j \leqslant t_1b_1 + t_2b_2$. Then to any $X=\{x_j\}\in T_2$

$$\sum_{w_j>0} x_j \leqslant \max_{w_j>0}\{(t_1b_1+t_2b_2)/(t_1a_{1j}+t_2a_{2j})\} \leqslant \max\left\{b_1\Big/\min_{w_j>0} a_{1j},\ b_2\Big/\min_{w_j>0} a_{2j}\right\}.$$

hence,

$$\max_{X\in T_2}\sum_{w_j>0} x_j \leqslant \max\left\{\left\lfloor b_1\Big/\min_{w_j>0} a_{1j}\right\rfloor,\ \left\lfloor b_2\Big/\min_{w_j>0} a_{2j}\right\rfloor\right\}. \tag{3.5}$$

Therefore,

$$\max_{X\in T_2} S(X) \leqslant \max_{X\in T_2}\sum_{w_j>0} w_j x_j + w_0 \leqslant \left\{\max_{w_j>0} w_j\right\}\max_{X\in T_2}\sum_{w_j>0} x_j + w_0$$

$$\leqslant \left\{\max_{w_j>0} w_j\right\}\max\left\{\left\lfloor b_1\Big/\min_{w_j>0} a_{1j}\right\rfloor,\ \left\lfloor b_2\Big/\min_{w_j>0} a_{2j}\right\rfloor\right\} + w_0.$$

Similarly, we have

$$\max_{X\in T_2}\sum_{w_j<0} x_j \leqslant \max\left\{\left\lfloor b_1\Big/\min_{w_j<0} a_{1j}\right\rfloor,\ \left\lfloor b_2\Big/\min_{w_j<0} a_{2j}\right\rfloor\right\} \tag{3.6}$$

and

$$-\min_{X\in T_2} S(X) = \max_{X\in T_2} -S(X) \leqslant -w_0 + \left\{\max_{w_j<0}|w_j|\right\}$$

$$\times\max\left\{\left\lfloor b_1\Big/\min_{w_j<0} a_{1j}\right\rfloor,\ \left\lfloor b_2\Big/\min_{w_j<0} a_{2j}\right\rfloor\right\}.$$

Hence,

$$\min_{X\in T_2} S(X) \geqslant w_0 - \left\{\max_{w_j<0}|w_j|\right\}\cdot\max\left\{\left\lfloor b_1\Big/\min_{w_j<0} a_{1j}\right\rfloor,\ \left\lfloor b_2\Big/\min_{w_j<0} a_{2j}\right\rfloor\right\}.$$

Therefore, L_2 and U_2 can be used as L and U in condition (2.1) of Theorem 2.1. □

As a special case, if $c_i = b_i$, $(i\in I)$, then $w_j \triangleq v_j = b_1 a_{2j} - b_2 a_{1j}$, $(j\in N)$, and $w_0 = 0$. Condition (3.4) is written as

$$t_1 b_1 + t_2 b_2 > \max\{-L_2,\ U_2\}, \tag{3.7}$$

where

$$L_2 = -\left\{\max_{v_j<0}|v_j|\right\}\max_i\left\{\left\lfloor b_i\Big/\min_{v_j<0} a_{ij}\right\rfloor\right\},$$

$$U_2 = \left\{\max_{v_j>0} v_j\right\}\max_i\left\{\left\lfloor b_i\Big/\min_{v_j>0} a_{ij}\right\rfloor\right\}.$$

There is an aggregation procedure in Theorem 2 in [13]:

Theorem 2 (Greenberg [13]). *Given a_{ij} and b_i positive integer values for* (1.1), *take $\overline{U}$ as the greatest integer satisfying*

$$\overline{U} \leqslant \max\left\{b_1\Big/\min_{j\in N} a_{1j},\ b_2\Big/\min_{j\in N} a_{2j}\right\}.$$

If t_1 and t_2 are relatively prime positive integers that satisfy

$$t_1 b_1 + t_2 b_2 > \overline{U} \cdot \max_{j\in N} |b_1 a_{2j} - b_2 a_{1j}|, \tag{3.8}$$

then (1.1) *and* (1.2) *have equivalent solutions.*

Comparing condition (3.7) with condition (3.8), it is easy to see that

$$\overline{U} = \max\left\{ \left\lfloor b_1 \Big/ \min_{j\in N} a_{1j} \right\rfloor, \left\lfloor b_2 \Big/ \min_{j\in N} a_{2j} \right\rfloor \right\},$$

$$\max_{j\in N} |v_j| = \max\left\{ \max_{v_j>0} v_j, \max_{v_j<0} |v_j| \right\},$$

$$-L_2 \leqslant \left\{ \max_{j\in N} |v_j| \right\} \max\left\{ \left\lfloor b_1 \Big/ \min_{j\in N} a_{1j} \right\rfloor, \left\lfloor b_2 \Big/ \min_{j\in N} a_{2j} \right\rfloor \right\},$$

$$U_2 \leqslant \left\{ \max_{j\in N} |v_j| \right\} \max\left\{ \left\lfloor b_1 \Big/ \min_{j\in N} a_{1j} \right\rfloor, \left\lfloor b_2 \Big/ \min_{j\in N} a_{2j} \right\rfloor \right\},$$

thus,

$$\max\{-L_2, U_2\} \leqslant \overline{U} \cdot \max_{j\in N} |v_j|.$$

Hence, our Theorem 3.2 generalizes and improves upon the procedure in [12].

3.2. Case II

In this case, we assume $\{a_{1j}\}$ and b_1 are arbitrary integers, $\{a_{2j}\}$ and b_2 are positive integers, $c_i = b_i$, $(i \in I)$. We use the following notation:

$$\min_{j\in N} \left\{ \frac{a_{1j}}{a_{2j}} \right\} = \frac{a_{1k}}{a_{2k}}, \qquad \max_{j\in N} \left\{ \frac{a_{1j}}{a_{2j}} \right\} = \frac{a_{1l}}{a_{2l}}.$$

Then the $\{v_j\}$ defined above have the following property:

Lemma 3.2. *If there exist $\{x_j\}$ satisfying system* (1.1) *then*

$$v_k = b_1 a_{2k} - b_2 a_{1k} \geqslant 0 \quad \text{and} \quad v_l = b_1 a_{2l} - b_2 a_{1l} \leqslant 0, \tag{3.9}$$

where all $a_{2j} > 0$ and $b_2 > 0$.

Proof. We know that if $L > 0$ (or $U < 0$) then system (1.1) is infeasible. Thus, it is easy to see that if for all $j \in N$, $v_j > 0$ (or < 0), then the non-homogeneous system (1.1) is infeasible. In other words, if there exists $X = \{x_j\}$ satisfying (1.1) then

$$v_{j_1} \leqslant 0 \left(\Leftrightarrow b_1 - b_2 \cdot \frac{a_{1,j_1}}{a_{2,j_1}} \leqslant 0 \right) \quad \text{and} \quad v_{j_2} \geqslant 0 \left(\Leftrightarrow b_1 - b_2 \cdot \frac{a_{1,j_2}}{a_{2,j_2}} \geqslant 0 \right) \tag{3.10}$$

for at least one j_1 and at least one j_2.

Since $b_1 - b_2x$ is a decreasing function of x, for all $j \in N$ we have

$$b_1 - b_2 \cdot (a_{1k}/a_{2k}) \geqslant b_1 - b_2 \cdot (a_{1j}/a_{2j}) \geqslant b_1 - b_2 \cdot (a_{1l}/a_{2l}). \tag{3.11}$$

From (3.10) and (3.11) we obtain (3.9). □

Note that Lemma 3.2 is essentially the first part of the theorem of Mattews, presented in other words, see [11] (we appreciate this being pointed out by one referee).

Using Theorem 2.1, a sufficient condition for aggregating system (1.1) is proved.

Theorem 3.3. *Let* $\{a_{1j}\}$ *and* b_1 *be arbitrary integers, and let* $\{a_{2j}\}$ *and* b_2 *be positive integers. Then* (1.1) *is equivalent to* (1.2) *if*

$$t_1a_{1k} + t_2a_{2k} > \max\{v_k,\ 0\} \quad \text{and} \quad t_1a_{1l} + t_2a_{2l} > \max\{-v_l,\ 0\}, \tag{3.12}$$

where $\{t_i\}$ *are relatively prime integers.*

Proof. Let

$$t_1a_{1j} + t_2a_{2j} > 0, \quad \text{for all } j \in N, \tag{3.13}$$

and

$$t_1b_1 + t_2b_2 > 0. \tag{3.14}$$

Then the problems of $\min_{X \in T_2} S(X; b_1, b_2)$ and $\max_{X \in T_2} S(X; b_1, b_2)$ are equality constrained (t_1, t_2)-parameter KPs. Using the property of the continuous KP (see (3.2)), we have

$$\begin{aligned}
\max_{X \in T_2} S(X; b_1, b_2) = \max_{X \in T_2} \sum_{j \in N} v_jx_j &\leqslant (t_1b_1 + t_2b_2) \cdot \max_{j \in N}\{v_j/(t_1a_{1j} + t_2a_{2j})\} \\
&= (t_1b_1 + t_2b_2) \cdot \max_{j \in N}\{(b_1 - b_2 \cdot (a_{1j}/a_{2j}))/(t_1 \cdot (a_{1j}/a_{2j}) + t_2)\} \\
&= (t_1b_1 + t_2b_2) \cdot \{(b_1 - b_2 \cdot (a_{1k}/a_{2k}))/(t_1 \cdot (a_{1k}/a_{2k}) + t_2)\} \\
&\triangleq U_3.
\end{aligned}$$

The third equality above can be proved by a property of the function $f(x) = (b_1 - b_2x)/(t_1x + t_2)$. This function is decreasing in x since

$$f'(x) = -(t_1b_1 + t_2b_2)/(t_1x + t_2)^2 < 0.$$

Similarly, we have

$$\begin{aligned}
\min_{X \in T_2} S(X; b_1, b_2) = \min_{X \in T_2} \sum_{j \in N} v_jx_j &\geqslant (t_1b_1 + t_2b_2) \cdot \min_{j \in N}\{v_j/(t_1a_{1j} + t_2a_{2j})\} \\
&= (t_1b_1 + t_2b_2) \cdot \{(b_1 - b_2 \cdot (a_{1l}/a_{2l}))/(t_1 \cdot (a_{1l}/a_{2l}) + t_2)\} \\
&\triangleq L_3.
\end{aligned}$$

Therefore, L_3 and U_3 can be used as L and U in condition (2.1) of Theorem 2.1. Hence, under conditions (3.13) and (3.14), system (1.1) is equivalent to equation (1.2) if

$$t_1b_1 + t_2b_2 > \max\{-L_3,\ U_3\}. \tag{3.15}$$

Condition (3.15) is equivalent to the condition

$$t_1a_{1k} + t_2a_{2k} > b_1a_{2k} - b_2a_{1k} \quad \text{and} \quad t_1a_{1l} + t_2a_{2l} > b_2a_{1l} - b_1a_{2l}. \tag{3.16}$$

Since $t_1x + t_2$ is a monotonic function of x, it is easy to prove that

$$t_1a_{1j} + t_2a_{2j} > 0, \quad \forall j \in N \Leftrightarrow t_1a_{1k} + t_2a_{2k} > 0 \quad \text{and} \quad t_1a_{1l} + t_2a_{2l} > 0. \tag{3.17}$$

Therefore, condition (3.13) can be replaced by $t_1a_{1k} + t_2a_{2k} > 0$ and $t_1a_{1l} + t_2a_{2l} > 0$ in (3.12).

If system (1.1) is feasible, by condition (3.13) and the non-negativity assumption of $\{x_j\}$, condition (3.14) is naturally satisfied. Hence, from conditions (3.16) and (3.17) we have that when (1.1) is feasible, (1.1) is equivalent to (1.2) under condition (3.12).

If system (1.1) is infeasible, after using condition (3.12) $t_1b_1 + t_2b_2$ in system (1.2) is either >0 or $\leqslant 0$. In the first case, condition (3.14) is naturally satisfied and can be removed. In the last case, (1.1) and the aggregated equation (1.2) are also equivalent because both of them are infeasible. Therefore, additional condition (3.14), in fact, is not required to be specified at all. Thus, from conditions (3.16) and (3.17) we have that (3.12) is a sufficient condition for the equivalence of (1.1) and (1.2). □

There is an aggregation result in Theorem 1 in [9]:

Theorem 1 (Glover [9]). *Suppose all $\{a_{ij}\}$ and $\{b_i\}$ are integers with at least one of b_1 and b_2 not zero, and let t_1 and t_2 be relatively prime integers. Then* (1.1) *is equivalent to* (1.2) *provided*

$$t_1a_{1j} + t_2a_{2j} \geqslant |b_2a_{1j} - b_1a_{2j}| = |v_j|, \quad \text{for all } j \in N \tag{3.18}$$

and (3.18) *holds as a strict inequality for $j \in J$, where J is any* (*nonempty*) *subset of N such that all non-negative integer solutions to* (1.2) *satisfy $x_j > 0$ for at least one $j \in J$.*

Condition (3.18) is suitable for aggregating a general non-homogeneous system (1.1). It requires the $\{t_i\}$ to satisfy n inequalities, where n is the number of the variables in (1.1). For the case we discuss here, where $\{a_{1j}\}$ and b_1 are arbitrary integers and $\{a_{2j}\}$ and b_2 are positive integers, Theorem 3.3 requires that the $\{t_i\}$ satisfy two inequalities, and if $v_k < 0$ or $v_l > 0$, from condition (3.9) we have known that (1.1) is infeasible before using Theorem 3.3 to aggregate the system (1.1).

From (3.17) we know that after using Theorem 3.3, the aggregated equation (1.2) has positive coefficients $\{t_1a_{1j} + t_2a_{2j}\}$. Theorem 3.3 can be extended to aggregate system (1.1) with a parametric integer $b_1 \in [\underline{b_1}, \overline{b_1}]$, where $\underline{b_1}$ and $\overline{b_1}$ are given integers, to the form (1.2) with positive coefficients $\{t_1 a_{1j} + t_2 a_{2j}\}$.

Corollary 3.1. *For arbitrary integers $\{a_{1j}\}$ and parametric integer $b_1 \in [\underline{b_1}, \overline{b_1}]$, let $\{a_{2j}\}$ and b_2 be positive integers. Then* (1.1) *is equivalent to* (1.2) *if*

$$t_1 a_{1k} + t_2 a_{2k} > \max\{\overline{b_1} a_{2k} - b_2 a_{1k},\ 0\} \tag{3.19}$$

and

$$t_1 a_{1l} + t_2 a_{2l} > \max\{b_2 a_{1l} - \underline{b_1} a_{2l},\ 0\}, \tag{3.20}$$

where $\{t_i\}$ are relatively prime integers.

Proof. To any given integer $b_1^0 \in [\underline{b_1}, \overline{b_1}]$, from (3.19) and (3.20) it follows that

$$t_1 a_{1k} + t_2 a_{2k} > \max\{b_1^0 a_{2k} - b_2 a_{1k},\ 0\},$$

and

$$t_1 a_{1l} + t_2 a_{2l} > \max\{b_2 a_{1l} - b_1^0 a_{2l},\ 0\}.$$

Hence, condition (3.12) in Theorem 3.3 has been satisfied. □

As an application, Corollary 3.1 can be used to develop an approach for the knapsack problem (see Section 6).

4. The system with all upper bounded variables

In this section, let $p = n$, i.e. all of the individual variables have been given upper bounds in (1.1) and (1.2). We use the notation:

$$L_4 = w_0 - \sum_{w_j < 0} |w_j| \cdot u_j, \quad U_4 = w_0 + \sum_{w_j > 0} w_j \cdot u_j,$$

where all $j \in N$. We have

Theorem 4.1. *For arbitrary integer coefficients $\{a_{ij}\}$ and $\{b_i\}$, let $\{c_i\}$ be arbitrary given integers with $c_i \neq 0$ for at least one $i \in I$. Then the bounded system* (1.1) *is equivalent to the bounded single equation* (1.2) *if*

$$t_1 c_1 + t_2 c_2 > \max\{-L_4,\ U_4\}, \tag{4.1}$$

where $\{t_i\}$ are relatively prime integers.

Proof. Because $x_j \leqslant u_j$ for all $j \in N$, then

$$\max_{X \in T_2} S(X) = \max_{X \in T_2} \sum_{j \in N \cup \{0\}} w_j x_j \leqslant w_0 + \max_{X \in T_2} \sum_{w_j > 0} w_j x_j \leqslant w_0 + \sum_{w_j > 0} w_j \cdot u_j.$$

Similarly,

$$\min_{X\in T_2} S(X) \geqslant w_0 - \max_{X\in T_2} \sum_{w_j<0} (-w_j)x_j \geqslant w_0 - \sum_{w_j<0} |w_j| \cdot u_j.$$

Hence, L_4 and U_4 can be used as L and U in condition (2.1) of Theorem 2.1. □

There is a procedure in [6] for aggregating a 0–1 non-homogeneous system (1.1):

Corollary 2 (Elimam and Elmaghraby [6]). *For arbitrary integer coefficients $\{a_{ij}\}$, let $b_i \geqslant 0$ for $i=1,2$ with $b_i>0$ for at least one i. If the x_j are 0–1 variables, then* (1.1) *is equivalent to* (1.2) *provided*

$$t_1 b_1 + t_2 b_2 > \sum_{j\in N} |b_1 a_{2j} - b_2 a_{1j}| = \sum_{j\in N} |v_j|, \tag{4.2}$$

where $\{t_i\}$ are relatively prime positive integers.

To aggregate the 0–1 non-homogeneous system (1.1), letting $c_i = b_i$ and $u_j = 1$, from condition (4.1) we have

$$\max\{-L_4,\ U_4\} = \max\left\{\sum_{v_j<0} |v_j|,\ \sum_{v_j>0} v_j\right\} \leqslant \sum_{j\in N} |v_j|.$$

Hence, Theorem 4.1 generalizes and improves upon Corollary 2 in [6].

Theorem 4.1 is also suitable for directly aggregating a homogeneous system (1.1). Letting $c_i = \sum_{j\in N} a_{ij}$ (or 1), $b_i = 0$, $u_j = 1$, for all $i\in I$ and $j\in N$ in condition (4.1), it is easy to see that our result implies the following result (or condition (24) in [6]):

Theorem 4 (Elimam and Elmaghraby [6]). *For 0, 1 integer variables x_j, arbitrary coefficients a_{ij}, $i=1,2$ and $j\in N$; and $b_i=0$ for $i=1,2$, then* (1.1) *is equivalent to* (1.2) *provided*

$$t_1 \sum_j a_{1j} + t_2 \sum_j a_{2j} > \sum_j \left| a_{2j} \sum_k a_{1k} - a_{1j} \sum_k a_{2k} \right|. \tag{4.3}$$

To illustrate our procedure, consider the 0–1 homogeneous system, which is discussed in [6].

Example 5 (Elimam and Elmaghraby [6]). Aggregate

$$3x_1 + 8x_2 + 7x_3 + 4x_4 - 2x_5 + x_6 - 19x_7 = 0, \tag{4.4}$$

$$2x_1 + 7x_2 + 7x_3 + 3x_4 - x_5 + x_6 - 17x_7 = 0, \tag{4.5}$$

$$x_j = \{0, 1\}, \quad \text{for all } j\in N.$$

First we use Theorem 4 of [6] in this example. From condition (4.3), we have

$$2t_1 + 2t_2 > 12,$$

i.e.

$$t_1 + t_2 > 6. \tag{4.6}$$

In [6] values $t_1 = 3$ and $t_2 = 4$ are chosen and the aggregated equation is

$$17x_1 + 52x_2 + 49x_3 + 24x_4 - 10x_5 + 7x_6 - 125x_7 = 0, \tag{4.7}$$

$$x_j = \{0, 1\}, \quad \text{for all } j \in N.$$

It is also stated in [6] that, for this example, Theorem 4 in [6] appears to dominate Theorem 3 in [6].

Now, apply our Theorem 4.1 to Example 5 in [6]. Letting $c_1 = \sum_{j\in N} a_{1j} = 2$ and $c_2 = \sum_{j\in N} a_{2j} = 2$, from condition (4.1) we have $L_4 = -6$ and $U_4 = 6$, then

$$2t_1 + 2t_2 > 6,$$

i.e.

$$t_1 + t_2 > 3. \tag{4.8}$$

Putting $t_1 = 3$, we obtain $t_2 = 1$ and the aggregated equation:

$$11x_1 + 31x_2 + 28x_3 + 15x_4 - 7x_5 + 4x_6 - 74x_7 = 0, \tag{4.9}$$

$$x_j = \{0, 1\}, \quad \text{for all } j \in N,$$

whose coefficients are smaller than in (4.7) (in the sense of the absolute value).

5. The system with special conditions

In this section, we discuss the case of a general system (1.1) with some special conditions.

In [4], for the system (1.1) with arbitrary integer coefficients $\{a_{ij}\}$, $b_i \geqslant 0$ integer for $i \in I$ and the restriction $b_1 + b_2 > 0$, the following notation is used:

$$W_1 = \max_{X\in G} \sum_{v_j > 0} v_j x_j, \quad W_2 = \max_{X\in G} \sum_{v_j < 0} |v_j| x_j,$$

$$G = D \cap \left\{ X \,\middle|\, \sum_{j=1}^{n} x_j \leqslant \boldsymbol{U} \right\},$$

$$X = (x_1, x_2, \ldots, x_n) \in D = \{X \mid 0 \leqslant x_j \leqslant u_j, \text{ integer}\},$$

where u_j are given numbers, $\boldsymbol{U}$ is an upper bound on the sum $\sum_{j=1}^{n} x_j$, obtained either as a value that is imposed externally or is derived from system (1.1), and (1.1) is considered under the possible additional condition: $X \in D$. To the non-homogeneous system (1.1), there is

Theorem 2.2 (Babayev and Mardanov [4]). *For arbitrary integer coefficients* $\{a_{ij}\}$, $b_i \geqslant 0$, *integer for* $i=1,2$ *and* $b_1+b_2>0$, *system* (1.1) *and Eq.* (1.2) *are equivalent under the following assumptions:* t_1 *and* t_2 *are relatively prime positive integers, and*

$$t_1 b_1 + t_2 b_2 > \max\{W_1,\ W_2\}. \tag{5.1}$$

In the assumption of the theorem, the condition $\sum_{j\in N} x_j \leqslant U$ has been used. If U is obtained as a value that is derived from system (1.1), the final aggregated result has Eq. (1.2) and inequality $\sum_{j\in N} x_j \leqslant U$ as well.

In the following, we give a generalized result using Theorem 2.1 for the general system (1.1) having the first p individual integer variables bounded.

Notation:

$$L_5 = w_0 - \max_{X\in G_1} \sum_{w_j<0} |w_j| x_j, \quad U_5 = w_0 + \max_{X\in G_2} \sum_{w_j>0} w_j x_j,$$

$$G_1 = D \cap \left\{ X:\ \sum_{w_j<0} x_j \leqslant V_1 \right\}, \quad G_2 = D \cap \left\{ X:\ \sum_{w_j>0} x_j \leqslant V_2 \right\},$$

$$X = (x_1,\ldots,x_p,x_{p+1},\ldots,x_n) \in D,$$

$$D = \{X:\ 0 \leqslant x_j \leqslant u_j \text{ integer},\ j=1,\ldots,p;\ 0 \leqslant x_j \text{ integer},\ j=p+1,\ldots,n\},$$

where values V_1 and V_2 are derived from the aggregated equation (1.2). We give

Theorem 5.1. *For arbitrary integer coefficients* $\{a_{ij}\}$ *and* $\{b_i\}$, *let* $\{c_i\}$ *be arbitrary given integers with* $c_i \neq 0$ *for at least one* $i\in I$. *Then system* (1.1) *is equivalent to single Eq.* (1.2) *if*

$$t_1 c_1 + t_2 c_2 > \max\{-L_5,\ U_5\}, \tag{5.2}$$

where $\{t_i\}$ *are relatively prime integers.*

Since Eq. (1.2) has the solution set

$$T_2 = \{X:\ (x_1,\ldots,x_p,x_{p+1},\ldots,x_n) \text{ satisfying } (1.2)\},$$

then $T_2 \subseteq D$. At the same time, $\sum_{w_j<0} x_j \leqslant V_1$ and $\sum_{w_j>0} x_j \leqslant V_2$ are two fixed properties of Eq. (1.2). Therefore, using a similar method as in the proof of Theorem 4.1, it is easy to prove that L_5 and U_5 can be used as L and U in condition (2.1) of Theorem 2.1. The proof is omitted. Note that after using the procedure, the final aggregated result has no inequalities $\sum_{w_j<0} x_j \leqslant V_1$ and $\sum_{w_j>0} x_j \leqslant V_2$ since both of them come from (1.2) and can be considered as redundant constraints.

If $p=0$, $a_{ij}>0$ and $b_i>0$ for all $i\in I$ and $j\in N$, Theorem 3.2 in Section 3 may be regarded as a special case of the Theorem 5.1. If $p=n$, and the conditions $\sum_{w_j<0} x_j \leqslant V_1$ and $\sum_{w_j>0} x_j \leqslant V_2$ are ignored, i.e. $G_1 = G_2 = D$ are assumed, Theorem 4.1 in Section 4 may be regarded as another special case of Theorem 5.1.

Apply Theorem 5.1 to an example below, that is a slightly revised form of an example discussed in [6, 9, 13, 20].

Example 5.1. Aggregate

$$7x_1 + 9x_2 + 5x_3 + 10x_4 = 84, \quad 6x_1 + 7x_2 + 5x_3 + 9x_4 = 72, \tag{5.3}$$

$$x_1,\ x_2,\ x_3,\ x_4 \geqslant 0 \text{ integers.}$$

In this example, all the coefficients are positive integers. Assuming $c_1 = b_1 = 84$ and $c_2 = b_2 = 72$, we have $v_1 = 0$, $v_2 = -60$, $v_3 = 60$ and $v_4 = 36$. From the aggregated equation (1.2) of (5.3) and using formulae (3.5) and (3.6) in Section 3, we have

$$x_2 \leqslant \max\{\lfloor 84/9 \rfloor, \lfloor 72/7 \rfloor\} = 10 = V_1,$$

$$x_3 + x_4 \leqslant \max\{\lfloor 84/5 \rfloor, \lfloor 72/5 \rfloor\} = 16 = V_2.$$

It is easy to see that system (5.3) is equivalent to the bounded system (5.4):

$$7x_1 + 9x_2 + 5x_3 + 10x_4 = 84, \quad 6x_1 + 7x_2 + 5x_3 + 9x_4 = 72, \tag{5.4}$$

$$0 \leqslant x_1 \leqslant 12, \quad 0 \leqslant x_2 \leqslant 9, \quad 0 \leqslant x_3 \leqslant 14, \quad 0 \leqslant x_4 \leqslant 8, \quad \text{integers,}$$

where $u_1 = 12$, $u_2 = 9$, $u_3 = 14$, $u_4 = 8$, $p = n = 4$, and

$$D = \{X\colon\ 0 \leqslant x_j \leqslant u_j \text{ integer, } j \in N = \{1,2,3,4\}\}.$$

Thus,

$$G_1 = D \cap \{X\colon\ x_2 \leqslant 10\}, \quad G_2 = D \cap \{X\colon\ x_3 + x_4 \leqslant 16\}.$$

We have

$$-L_5 = \max_{X \in G_1}\{60x_2\} = 60 \cdot 9 = 540,$$

$$U_5 = \max_{X \in G_2}\{60x_3 + 36x_4\} = 60 \cdot 14 + 36 \cdot 2 = 912.$$

Hence, if the relatively prime positive integers $\{t_i\}$ satisfy condition

$$84t_1 + 72t_2 > 912,$$

i.e.

$$7t_1 + 6t_2 > 76, \tag{5.5}$$

then the bounded system (5.4), i.e. system (5.3), is always equivalent to the bounded single Eq. (1.2).

6. Conclusions

In this paper, starting with ideas given in [4, 6, 9, 13, 20], we develop some procedures based on a new technique (Theorem 2.1) for the aggregation problem. Our work generalizes and improves upon some aggregation procedures devised in [6] and the procedure provided in [13]. The aggregated result is always guaranteed to be a single linear Diophantine equation.

Our results have a number of general properties. Letting arbitrary integer parameters $\{c_i\}$, that satisfy $c_i \neq 0$ for at least one $i \in I = \{1,2\}$, be taken as given, a corresponding sufficient condition on the multipliers $\{t_i\}$ for the aggregation problem may be obtained. In [6], some aggregation criteria are provided. Hence, for aggregating a non-homogeneous system, we may choose $c_i = b_i$; for aggregating a homogeneous system, we may choose $c_i = \sum_{j \in N} a_{ij}$ or 1 for all $i \in I$. In addition, we have dealt with two special cases: on the one hand, the case in which is all of the individual variables have been given upper bounds and on the other, no upper bounds. The given procedures may also synthetically be applied to the general case wherein a subset of the individual variables have been given upper bounds.

Recently, Glover and Babayev [10] integrated some ideas expressed in [5, 11, 19], and obtained some new results for aggregating a general linear or non-linear system. They derive cases wherein individual bounds on multipliers t_1 and t_2 are required. Our basic result on the aggregation problem discussed in this paper is that t_1 and t_2 are required to satisfy a single inequality.

One of the aggregation results in [10] is used to develop a new and efficient approach to solve the inequality constrained knapsack problem, see Babayev et al. [3]. The efficiency of the presented method in [3] is mainly due to the efficiency of algorithms used to test the consistency of the resulting aggregated equation. Aggregation conditions (finding suitable multipliers t_1 and t_2 to aggregate the objective function and knapsack constraint) for the KP, that is similar to aggregation conditions for the KP given in [3], can be directly derived from Corollary 3.1 (based on Theorem 2.1) in this paper. It is easy to see that both of these aggregation conditions for the KP, which are obtained by different methods, dominate aggregation conditions for the KP given by Greenberg (see Theorem 1 in [12]).

Acknowledgements

The authors sincerely thank Professor F. Glover and Professor D.A. Babayev for their valuable papers. The authors would also like to thank the referees for their helpful comments and suggestions.

References

[1] J.M. Anthonisse, A note on equivalent systems of linear Diophantine equations, Z. Oper. Res. 17 (1973) 167–177.

[2] D.A. Babayev, F. Glover, Aggregation of nonnegative integer-valued equations, Discrete Appl. Math. 8 (1984) 125–130.

[3] D.A. Babayev, F. Glover, J. Ryan, A new knapsack solution approach by integer equivalent aggregation and consistency determination, INFORMS J. Comput. 9 (1) (1997) 43–50.

[4] D.A. Babayev, S.S. Mardanov, Sequential and simultaneous aggregation of Diophantine equations, Discrete Appl. Math. 50 (3) (1994) 209–220.
[5] G.H. Bradley, Transformation of integer programs to knapsack problems, Discrete Math. 1 (1) (1971) 29–45.
[6] A.A. Elimam, S.E. Elmaghraby, On the reduction method for the integer linear programs, II, Discrete Appl. Math. 12 (1985) 241–260.
[7] S.E. Elmaghraby, M.K. Wig, On the treatment of stock cutting problems as Diophantine programs, Operations Research Report No. 61, NC State University, Raleigh, NC, 1971.
[8] R.S. Garfinkel, G.L. Nemhauser, Integer Programming, Wiley, New York, 1972.
[9] F. Glover, New results on equivalent integer programming formulations, Math. Programming 8 (1975) 84–90.
[10] F. Glover, D.A. Babayev, New results for aggregating integer-valued equations, Ann. Oper. Res. 58 (1995) 227–242.
[11] F. Glover, R.E. Woolsey, Aggregating Diophantine equations, Z. Oper. Res. 16 (1972) 1–10.
[12] H. Greenberg, On equivalent knapsack problems, Discrete Appl. Math. 14 (1986) 263–268.
[13] H. Greenberg, A new reduction method in integer programming, Discrete Appl. Math. 21 (2) (1988) 169–172.
[14] P.L. Hammer, I.G. Rosenberg, Equivalent forms of zero-one programs, in: S.K. Zaremba (Ed.), Applications of Number Theory to Numerical Analysis, Academic Press, New York, 1972, pp. 453–463.
[15] K.E. Kendall, S. Zionts, Solving integer programming problem by aggregating constraints, Oper. Res. 25 (2) (1977) 346–351.
[16] G.D. Lin, A note on aggregating constraints in integer programming, Z. Oper. Res. 30 (1986) A79–A82.
[17] G.B. Mathews, On the partition of numbers, Proc. London Math. Soc. 28 (1897) 486–490.
[18] D.C. Onyekwelu, Computational viability of a constraint aggregation scheme for integer linear programming problems, Oper. Res. 31 (4) (1983) 795–801.
[19] M.W. Padberg, Equivalent knapsack-type formulations of bounded integer linear programs: an alternative approach, Naval Res. Logist. Quart. 19 (1972) 699–708.
[20] N. Zhu, A reduction method in integer linear programming, Math. Practice Theory (4) (1993) 66–73 (in Chinese).