A linear multistep numerical integration scheme for solving systems of ordinary differential equations with oscillatory solutions

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ABSTRACT

In [1], a set of convergent and stable two-point formulae for obtaining the numerical solution of ordinary differential equations having oscillatory solutions was formulated. The derivation of these formulae was based on a non-polynomial interpolant which required the prior analytic evaluation of the higher order derivatives of the system before proceeding to the solution.
In this paper, we present a linear multistep scheme of order four which circumvents this (often tedious) initial preparation. The necessary starting values for the integration scheme are generated by an adaptation of the variable order Gragg-Bulirsch-Stoer algorithm as formulated in [2].

1. INTRODUCTION

Linear multistep integration formulae to obtain the numerical solution of an initial value problem of the form

$$y' = f(x, y), \quad y(a) = \eta , \tag{1.1}$$

based on non-polynomial interpolants have been proposed by Lambert & Shaw [4], and Shaw [5]. It can be shown that these are particularly well suited to solving initial value problems whose solutions contain singularities.
As in [1], our numerical integration procedure will be based on the representation of the solution to problem (1.1) on a certain sub-interval by either the interpolating function

$$F_t(x) = \sum_{r=0}^{L} a_r x^r + b_t \sin(N_t x + A_t) \tag{1.2}$$

or

$$F_t(x) = \sum_{r=0}^{L} a_r x^r + b_t \sinh(N_t x + A_t), \tag{1.3}$$

where L, the order of the polynomial part of the interpolating function is a non-negative integer; b_t and $\{a_r, \ r = 0, 1, \ldots, L\}$ are real undetermined coefficients, whilst N_t and A_t are real oscillatory parameters which need to be evaluated at each step of the integration procedure. Rather than obtain the higher derivatives of $f = f(x, y)$ analytically as in [1] (which could at times be very cumbersome), we shall obtain the oscillatory parameters N_t and A_t by solving a pair of non-linear trigonometric (or hyperbolic) equations by a Newton-Raphson iteration procedure. Good initial estimates of the parameters for this iterative procedure can be obtained and the proposed scheme yields an adaptive convergent and zero-stable integration formula for oscillatory problems.

2. PRELIMINARIES

We shall consider the initial value problem (1.1) over a closed and finite interval $I : [a \leq x \leq b]$ with the assumption that the function $f = f(x, y)$ satisfies the conditions of the existence theorem as in Henrici (1962).
An integer N is chosen to define a uniform mesh-size h given as

$$h = \frac{b - a}{N} , \tag{2.1}$$

and a sequence of mesh points is then defined as

$$\{x_t : x_t = a + th, \quad t = 0, 1, \ldots, N\} . \tag{2.2}$$

If we define the sequence of sub-intervals as

$$\{I_t : x_t \leq x \leq x_{t+1}; \quad t = 0, 1, \ldots, N-1\} , \tag{2.3}$$

then the interval $I : [a \leq x \leq b]$ can be expressed as

$$I = \bigcup_{t=0}^{N-1} I_t . \tag{2.4}$$

We shall denote by k, the step-number of the multistep integration formula and any necessary additional starting values (i.e., $y_1, y_2, \ldots, y_{k-1}$) are obtained with the variable order Gragg-Bulirsch-Stoer algorithm as discussed in [2].

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Let us assume that the numerical solutions $y_t, y_{t+1}, \ldots, y_{t+k-1}$ have been obtained respectively at the points $x_t, x_{t+1}, \ldots, x_{t+k-1}$ and the numerical solution y_{t+k} at the points $x = x_{t+k}$ is sought.
Let

$$I^* = \bigcup_{i=0}^{k-1} I_{t+i}\, ; \quad 0 \leqslant t \leqslant N-k\,, \tag{2.5}$$

denote the union of the sub-intervals $I_t, I_{t+1}, \ldots, I_{t+k-1}$ defined by equations (2.2) and (2.3).
Over the interval I^*, the solution to the initial value problem (1.1) is represented by the interpolating function (1.2).

3. DETERMINATION OF THE OSCILLATORY PARAMETERS N_T AND A_T

Let f_{t+j} denote the value of the function $f=f(x, y)$ at the point $(x=x_{t+j}, y=y_{t+j})$. In an attempt to eliminate the undetermined coefficients in equation (1.2), the following constraints are imposed on the interpolant (1.2) :

(a) the interpolating function should pass through the points

$\{x_{t+j}, y_{t+j},\ j = 0, 1, \ldots, k\}$ i.e.,

$$F_t(x_{t+j}) = y_{t+j}\,, \quad j = 0, 1, \ldots, k. \tag{3.1}$$

(b) the first derivative of the interpolant must also satisfy the differential equation (1.1) at those points specified in equation (3.1), that is

$$\left.\frac{dF_t(x)}{dx}\right|_{\substack{x = x_{t+j} \\ y = y_{t+j}}} = f_{t+j}\, ; \; j = 0, 1, \ldots, k. \tag{3.2}$$

Equations (3.1) and (3.2) respectively imply that the relationships

$$\sum_{i=0}^{L} a_i x_{t+j}^{i} + b_t \sin(N_t x_{t+j} + A_t) = y_{t+j}\, ; \quad j = 0, 1, \ldots, k, \tag{3.3}$$

and

$$\sum_{i=0}^{L} i a_i x_{t+j}^{i-1} + b_t N_t \cos(N_t x_{t+j} + A_j) = f_{t+j}\, ; \quad j = 0, 1, \ldots, k\,, \tag{3.4}$$

hold.
As the polynomial part of the equation (3.4) is of degree at most $L-1$, the application of the L^{th} forward difference operator denoted by Δ^L to both sides of the equation (3.4) will annihilate the polynomial part. This gives the relationships,

$$b_t N_t \Delta^L \cos(N_t x_{t+j} + A_j) = \Delta^L f_{t+j}$$

The last equation implies that

$$b_t = \frac{\Delta^L f_{t+j}}{N_t \Delta^L \cos(N_t x_{t+j} + A_t)}\,, \; j = 0, 1, \ldots, k-1. \tag{3.5}$$

In particular, by setting $j = 0, 1, 2$, in equation (3.5), the undetermined coefficient b_t can be obtained either as

$$b_t = \frac{\Delta^L f_t}{N_t \Delta^L \cos(N_t x_t + A_t)}\,, \tag{3.6}$$

$$b_t = \frac{\Delta^L f_{t+1}}{N_t \Delta^L \cos(N_t x_{t+1} + A_t)}\,, \tag{3.7}$$

or

$$b_t = \frac{\Delta^L f_{t+2}}{N_t \Delta^L \cos(N_t x_{t+2} + A_t)} \tag{3.8}$$

By pairing up the equations (3.6) to (3.8), we can obtain the following equations

$$R_1(N_t, A_t) = \Delta^L \cos(N_t x_t + A_t)\, \Delta^L f_{t+1} - \Delta^L \cos(N_t x_{t+1} + A_t)\, \Delta^L f_t = 0\,, \tag{3.9}$$

$$R_2(N_t, A_t) = \Delta^L \cos(N_t x_{t+1} + A_t)\, \Delta^L f_{t+2} - \Delta^L \cos(N_t x_{t+2} + A_t)\, \Delta^L f_{t+1} = 0\,, \tag{3.10}$$

and finally,

$$R_3(N_t, A_t) = \Delta^L \cos(N_t x_{t+2} + A_t)\, \Delta^L f_t - \Delta^L \cos(N_t x_t + A_t)\, \Delta^L f_{t+2} = 0\,. \tag{3.11}$$

Any suitable pairs of the equations (3.9) to (3.11) can be solved for the parameters N_t and A_t.
We shall now give a detailed discussion of the determination of the parameters N_t and A_t by using equations (3.9) and (3.10) for the case when the polynomial part of the interpolant (1.2) is of degree one (i.e. $L=1$). We use the Newton iteration scheme to solve the relevant equations.
Equation (3.9) now becomes,

$$R_1(N_t, A_t) = (f_{t+2} - f_{t+1})\,[\cos(N_t x_{t+1} + A_t) - \cos(N_t x_t + A_t)] - (f_{t+1} - f_t)[\cos(N_t x_{t+2} + A_t) - \cos(N_t x_{t+1} + A_t)] = 0\,, \tag{3.12}$$

whilst equation (3.10) yields,

$$R_2(N_t, A_t) = (f_{t+3} - f_{t+2})[\cos(N_t x_{t+2} + A_t) - \cos(N_t x_{t+1} + A_t)] - (f_{t+2} - f_{t+1})[\cos(N_t x_{t+3} + A_t) - \cos(N_t x_{t+2} + A_t)] = 0\,. \tag{3.13}$$

With the aim of obtaining the approximate roots N_t^* and A_t^* of the functions $R_1(N_t, A_t)$ and $R_2(N_t, A_t)$ we denote the partial derivatives of $R_1(N_t, A_t)$ and $R_2(N_t, A_t)$ with respect to the parameters N_t and A_t as follows,

$$R_{1,N_t} = \frac{\partial R_1(N_t, A_t)}{\partial N_t} \; ; \; R_{2,N_t} = \frac{\partial R_2(N_t, A_t)}{\partial N_t}$$

$$R_{1,A_t} = \frac{\partial R_1(N_t, A_t)}{\partial A_t} \; ; \; R_{2,A_t} = \frac{\partial R_2(N_t, A_t)}{\partial A_t} \qquad (3.14)$$

By replacing the higher order derivatives of $f(x, y)$ by their equivalent forward differences, i.e.,

$$f^{(s)}(x_0, y_0) \approx \frac{1}{h^s} \sum_{r=0}^{s+1} (-1)^r \binom{s+1}{r} y_{s-r+1}, \qquad (3.15)$$

the initial estimates $N_0^{[0]}$, $A_0^{[0]}$ at x are obtained from [1] by either the equations,

$$N_0^{[0]} = \left[-\frac{\Delta^4 y_0}{\Delta^2 y_0} \right]^{1/2} \qquad (3.16)$$

and

$$A_0^{[0]} = \cot^{-1} \left[\frac{\Delta^3 y_0}{N_0^{[0]} \Delta^2 y_0} \right] - N_0^{[0]} x_0 ; \qquad (3.17)$$

or by the following equations,

$$N_0^{[0]} = \left[-\frac{\Delta^5 y_0}{\Delta^3 y_0} \right]^{1/2}, \qquad (3.18)$$

and

$$A_0^{[0]} = \tan^{-1} \left[-\frac{\Delta^4 y_0}{N_0^{[0]} \Delta^3 y_0} \right] - N_0^{[0]} x_0 . \qquad (3.19)$$

At the i^{th} iteration of the Newton Raphson scheme, the new estimates $N_t^{[i+1]}$, $A_t^{[i+1]}$ of the oscillatory parameters N_t^*, A_t^* are given by

$$N_t^{[i+1]} = N_t^{[i]} + \delta N_t^{[i]},$$

$$A_t^{[i+1]} = A_t^{[i]} + \delta A_t^{[i]}, \qquad (3.20)$$

where the correction terms $\delta N_t^{[i]}$, $\delta A_t^{[i]}$ are given by

$$\begin{bmatrix} \delta N_t^{[i]} \\ \delta A_t^{[i]} \end{bmatrix} = J^{-1} \begin{bmatrix} R_{2,A_t}^{[i]} & -R_{2,N_t}^{[i]} \\ -R_{1,A_t}^{[i]} & R_{1,N_t}^{[i]} \end{bmatrix} \begin{bmatrix} R_1^{[i]} \\ R_2^{[i]} \end{bmatrix} \qquad (3.21)$$

and J denotes the determinant of the Jacobian of the functions $R_1(N_t, A_t)$ and $R_2(N_t, A_t)$ at $N_t = N_t^{[i]}$, $A_t = A_t^{[i]}$.

We denote the terminal values obtained by the iterative Newton method as N_t^* and A_t^* and are given by

$$N_t^* = \lim_{i \to \infty} N_t^{[i]}$$

and

$$A_t^* = \lim_{i \to \infty} A_t^{[i]} .$$

4. DERIVATIVE OF THE INTEGRATION FORMULAE

To derive the required integration formulae, it is now necessary to eliminate the remaining undetermined coefficients $\{a_r, \; r = 0, 1, \ldots, L\}$ in the interpolating function (1.2). We shall achieve this objective with the view of obtaining a consistent and zero-stable (and hence convergent) linear multistep scheme.

We introduce a function z_{t+i} defined by :

$$z_{t+i} = y_{t+i} - b_t \sin(N_t^* x_{t+i} + A_t^*), \qquad (4.1)$$

whose derivative z'_{t+i} with respect to x is given by :

$$z'_{t+i} = f_{t+i} - N_t^* b_t \cos(N_t^* x_{t+i} + A_t^*) . \qquad (4.2)$$

The application of the equations (4.1) and (4.2) in equation (1.2) yields the results,

$$z_{t+j} = \sum_{i=0}^{L} a_i x_{t+j}^i , \qquad (4.3)$$

and

$$z'_{t+j} = \sum_{i=0}^{L} i a_i x_{t+j}^{i-1} . \qquad (4.4)$$

We now introduce the consistency parameters $\{\alpha_j, \beta_j; \; j = 0, 1, \ldots, k\}$ such that α_0, β_0 are not both zero and as we are only interested in an explicit integration scheme, β_k is allowed to vanish.

For $j= 0, 1, \ldots, k$; equation (4.3) is multiplied by α_j and equation (4.4) is multiplied by $-h\beta_j$. We now add columnwise to obtain the result

$$\sum_{j=0}^{k} \alpha_j z_{t+j} - h \sum_{j=0}^{k} \beta_j z'_{t+j} = \sum_{i=0}^{L} a_i \left[\sum_{j=0}^{k} \alpha_j x_{t+j}^i - ih \sum_{j=0}^{k} \beta_j x_{t+j}^{i-1} \right] \qquad (4.5)$$

Since we are interested in an Adam's type formula, we assign the following values to the consistency parameters:

$$\alpha_0 = 0, \quad \alpha_1 = -1, \quad \alpha_k = +1; \text{ and}$$

$$\alpha_j = 0 \quad \text{for} \quad j = 2, 3, \ldots, k-1. \qquad (4.6)$$

This choice of parameters gives

$$\sum_{j=0}^{k} \alpha_j = 0 \qquad (4.7)$$

which is the first consistency condition for a general linear multistep scheme.
The application of equation (4.6) in equation (4.5) gives

$$\sum_{j=0}^{k} \alpha_j z_{t+j} - h \sum_{j=0}^{k-1} \beta_j z'_{t+j}$$

$$= \sum_{i=1}^{L} a_i \left[\sum_{j=0}^{k} \alpha_j x^i_{t+j} - ih \sum_{j=0}^{k-1} \beta_j x^{i-1}_{t+j} \right]. \tag{4.8}$$

Now it can be shown that there is no loss of generality in assigning the following values,

$$x_t = 0 \quad \text{and} \quad h = 1 , \tag{4.9}$$

after allowing the coefficients of a_i to vanish in equation (4.8) (for $i = 1, 2, \ldots, L$). This gives the result,

$$\sum_{j=0}^{k} j^i \alpha_j - i \sum_{j=0}^{k-1} j^{i-1} \beta_j = 0 . \tag{4.10}$$

By choosing,

$$L = k - 3 , \tag{4.11}$$

the equation (4.10) will give a set of k−3 equations in k unknowns $\beta_0, \beta_1, \ldots, \beta_{k-1}$ and thus allowing three degrees of freedom.
If we now choose $L = 1$ and hence $k = 4$ and set $i = 2, 3, 4$, in equation (4.10) we obtain the following values for β_1, β_2, and β_3, i.e.,

$$\beta_1 = 1.875 ,$$
$$\beta_2 = -1.125 ,$$
$$\beta_3 = 2.625 , \tag{4.12}$$

and hence for β_0 the result −0.375.

The above procedure makes equation (4.8) a linear multistep formula and by allowing the coefficient of a_1 to vanish and then apply equations (4.3) and (4.4), we obtain the integration formula as

$$\sum_{j=0}^{k} \alpha_j y_{t+j} - h \sum_{j=0}^{k-1} \beta_j f_{t+j}$$

$$= \sum_{j=0}^{k} \alpha_j [y_{t+j} - b_t \sin (N_t^* x_{t+j} + A_t^*)]$$
$$-h \sum_{j=0}^{k} \beta_j [f_{t+j} - N_t^* b_t \cos (N_t^* x_{t+j} + A_t^*)] . \tag{4.13}$$

Finally, from equations (4.8), the undetermined coefficient b_t is

$$b_t = \frac{f_{t+3} - f_{t+2}}{N_t^* [\cos (N_t^* x_{t+3} + A_t^*) - \cos (N_t^* x_{t+2} + A_t^*)]} \tag{4.14}$$

and by using equation (4.14) in equation (4.13) and re-arranging terms, we obtain the final integration formula as :

$$y_{t+k} = -\sum_{j=0}^{k-1} \alpha_j y_{t+j} + h \sum_{j=0}^{k-1} \beta_j f_{t+j}$$

$$+ \frac{f_{t+3} - f_{t+2}}{N_t^* [\cos (N_t^* x_{t+3} + A_t^*) - \cos (N_t^* x_{t+2} + A_t^*)]}$$

$$\times \left[\sum_{j=0}^{k} \alpha_j \sin (N_t^* x_{t+j} + A_t^*) - N_t^* h \sum_{j=0}^{k-1} \beta_j \cos (N_t^* x_{t+j} + A_t^*) \right] \tag{4.15}$$

The consistency and zero-stability of equation (4.15) was established in [3].
The parameters N_t^*, A_t^* are used as the initial estimates over the next interval I_{t+1} of the integration procedure, i.e.,

$$N_{t+1}^{[0]} = N_t^*$$

and

$$A_{t+1}^{[0]} = A_t^* , \qquad t = 0, 1, \ldots, N-k. \tag{4.16}$$

In the eventuality that the Jacobian of the functions $R_1(N_t, A_t)$ and $R_2(N_t, A_t)$ is singular, a new pair of equations could be chosen from equations (3.9) to (3.11). However, if all possible choices of the pairs yield unsatisfactory results, we can switch to the alternative interpolant (1.3). This technique is fully discussed in [4].

5. APPLICATIONS AND NUMERICAL RESULTS

Example 5.1

We first consider the initial value problem of Schweitzer (1974) given as :

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} \sin x \\ 2(\cos x - \sin x) \end{bmatrix}$$

with initial values (5.1)

$$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

over the interval $0 \leqslant x \leqslant \pi$ whose theoretical solution in the specified range is given by,

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} \sin x \\ \cos x \end{bmatrix}$$

The numerical solution was obtained with a uniform mesh-size of $h = \frac{\pi}{20}$. The initial estimates of the oscillatory parameters are obtained as

Table 5.1a.

Time Step t	No. of Newton Iterations	Abscissae x_t	Oscillatory parameters $N^*_{t,1}$	$A^*_{t,1}$	Computed soln. $y_{t,1}$	Truncation Error $10^7.T_{t+1,1}$
0	6	0.00000000	1.0000019	0.00000003	0.00000000	0.00000
1	2	0.15707963	1.0000053	-0.00000003	0.15643446	0.04219
2	2	0.31415926	0.9999848	0.00000130	0.30901699	0.09763
3	2	0.47123889	1.0000195	-0.00000393	0.45399050	0.03936
4	2	0.62831852	0.9999779	0.00000785	0.58778524	0.02126
5	2	0.78539815	1.0000246	-0.00001432	0.70710678	0.03520
6	2	0.94247778	0.9999699	0.00002432	0.80901698	0.04340
7	2	1.09955741	1.0000356	-0.00003888	0.89100653	0.05248
8	2	1.25663704	0.9999592	0.00005722	0.95105650	0.06660
9	2	1.41371667	1.0000398	-0.00006932	0.98768835	0.07632
10	2	1.57079630	0.9999651	0.00007132	0.99999998	0.09684
11	2	1.72787593	1.0000469	-0.00010703	0.98768837	0.10978
12	2	1.88495556	0.9999144	0.00021732	0.95105650	0.11948
13	2	2.04303519	1.0001499	-0.00041286	0.89100655	0.25810
14	2	2.19911482	0.9997400	0.00075868	0.80901700	0.11045
15	3	2.35619445	1.0004633	-0.00140147	0.70710683	0.14257
16	3	2.51327408	0.9991316	0.00269235	0.58778521	0.48340
17	-	2.67035371	-	-	0.45399064	0.69677
18	-	2.82743334	-	-	0.30901689	1.13746
19	-	2.98451297	-	-	0.15643467	1.13681
20	-	3.14159265	-	-	0.00000000	1.96611

Table 5.1b.

Time Step t	No. of Newton Iterations	Abscissae x_t	Oscillatory parameters $N^*_{t,2}$	$A^*_{t,2}$	Computed soln. $y_{t,2}$	Truncation Error $10^7.T_{t+1,2}$
0	4	0.00000000	0.9999996	1.5707963	1.00000000	0.00000
1	2	0.15707963	0.9999962	1.5707984	0.98768834	0.09396
2	2	0.31415926	1.0000151	1.5707818	0.95105651	0.08331
3	2	0.47123889	0.9999735	1.5708279	0.89100652	0.07095
4	2	0.62831852	1.0000408	1.5707412	0.80901700	0.01963
5	2	0.78539815	0.9999343	1.5708924	0.70710678	0.06018
6	2	0.94247778	1.0001147	1.5706209	0.58778527	0.05531
7	2	1.09955741	0.9997664	1.5711613	0.45399050	0.10031
8	2	1.25663704	1.0008089	1.5695258	0.30901703	0.12626
9	2	1.41371667	1.0007450	1.5696261	0.15643447	0.19219
10	3	1.57079630	0.9997980	1.5711155	0.00000005	0.23836
11	2	1.72787593	1.0001824	1.5705043	-0.15643446	0.19378
12	2	1.88495556	0.9997519	1.5712085	-0.30901698	0.11378
13	3	2.04203519	0.0003081	1.5702510	-0.43599052	0.38254
14	3	2.19911482	0.9996277	1.5715125	-0.58778516	0.38153
15	3	2.35619445	1.0004596	1.5698189	-0.70710686	0.62036
16	3	2.51327408	0.9994182	1.5721769	-0.80901686	0.62575
17	-	2.67035371	-	-	-0.89100667	0.88692
18	-	2.82743334	-	-	-0.95105628	1.11933
19	-	2.98451297	-	-	-0.98768865	1.61092
20	-	3.14159265	-	-	-1.00000000	2.10282

Table 5.2a.

Time Step t	No. of Newton Iterations	Abscissae x_t	Oscillatory parameters $N^*_{t,1}$	$A^*_{t,1}$	Computed Soln. $y_{t,1}$	Truncation Error $10^6.T_{t+1,1}$
0	4	7.38905610	0.4356458	-1.5280714	1.00000000	0.00000
1	6	7.48905610	0.4119962	-1.3370577	1.04132752	0.00418
2	9	7.58905610	0.3998252	-1.2359248	1.08028669	0.00527
3	9	7.68905610	0.3700663	-0.9810982	1.11689773	0.00623
4	13	7.78905610	0.3517235	-0.8190168	1.15118325	0.06750
5	16	7.88905610	0.3426462	-0.7366731	1.18317107	0.00196
6	8	7.98905610	0.3113717	-0.4442881	1.21289184	0.03444
7	15	8.08905610	0.2944521	-0.2809082	1.24037929	0.58150
8	3	8.18905610	0.2883406	-0.2205216	1.26566957	0.04245
9	9	8.28905610	0.2531217	0.1379613	1.28880209	0.03104
10	4	8.38905610	0.2371443	0.3060069	1.30981705	0.01801
11	11	8.48905610	0.2341942	0.3376040	1.32875693	0.11055
12	8	8.58905610	0.1905928	0.8182852	1.34566660	0.00710
13	-	8.68905610	-	-	1.36059045	0.03971
14	-	8.78905610	-	-	1.37357500	0.19191
15	-	8.88905610	-	-	1.38466850	0.02773
16	-	8.98905510	-	-	1.39391758	0.10814

Initial oscillatory parameters = $N^{[0]}_{0,1}$ = 0.45730571, $A^{[0]}_{0,1}$ = -1.68002638,

Table 5.2b.

Time Step t	No. of Newton Iterations	Abscissae x_t	Oscillatory parameters $N^*_{t,2}$	$A^*_{t,2}$	Computed Soln. $y_{t,2}$	Truncation Error $10^6.T_{t+1,2}$
0	6	7.38905610	1.5760359	-11.2858253	0.42516833	0.00000
1	5	7.48905610	1.3724154	-9.8070483	0.40140655	0.00329
2	5	7.58905610	1.2262821	-8.7393236	0.37781517	0.00329
3	5	7.68905610	1.1109863	-7.8907389	0.35443893	0.00329
4	5	7.78905610	1.0175952	-7.1975520	0.33131763	0.36287
5	4	7.88905610	0.9436730	-6.6435050	0.30848809	0.26902
6	4	7.98905610	0.8771144	-6.1393280	0.28598162	0.21651
7	4	8.08905610	0.8182216	-5.6876621	0.26382606	0.55344
8	4	8.18905610	0.7721938	-5.3301387	0.24204741	0.41530
9	4	8.28905610	0.7261200	-4.9671397	0.22066680	0.34360
10	4	8.38905610	0.6827084	-4.6200680	0.19970275	0.68269
11	10	8.48905610	0.6514154	-4.3655428	0.17917270	0.50778
12	4	8.58905610	0.6159374	-4.0726420	0.15908980	0.42887
13	-	8.68905610	-	-	0.13946534	0.77925
14	-	8.78905610	-	-	0.12031009	0.56714
15	-	8.88905610	-	-	0.10163100	0.48782
16	-	8.98905610	-	-	0.08343378	0.85443

Initial oscillatory parameters = $N^{[0]}_{0,2}$ = 1.96633719, $A^{[0]}_{0,2}$ = -14.04732875

$N^{[0]}_{0,1} = 1.41197420$

$A^{[0]}_{0,1} = 0.11112352$

and

$N^{[0]}_{0,2} = 0.99505019$

$A^{[0]}_{0,2} = 1.68870386$

Details of the numerical results are given in tables (5.1a) and 5.1b).

Example 5.2

We also consider the example of Amdursky and Ziv (1974). The initial value problem is given by the equations,

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(\beta^2/x^2) & -(1/x) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} , \tag{5.2}$$

where β is a real constant. The general solution to this system of differential equations is

$y_1(x) = C \sin (\beta \log x) + D \cos (\beta \log x)$,

$y_2(x) = \beta\,[C \cos (\beta \log x) - D \sin (\beta \log x)]\,/x$,

for $x > 0$, where C and D are arbitrary constants. The numerical solution to problem (5.2) was obtained in the interval $e^2 \leqslant x \leqslant 9$, where $e = 2.7182818$ with a uniform mesh-size of $h = 0.1$ for the values $C = D = 1$ and $\beta = \pi$. This gives the initial conditions as

$y_1(e^2) = 1$,

and

$y_2(e^2) = \frac{\pi}{e^2}$.

Details of the numerical results are given in tables (5.2a) and (5.2b).

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