Rectifiable sets and coarea formula for metric-valued mappings

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Abstract

We study Lipschitz mappings defined on an $H^n$-rectifiable metric space with values in an arbitrary metric space. We find necessary and sufficient conditions on the image and the preimage of a mapping for the validity of the coarea formula. As a consequence, we prove the coarea formula for some classes of mappings with $H^k$-$\sigma$-finite image. We also obtain a metric analog of the Implicit Function Theorem. All these results are extended to large classes of mappings with values in a metric space, including Sobolev mappings and BV-mappings.

Keywords: Rectifiable metric space; Arbitrary metric space; Metric differential; Coarea formula

1. Introduction

It is well known that, for $C^1$-mappings $\varphi: \mathbb{R}^n \to \mathbb{R}^k$, $n \geq k$, the Implicit Function Theorem and the coarea formula

$$\int_U \sqrt{\det(D\varphi(x)D\varphi^*(x))} \, dx = \int_{\mathbb{R}^k} dz \int_{\varphi^{-1}(z)} dH^{n-k}(u),$$

(1.1)

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where \( U \subset \mathbb{R}^n \) is an open set, hold.

This formula was first stated in 1950 by A.S. Kronrod [18] in a special case of \( \varphi : \mathbb{R}^2 \to \mathbb{R} \). Later, in 1959, H. Federer [7] generalized it to Lipschitz mappings of Riemannian manifolds, and in 1969 this result was extended to Lipschitz mappings of rectifiable sets of Euclidean spaces [8]. Results on the coarea formula can also be found in [6,9,19]. In 1978, M. Ohtsuka proved the coarea formula in the case of \( \varphi \in \text{Lip}(\mathbb{R}^n, \mathbb{R}^m), n, m \geq k \), when the image \( \varphi(\mathbb{R}^n) \) is \( \mathcal{H}^k \)-finite [24]. An infinite-dimensional analog of the coarea formula was proved by H. Airault and P. Malliavin in 1988 [1] for the case of Wiener spaces. This result can be found in the book [20].

The generalization of the Implicit Function Theorem to Lipschitz mappings asserts [8] (see also [11]) that the preimage \( \varphi^{-1}(z) \) of \( \mathcal{H}^k \)-almost every \( z \in \mathbb{R}^k \) is an \( \mathcal{H}^n-k \)-rectifiable set. Note that the statement about the preimage of a point can be derived from the coarea formula for Lipschitz mappings. Recall that a subset \( A \subset X \) of a metric space \( X \) is \( \mathcal{H}^l \)-rectifiable if there exists an at most countable collection of Lipschitz mappings \( \alpha_i \) defined on measurable sets \( A_i \subset \mathbb{R}^l \) such that \( \mathcal{H}^l(A \setminus \bigcup_{i \in \mathbb{N}} \alpha_i(A_i)) = 0 \).

In 2000–2003, some coarea properties and level sets of Sobolev mappings were studied in [11,21–23].

In 2000, L. Ambrosio and B. Kirchheim [3] proved an analog of coarea formula for Lipschitz mappings defined on \( \mathcal{H}^m \)-rectifiable metric space with values in \( \mathbb{R}^k \). The main contribution of these authors is the linearization of a pure metric problem.

Using the methods of [28], in [15] we prove the coarea formula under different assumptions, namely, when a Lipschitz mapping \( \varphi \) is defined on a measurable set \( E \subset \mathbb{R}^n \) and takes values in an \( \mathcal{H}^k \)-rectifiable metric space \( X, n \geq k \),

\[
\int_E f(x)J_k(MD(\varphi, x)) \, dx = \int_X d\mathcal{H}^k(z) \int_{\varphi^{-1}(z)} f(u) \, d\mathcal{H}^n-k(u). 
\]

(1.2)

(Here \( f : E \to \mathbb{E} \), where \( \mathbb{E} \) is an arbitrary Banach space, is such that the mapping \( f(x) \cdot J_k(MD(\varphi, x)) \) is integrable.) In the same paper, we also generalize the coarea formula to the case of a Lipschitz mapping \( \varphi \) defined on an \( \mathcal{H}^m \)-rectifiable metric space \( Y \) with values in an \( \mathcal{H}^k \)-rectifiable metric space \( X, n \geq k \). Thus, the result of [15] covers the result of [3].

In (1.2), the geometrical sense of the \( m \)-coarea factor \( J_k(MD(\varphi, x)) \) is the same as in the Euclidean case, i.e., in the case of an open set \( E \),

\[
J_k(MD(\varphi, x)) = \lim_{r \to 0} \frac{\mathcal{H}^k(\varphi(B_k(x, r)))}{\omega_k r^k},
\]

where \( B_k(x, r) \) is a \( k \)-dimensional ball in \( x + (\ker(MD(\varphi, x)))^\perp \) and \( \omega_k \) denotes the volume of a unit ball in \( \mathbb{R}^k \). In the case of an arbitrary measurable set \( E \), we approximate \( (\ker(MD(\varphi, x)))^\perp \) with \( k \)-dimensional planes. For the analytic description of the \( m \)-coarea factor, we need the concepts of metric differentiability and metric differential.

**Definition 1.1.** (See [17].) Let \( E \subset \mathbb{R}^n \) be a measurable set and let \( (X, d_X) \) be a metric space. A mapping \( \varphi : E \to (X, d_X) \) is **metrically differentiable**, or **m-differentiable**, at a point \( x \in E \) if there exists a seminorm \( L(x) \) on \( \mathbb{R}^n \) such that, in some neighborhood of \( x \),

\[
d_X(\varphi(z), \varphi(y)) - L(x)(z - y) = o(|z - x| + |y - x|)
\]
as \( z, y \to x \), where \( z, y \in E \). The seminorm \( L(x) \) is called the metric differential, or the \( m \)-differential of the mapping \( \varphi \) at the point \( x \) and is denoted by \( MD(\varphi, x) \).

Notice that in (1.2) the dimension of a kernel of the \( m \)-differential is at least \( n - k \) on \( E \) almost everywhere. The \( m \)-coarea factor is defined in [12,15] as

\[
J_k(MD(\varphi, x)) = \omega_k k \left( \int_{S^{n-1} \cap (\ker(MD(\varphi, x)))^\perp} [MD(\varphi, x)(u)]^{-k} d\mathcal{H}^{k-1}(u) \right)^{-1},
\]

where \( S^{n-1} \) is a unit sphere in \( \mathbb{R}^n \).

Assume that (1.2) holds for Lipschitz mappings \( \varphi : E \to \mathbb{X} \), where \( \mathbb{X} \) is an arbitrary metric space. The question arises of the geometrical properties of the metric space \( \mathbb{X} \) caused by the validity of (1.2). In this paper, we find necessary and sufficient conditions on the image and the preimage of Lipschitz mappings under which (1.2) holds. We also prove a metric analog of the Implicit Function Theorem independently of the coarea formula.

It is known [17] (see also [13] for a new proof) that a Lipschitz mapping \( \varphi : E \to \mathbb{X} \) is \( m \)-differentiable on \( E \) almost everywhere. We consider Lipschitz mappings defined on a measurable set \( E \subset \mathbb{R}^n \) that take values in an arbitrary metric space \( \mathbb{X} \) and are such that \( \dim \ker(MD(\varphi, x)) \geq n - k \) for \( \mathcal{H}^n \)-almost all \( x \in E \). Put \( Z = \{ x \in E : \dim \ker(MD(\varphi, x)) > n - k \} \). These requirements on the Lipschitz mapping \( \varphi \) are minimal, since, first, the definition of the \( m \)-coarea factor (1.3) implies that the dimension of the orthogonal complement of the kernel of the \( m \)-differential should not exceed \( k \) on \( E \) almost everywhere, and, second, the set of \( \mathcal{H}^n \)-measure zero does not influence the integrals in the coarea formula (see Lemma 3.8 below).

Note that from Lemma 3.7 it follows that the \( \mathcal{H}^k \)-\( \sigma \)-finiteness of \( \varphi(E) \) implies \( \dim \ker(MD(\varphi, x)) \geq n - k \) \( \mathcal{H}^n \)-almost everywhere. It means that the class of mappings studied in our paper includes that considered in [24].

In this paper, we denote the subsets of \( \mathbb{R}^n \) of the definition of an \( \mathcal{H}^n \)-rectifiable metric space \( \mathbb{Y} \) by \( B_m \), and the corresponding Lipschitz mappings, by \( \beta_m \), \( m \in \mathbb{N} \). Designate the set \( \{ x \in B_m : \dim \ker(MD(\varphi \circ \beta_m, x)) > n - k \} \) as \( Z_m \), \( m \in \mathbb{N} \), and denote the set \( \bigcup_{m \in \mathbb{N}} \beta_m(Z_m) \) by \( Z \).

The following three theorems constitute the main result of the paper.

**Theorem 1.2 (The Preimage-of-a-Point Theorem).** Let \( (\mathbb{Y}, d_\mathbb{Y}) \) be an \( \mathcal{H}^n \)-rectifiable metric space and let \( \mathbb{X} \) be an arbitrary metric space. Suppose that \( \varphi \in \text{Lip}(\mathbb{Y}, \mathbb{X}) \) and \( \dim \ker(MD(\varphi \circ \beta_m, y)) \geq n - k \) for \( \mathcal{H}^n \)-almost all \( y \in B_m \), \( m \in \mathbb{N} \). Then there exists a set \( \Sigma \subset \mathbb{Y} \) such that \( \mathcal{H}^n(\Sigma) = 0 \) and \( \varphi^{-1}(z) \setminus (Z \cup \Sigma) \) is a subset of the union of countably many images of measurable subsets of \( \mathbb{R}^{n-k} \) under Lipschitz mappings, in particular, it is an \( \mathcal{H}^{n-k} \)-rectifiable metric space for all \( z \in \mathbb{X} \).

**Theorem 1.3 (The Coarea Formula Validity Criterion).** Let \( (\mathbb{Y}, d_\mathbb{Y}) \) be an \( \mathcal{H}^n \)-rectifiable metric space, let \( \mathbb{X} \) be an arbitrary metric space, and let a mapping \( \varphi \in \text{Lip}(\mathbb{Y}, \mathbb{X}) \). Suppose also that \( \dim \ker(MD(\varphi \circ \beta_m, y)) \geq n - k \) for \( \mathcal{H}^n \)-almost all \( y \in B_m \), \( m \in \mathbb{N} \). Then the following are equivalent:
1. The coarea formula
\[ \int_Y f(x) J_k(\varphi, x) \, dx = \int_X dH^k(z) \int_{\varphi^{-1}(z)} f(u) \, dH^{n-k}(u) \]
holds for every function \( f : Y \to \mathbb{E} \) such that \( f(x) J_k(\varphi, x) \) is integrable, where \( \mathbb{E} \) is an arbitrary Banach space.

2. The coarea formula holds for any \( f(x) = \chi_A(x) \), where \( A \subset Y \) is such that \( A \cap (Y \setminus Z) \) is measurable.

3. \( \varphi(Y) = X_\Sigma \cup X_Z \cup X_0 \), where \( X_\Sigma = \varphi(\Sigma) \), \( H^n(\Sigma) = 0 \), \( X_Z = \varphi(Z) \); moreover,
\[ \int \chi_{\varphi^{-1}(z) \cap Z} \, dH^{n-k}(u) = 0 \]
for \( H^k \)-almost all \( z \in X \), and \( X_0 \) is a subset of the union of countably many images of measurable subsets of \( \mathbb{R}^k \) under Lipschitz mappings, in particular, \( X_0 \) is an \( H^k \)-rectifiable metric space.

4. \( \varphi(Y) = X_\Sigma \cup X_Z \cup X_0 \), where the sets \( X_\Sigma \) and \( X_Z \) are the same as in 3, and \( X_0 \) is an arbitrary \( H^k \)-rectifiable metric space.

5. In the case of \( Y = E \subset \mathbb{R}^n \), where \( E \) is measurable, the set \( E \) possesses the following properties:
   
   a) for each intersection \( E \cap B(0,s), s \in \mathbb{R}, \) and each \( \varepsilon > 0 \), there exists a measurable set \( \Sigma_\varepsilon \subset E \cap B(0,s) \) and a collection of compact sets \( \{ K_i \}_{i \in \mathbb{N}} \) such that \( H^n(\Sigma_\varepsilon) < \varepsilon \), \( (E \cap B(0,s)) \setminus \Sigma_\varepsilon \subset \bigcup_{i \in \mathbb{N}} K_i \), and the equality
   \[ J_k(\text{MD}(\varphi|_{K_i}, x)) = \lim_{r \to 0} \frac{H^k(\varphi|_{K_i}(B(x, r) \cap (E \setminus (\Sigma_\varepsilon \cup Z))))}{H^k(B_k(x, r))} \]
   holds everywhere on \( (E \cap B(0,s) \cap K_i) \setminus (\Sigma_\varepsilon \cup Z), \) \( i \in \mathbb{N} \), where \( B_k(x, r) \) is a \( k \)-dimensional ball;
   
   b) the set \( Z \) equals the union of two disjoint measurable sets \( Z_1 \) and \( Z_2 \), where \( Z_2 \) is such that
   \[ \int \chi_{\varphi^{-1}(z) \cap Z_2} \, dH^{n-k}(u) = 0 \]
   for all \( z \in X \) and \( H^k(\varphi(Z_1)) = 0 \).

**Theorem 1.4** (The coarea formula for mappings with \( H^k \)-\( \sigma \)-finite image). Assume that \( (Y, d_Y) \) is an \( H^n \)-rectifiable metric space, \( X \) is an arbitrary metric space, and \( \varphi \in \text{Lip}(Y, X) \). Suppose also that \( \dim \ker(\text{MD}(\varphi \circ \beta_m, y)) \leq n - k \) almost everywhere and \( \varphi(Y) \) is \( H^k \)-\( \sigma \)-finite. Then the formula
\[ \int_Y f(x) J_k(\varphi, x) \, dH^n(x) = \int_X dH^k(z) \int_{\varphi^{-1}(z)} f(u) \, dH^{n-k}(u) \]
holds for every function \( f : \mathbb{Y} \to \mathbb{E} \) (where \( \mathbb{E} \) is an arbitrary Banach space) such that \( f(x)J_k(\varphi, x) \) is integrable.

**Remark 1.5.** Note that in item 1 of Theorem 1.3, the function \( f : \mathbb{Y} \to \mathbb{E} \) need not be measurable. The same is true for a set \( A \subset \mathbb{Y} \) in item 2.

Item 5(b) also holds in the case of an arbitrary \( \mathcal{H}^n \)-rectifiable metric space \( \mathbb{Y} \).

**Remark 1.6.** The equivalence 1 \( \iff \) 2 is established by standard arguments. It is enough to consider the case of a compact set \( A \subset \mathbb{Y} \setminus \mathbb{Z} \), a real-valued bounded function \( f \) and sequences \( \{f_n\}_{n \in \mathbb{N}} \) of simple functions converging everywhere to \( f \) and apply Lebesgue’s theorem. The general case is proved by applying Lemma 3.8 and Lebesgue’s and B. Levi’s theorems.

**Remark 1.7.** Note that Theorem 1.3 is new even if \( \mathbb{Y} = \mathbb{R}^n \), \( \mathbb{X} = \mathbb{R}^m \), \( n,m \geq k \). In this case, the coarea formula is of the following type:

\[
\int_{\mathbb{R}^n} J_k(\varphi, x) \, dx = \int_{\mathbb{R}^m} dH^k(z) \int_{\varphi^{-1}(z)} dH^{n-k}(u).
\]

**Remark 1.8.** Theorem 1.4 can be interpreted as a generalization of a result of [24].

In Sections 2–6, we consider a Lipschitz mapping \( \varphi \) defined on a measurable subset of \( E \subset \mathbb{R}^n \), and in Section 7, we generalize our results to mappings defined on \( \mathcal{H}^n \)-rectifiable metric spaces.

The main result of the paper and a sketch of its proof was published in [14].

### 2. \( \mathcal{H}^{n-k} \)-rectifiability of the preimage of a point

**Definition 2.1.** A metric space \( \mathbb{X} \) is called \( \mathcal{H}^k \)-rectifiable if there exists an at most countable collection of Lipschitz mappings \( \alpha_i : A_i \to \mathbb{X} \) defined on measurable sets in \( \mathbb{R}^k \) such that \( \mathcal{H}^k(\mathbb{X} \setminus \bigcup_{i \in \mathbb{N}} \alpha_i(A_i)) = 0 \).

**Definition 2.2.** (See [13].) A point \( x \in E \) is called a linear density point in a direction \( u \), where \( u \in \mathbb{S}^{n-1} \), if \( x \) is a density point of the set \( E \cap (x + \mathbb{R}u) \).

Henceforth, denote by \( \mathcal{U} \) a countable dense set on \( \mathbb{S}^{n-1} \).

**Theorem 2.3 (The Preimage-of-a-Point Theorem: Regular case).** Suppose that \( E \) is a measurable set in \( \mathbb{R}^n \), \( \mathbb{X} \) is a metric space, and \( \varphi \in \text{Lip}(E, \mathbb{X}) \). Assume that:

1. \( E \subset E \) is a compact set and every point \( x \in E \) is a linear density point of the set \( E \) in all directions \( u_m \in \mathcal{U} \);
2. \( \varphi \) is \( m \)-differentiable at every \( x \in E \);
3. the \( m \)-differential \( MD(\varphi, x)(u_m) \) is continuous on \( E \) for each fixed \( u_m \in \mathcal{U} \);
4. the “difference quotients” \( d_{\mathbb{X}}(\varphi(x + ru_m), \varphi(x))/r \), \( x \in E \), converge uniformly on \( E \) to the \( m \)-differential \( MD(\varphi, x)(u_m) \) for each fixed \( u_m \in \mathcal{U} \);
5. \( \dim \ker(MD(\varphi, x)) = n - k \) for all \( x \in E \);
(6) the “difference quotients” \(|(x, x + ru_m) \cap E|/r, x \in E, u_m \in U, m \in \mathbb{N}\), converge uniformly to the unity on \(E\).

Then each set \(\varphi^{-1}(z) \cap E\) is a subset of the union of countably many images of measurable subsets of \(\mathbb{R}^{n-k}\) under Lipschitz mappings for all \(z \in \mathbb{X}\). In particular, it is \(\mathcal{H}^{n-k}\)-rectifiable.

**Proof.** I. Observe that, for each fixed \(u \in \mathbb{S}^{n-1}\), the \(m\)-differential \(MD(\varphi, x)(u)\) is uniformly continuous on \(x\), since \(E\) is the compact set.

II. Prove that the \(m\)-differential \(MD(\varphi, x)(u)\) is uniformly continuous on \(E\) uniformly in all \(u \in \mathbb{S}^{n-1}\). Indeed, let \(\varepsilon > 0\). Choose a finite subset \(\{u_m\}^M_1 \subset U\) such that, for each \(u \in \mathbb{S}^{n-1}\), there exists \(u_l \in \{u_m\}^M_1\) satisfying \(|u - u_l| < \frac{\varepsilon}{5 \text{Lip}(\varphi)}\). For each \(u_l \in \{u_m\}^M_1\), there exists \(\delta_l > 0\) such that \(|MD(\varphi, x)(u_l) - MD(\varphi, y)(u_l)| < \frac{\varepsilon}{2}\) whenever \(|x - y| < \delta_l\). Put \(\delta = \min_{1 \leq l \leq M} \delta_l\). Hence, if \(|x - y| < \delta\), we have

\[
\frac{|MD(\varphi, x)(u) - MD(\varphi, y)(u)|}{r} \\
\leq \frac{|MD(\varphi, x)(u) - MD(\varphi, x)(u_l)|}{r} + \frac{|MD(\varphi, x)(u_l) - MD(\varphi, y)(u_l)|}{r} + \frac{\varepsilon}{2} + \text{Lip}(\varphi)|u - u_l| < \varepsilon
\]

for every \(u \in \mathbb{S}^{n-1}\).

III. Similar arguments show that, for all \(x \in E\), the “difference quotients” \(d_\varphi(x + ru), \varphi(x))/r, x \in E\), converge uniformly on \(E\) to the metric differential \(MD(\varphi, x)(u)\) uniformly in all \(u \in \mathbb{S}^{n-1}\). Fix \(\varepsilon > 0\) and consider a subset \(\{u_m\}^M_1 \subset U\) such that, for each \(u \in \mathbb{S}^{n-1}\), there exists \(u_l \in \{u_m\}^M_1\) such that \(|u - u_l| < \frac{\varepsilon}{5 \text{Lip}(\varphi)}\). Condition (4) implies that, for each \(u_l\), there exists \(\rho_l > 0\) such that if \(r < \rho_l\) and \(x + ru \in E\), then the relation

\[
\left| \frac{d_\varphi(x + ru, \varphi(x))}{r} - MD(\varphi, x)(u_l) \right| < \frac{\varepsilon}{2}
\]

holds for all \(x \in E\). Next, it follows from condition (6) that there exists \(\rho_l > 0\) such that

\[
\frac{|(x, x + ru)| \cap E|}{r} > 1 - \frac{\varepsilon}{20 \text{Lip}(\varphi)}
\]

if \(r < \rho_l\), for all \(x \in E\). Put \(r_0 = \min_{1 \leq l \leq M} \{\delta_l, \rho_l\}\). Take \(u \in \mathbb{S}^{n-1}\). By choice of \(r_0\) we have that, for each \(r > 0, r < r_0\), there exists \(r' > 0\) such that \(x + r'u \in E\) and \(|r - r'| < \frac{\varepsilon r}{20 \text{Lip}(\varphi)}\) for all \(x \in E\). Hence, for \(r < r_0\) and \(x + ru \in E\), we have

\[
\frac{|d_\varphi(x + ru, \varphi(x))}{r} - MD(\varphi, x)(u)}{r} \\
\leq \frac{|d_\varphi(x + ru, \varphi(x))}{r} - \frac{d_\varphi(x + ru', \varphi(x))}{r} \leq \frac{d_\varphi(x + ru', \varphi(x))}{r} - MD(\varphi, x)(u_l)}{r'}
\]

for all \(x \in E\). Next, it follows from condition (6) that there exists \(\rho_l > 0\) such that

\[
\frac{|(x, x + ru)| \cap E|}{r} > 1 - \frac{\varepsilon}{20 \text{Lip}(\varphi)}
\]

if \(r < \rho_l\), for all \(x \in E\). Put \(r_0 = \min_{1 \leq l \leq M} \{\delta_l, \rho_l\}\). Take \(u \in \mathbb{S}^{n-1}\). By choice of \(r_0\) we have that, for each \(r > 0, r < r_0\), there exists \(r' > 0\) such that \(x + r'u \in E\) and \(|r - r'| < \frac{\varepsilon r}{20 \text{Lip}(\varphi)}\) for all \(x \in E\). Hence, for \(r < r_0\) and \(x + ru \in E\), we have
\[ + \frac{d\gamma(x + r'u_t), \varphi(x)}{r'} - \frac{d\gamma(x + r'u_t), \varphi(x)}{r} + |MD(\varphi, x)(u) - MD(\varphi, x)(u_t)| \leq \text{Lip}(\varphi)\left|u - \frac{r'}{r}u_t\right| + \varepsilon + \text{Lip}(\varphi)\left(1 - \frac{r'}{r}\right) + \text{Lip}(\varphi)|u - u_t| < \varepsilon. \] (2.1)

**IV.** Consider \( z \in \mathbb{X} \) and the level set \( \varphi^{-1}(z) \). Assume that it is not empty and fix \( x \in \varphi^{-1}(z) \). Item II implies that for each set \((\ker(MD(\varphi, x)))^\perp \cap S^{k-1}) \times (B(x, r) \cap E)\), there exist \( u_0 \in (\ker(MD(\varphi, x)))^\perp \cap S^{k-1} \) and \( y_0 \in B(x, r) \cap E \) on which \( MD(\varphi, y)(u) \) takes the minimal value \( c \). Choose \( r > 0, r < r_0 \), so that \( c \) is positive (the existence of such \( r > 0 \) follows from what has been said above). Moreover, fix \( 0 < 2\rho < r \) so that inequality (2.1) holds for \( \varepsilon < c/2 \). Denote \( \mathbb{I}_k = (\ker(MD(\varphi, x)))^\perp \) and consider the correspondence \( \psi \) defined as follows:

\[ \psi : \mathbb{R}^n \ni y \mapsto \{ \varphi^{-1}(z) \cap (y + \mathbb{I}_k) \cap E \}. \]

Denote by \( B^{n-k} \) the set \( B^{n-k}(x, \rho) \subset x + \ker(MD(\varphi, x)) \), where \( B^{n-k}(x, \rho) \) is an \((n-k)\)-dimensional ball. Note that the restriction \( \psi \) on the set \( B^{n-k} \) is a mapping. Indeed, suppose that there are two different points \( w_1 \neq w_2 \) in the image \( \psi(y) \) of some point \( y \), \( |w_1 - w_2| < 2\rho \) and \( \varphi(w_1) = \varphi(w_2) = z \). By the definition of \( \psi \) we have \( w_1 - w_2 \in (\ker(MD(\varphi, x)))^\perp \). From (2.1), it follows that

\[ 0 - MD(\varphi, w_1)\left(\frac{w_1 - w_2}{|w_1 - w_2|}\right) = MD(\varphi, w_1)\left(\frac{w_1 - w_2}{|w_1 - w_2|}\right) < c/2. \]

The so-obtained contradiction shows that the restriction \( \psi|_{B^{n-k}} \) is a mapping.

**V.** To simplify the notations, we assume from now on that \( x = 0 \). Consider the mapping \( \pi : \varphi^{-1}(z) \cap B(0, \rho) \to B^{n-k} \subset \ker(MD(\varphi, 0)) \) such that \( \pi(w), w \in \varphi^{-1}(z) \), is the orthogonal projection of \( w \) onto \( B^{n-k} \), i.e., \( \pi(w) \perp (w - \pi(w)) \). The definition of \( \pi \) implies that it is a Lipschitz mapping. Show that there exists \( \alpha > 0 \) such that

\[ \lim_{v \to y} \frac{|\pi(v) - \pi(y)|}{|v - y|} \geq \alpha > 0 \] (2.2)

everywhere on \( \varphi^{-1}(z) \cap B(x, \rho) \). Suppose the contrary. For each \( \varepsilon > 0 \), there exist \( y, v \in \varphi^{-1}(z) \cap B(x, \rho) \) such that

\[ \frac{|\pi(v) - \pi(y)|}{|v - y|} < \varepsilon. \] (2.3)

Consider \( \varepsilon < \min\left\{ \frac{\varepsilon}{2 \text{Lip}(\varphi)}, \frac{1}{2} \right\} \). Then \( 2\rho \geq |v - y| > \frac{1}{\varepsilon}|\pi(v) - \pi(y)| \). Involving the fact that \( \pi \) takes values in \( \ker(MD(\varphi, 0)) \), we see that \( \pi(v) - \pi(y) = \pi(v - y) \in \ker(MD(\varphi, 0)) \) (here, in the case of \( v - y \notin \varphi^{-1}(t) \), the value \( \pi(v - y) \) means the orthogonal projection of \( v - y \) onto \( B^{n-k} \)). Denote by \( k \) the orthogonal projection of a vector \( v - y \) onto \( (\ker(MD(\varphi, 0)))^\perp \). Then

\[ |v - y|^2 = |k|^2 + |\pi(v) - \pi(y)|^2, \] (2.4)
and (2.3) yields $|k|^2 > |v - y|^2(1 - \varepsilon^2)$. Since $MD(\varphi, y)(\cdot)$ is a seminorm, we have

$$MD(\varphi, y)(k) \leq MD(\varphi, y)(v - y) + MD(\varphi, y)(\pi(v) - \pi(y)).$$

(2.5)

From (2.1) and the equality $\varphi(v) = \varphi(y) = z$, we deduce that $MD(\varphi, y)(v - y) \leq c/2|v - y|$. Put

$$u_{v,y} = \frac{|v - y|}{|\pi(v) - \pi(y)|}(\pi(v) - \pi(y))$$

and estimate $MD(\varphi, y)(\pi(v) - \pi(y))$. Note that $|u_{v,y}| = |v - y|$. Relation (2.1) and the fact that $MD(\varphi, y)(u)$ is a Lipschitz mapping on $S^{n-1}$ [13,17] imply that

$$MD(\varphi, y)(u_{v,y}) \leq MD(\varphi, y)(v - y) + 2Lip(\varphi)|v - y|.$$

Now, from (2.3) we obtain

$$MD(\varphi, y)(k) \leq \frac{c}{2}|v - y| + \varepsilon 3 Lip(\varphi)|v - y| = \left(\frac{c}{2} + 3\varepsilon Lip(\varphi)\right)|v - y|.$$

Finally,

$$MD(\varphi, y)\left(\frac{k}{|k|}\right) \leq \frac{\left(\frac{c}{2} + 3\varepsilon Lip(\varphi)\right)|v - y|}{|v - y|\sqrt{1 - \varepsilon^2}} \leq \frac{3c}{4\sqrt{1 - \varepsilon^2}} < c.$$

The so-obtained contradiction with the assumption on $r$ of item IV shows that the lower limit (2.2) is strictly positive.

Thus, $\pi$ is a bi-Lipschitz mapping. Hence, the restriction of the mapping $\psi$ on the compact set $\pi(\varphi^{-1}(z) \cap B(x, \rho))$ is also bi-Lipschitz. The balls $\{B(x, \rho): x \in \varphi^{-1}(z), \rho > 0\}$ constitute an open covering of the compact set $\varphi^{-1}(z) \cap E$. Choose a finite subcovering $B(x_1, \rho_1), \ldots, B(x_m, \rho_m)$ and consider the intersection $B(x_l, \rho_l) \cap \varphi^{-1}(z) \cap E$, $l = 1, \ldots, m$. Thus, the level set $\varphi^{-1}(z) \cap E$ is a subset of some finite union of the images of measurable sets in $\mathbb{R}^{n-k}$ under Lipschitz mappings. In particular, it is an $\mathcal{H}^{n-k}$-rectifiable set. The theorem follows. $\square$

**Proof of Theorem 1.2 (The case $Y = E$).** I. We may assume without loss of generality that $E$ is compact. Show that, for $\varepsilon > 0$, there exists a measurable set $E_\varepsilon$, $\mathcal{H}^n(E_\varepsilon) < \varepsilon$, such that $E \setminus (E_\varepsilon \cup Z)$ is a compact set and conditions (1)–(6) of Theorem 2.3 hold on $E \setminus (E_\varepsilon \cup Z)$. 

In view of the properties of measurable sets and Egorov’s theorem, for each \( \varepsilon > 0 \) and the bounded measurable set \( E \), there exists a compact set \( K_\varepsilon \subset (E \setminus Z) \), \( \mathcal{H}^n (E \setminus (K_\varepsilon \cup Z)) < \frac{\varepsilon}{2} \), on which conditions (2), (3) and (5) of Theorem 2.3 hold.

**II.** 1. For obtaining condition (4), fix \( u_m \in \mathcal{U} \) and consider the functions

\[
r_m(x) = \sup_{\alpha} \{ \alpha \in [0, 1/n]: x + \alpha u_m \in E \}, \quad n \in \mathbb{N}.
\]

Show that each \( r_n(x) \), \( n \in \mathbb{N} \), is measurable on \( E \). Fix \( n \in \mathbb{N} \) and consider a Lebesgue set \( E_a = \{ x \in E: r_n(x) < a \} \). If \( a > 1/n \) then \( E_a = E \), and if \( a < 0 \) then \( E_a = \emptyset \). For \( 0 \leq a \leq 1/n \), it is easy to see in view of compactness of \( E \) that

\[
y \in E_a \Leftrightarrow [y + au_m, y + u_m/n] \subset E^c = \mathbb{R}^n \setminus E.
\]

If \( y \in E_a \) then, since \( E^c \) is open, for each \( \xi \in [y + au_m, y + u_m/n] \), there exists a ball \( B(\xi, r_\varepsilon) \subset E^c \). These balls form an open covering of the compact set \( [y + au_m, y + u_m/n] \). Choose a finite subcovering \( \{ B(\xi_k, r_k) \}_{k=1}^K \) of \( [y + au_m, y + u_m/n] \). Then there exists a positive number \( r_0 \leq \min_{k=1,\ldots,K} r_k \) such that if \( |y - y'| < r_0 \) then \( [y' + au_m, y' + u_m/n] \subset E^c \), or, in other words, \( y' \in E_a \). Thus, the set \( E_a \) is open and, hence, it is measurable, and the function \( r_n(x) \) is measurable.

2. Making use of Egorov’s theorem, remove a set of measure not exceeding \( \frac{\varepsilon}{2} \) from \( E \) on whose complement \( E \setminus E \) the functions \( r_n(x) \) are continuous. On this complement the functions

\[
\varphi_n(x) = \frac{d_{\mathcal{H}}(\varphi(x), \varphi(x + r_n(x))u_m))}{r_n(x)}
\]

are measurable. Applying Egorov’s theorem once again, we remove the set of measure not exceeding \( \frac{\varepsilon}{2} \) on whose complement we have \( \varphi_n(x) \to_{n \to \infty} \) \( MD(\varphi, x)(u_m) \) uniformly in \( x \), for all \( m \in \mathbb{N} \). Denote the so-obtained set by \( D_\varepsilon \). By choice, \( \mathcal{H}^n (E \setminus D_\varepsilon) < \frac{\varepsilon}{2} \). It is left to show, that the “difference quotients” \( d_{\mathcal{H}}(\varphi(x + ru_m), \varphi(x))/r, x \in D_\varepsilon, x + ru_m \in E \), converge uniformly on \( D_\varepsilon \) to the metric differential \( MD(\varphi, x)(u_m) \) for each fixed \( u_m \in \mathcal{U} \).

Take arbitrary \( \sigma > 0 \) and \( n_0 = n_0(m) \in \mathbb{N} \) such that \( |\varphi_n(x) - MD(\varphi, x)(u_m)| < \sigma \) for \( n > n_0 \) for all \( x \in D_\varepsilon \). Let \( 0 < r < r_n(x) \) be such that \( x + ru_m \in E \) and \( r_n(x) \geq r \) be closest to \( r \). Consequently, \( r \geq 1/(n + 1) \), and

\[
|r_n(x) - r| \leq \frac{1}{n} - \frac{1}{n + 1} = o(r), \quad \text{where } o(\cdot) \text{ is uniform on } D_\varepsilon.
\]

Applying arguments similar to those in (2.1), we have condition (4).

**III.** To obtain condition (6), consider functions

\[
f_n(x) = \frac{1}{n} \left[ x, x + \frac{1}{n} u_m \right] \cap E,
\]

\( u_m \in \mathcal{U} \). Show that they are measurable. Indeed, we may assume without loss of generality that \( E \) is a closed set. Note that \( f_n \) is measurable if and only if each function

\[
g_n(x) = \frac{1}{n} \left[ x, x + \frac{1}{n} u_m \right] \cap E^c
\]
is measurable. Take $\alpha \in [0, \frac{1}{n}]$ and prove that the set $G_\alpha = \{x: g_n^m(x) > \alpha\}$ is open (the cases of $\alpha < 0$ and $\alpha > \frac{1}{n}$ are trivial). Fix $x \in G_\alpha$. Since the set $(x, x + \frac{1}{n} u_m) \cap E^c$ is open and it equals a union of disjoint intervals, we may choose a finite collection of closed intervals lying in this set sum of lengths of which is also greater than $\alpha$. The obtained set $G$ is compact. The set $E^c$ is open, consequently, for each $y \in G$ there exists a cube $Q(y, \delta) \subseteq E^c$ faces of which are either parallel or orthogonal to the direction $u_m$. Choose a finite subcovering of $G$. Obviously, there exists $\delta > 0$ small enough, such that, for each $y \in B(x, \delta)$, we have $g_n^m(y) > \alpha$, or, equivalently, $y \in G_\alpha$. Thus, $g_n^m$ and $f_n^m$ are measurable.

On each set $y + \mathbb{R}u_m$, we have [8] $f_n^m \to 1$ as $n \to \infty$ a.e. Taking the measurability of the limit function and Fubini theorem into account, we have $f_n^m \to 1$ as $n \to \infty$ a.e. Repeating the above arguments, we obtain a compact set $C_\varepsilon \subseteq E$, $\mathcal{H}^n(E \setminus C_\varepsilon) < \varepsilon$, on which the “difference quotients” $||(x, x + ru_m) \cap E||/r$ converge uniformly to 1 for each fixed $u_m \in \mathcal{U}$.

Since almost every point $x \in E$ is its linear density point in each direction $u_m \in \mathcal{U}$ [8], then, without loss of generality, we can suppose that the set $C_\varepsilon$ enjoys property (1).

Put $E_\varepsilon = E \setminus ((K_\varepsilon \cap D_\varepsilon \cap C_\varepsilon) \cup Z)$. By the choice of $K_\varepsilon$, $D_\varepsilon$ and $C_\varepsilon$, the measure of $E_\varepsilon$ is at most $\varepsilon$. Then, for all $z \in \mathcal{X}$, the preimage $\varphi^{-1}(z) \setminus (Z \cup E_\varepsilon)$ is an $\mathcal{H}^{n-k}$-rectifiable set. Consider the sequence $\{E_m = \frac{1}{m}\}_{m \in \mathbb{N}}$ and the corresponding sets $\{E_m\}_{m \in \mathbb{N}}$. Put $\Sigma = \bigcap_{m \in \mathbb{N}} E_m$. From the properties of the sets $E_m$, we have $\mathcal{H}^n(\Sigma) = 0$.

Thus, the Preimage-of-a-Point Theorem is proved. $\square$

3. Sets which do not influence the values of integrals

Hereinafter, by $E$ we denote the part of the set $E$ satisfying the conditions of Theorem 2.3.

**Definition 3.1.** (See [12,15].) Suppose that $E \subset \mathbb{R}^n$ is a measurable set, $(\mathcal{X}, d_\mathcal{X})$ is a metric space, and $\varphi : E \to \mathcal{X}$ is a mapping $m$-differentiable at a point $x \in E$ with $\dim \ker(MD(\varphi, x)) = n - k$. Define

$$J_k(MD(\varphi, x)) = \omega_k k \left( \int_{\mathcal{S}^{k-1}} [MD(\varphi, x)(u)]^{-k} d\mathcal{H}^{k-1}(u) \right)^{-1},$$

where $\mathcal{S}^{k-1}$ is the $(k - 1)$-dimensional unit sphere in the space $(\ker(MD(\varphi, x)))^\perp$.

If $\dim \ker(MD(\varphi, x)) > n - k$, then we set $J_k(MD(\varphi, x)) = 0$.

Recall that, throughout the paper, the zero set of the $m$-coarea factor is denoted by $Z$, i.e.,

$$Z = \{x \in E: J_k(MD(\varphi, x)) = 0\} = \{x \in E: \dim \ker(MD(\varphi, x)) > n - k\}.$$

Denote by $B(\mathcal{X})$ the family of all Borel subsets of $\mathcal{X}$.

**Theorem 3.2.** (See [8].) The Hausdorff measure $\mathcal{H}^k$ is a regular outer Borel measure, i.e., for each set $A \subset \mathcal{X}$, there exists $B \in B(\mathcal{X})$ such that $B \supseteq A$ and $\mathcal{H}^k(B) = \mathcal{H}^k(A)$.

**Lemma 3.3.** (See [13].) Suppose that $E \subset \mathbb{R}^n$ is a measurable set, $(\mathcal{X}, d_\mathcal{X})$ is a metric space, and $\varphi \in \text{Lip}(E, \mathcal{X})$. Then, for each $u \in \mathcal{S}^{n-1}$, the function $MD(\varphi, x)(u)$ is measurable on $E$. 

Definition 3.4. Let $E \subset \mathbb{R}^n$ be a measurable set and let $X$ be a metric space. The mapping $\varphi : E \to X$ is \textit{continuously $m$-differentiable} on $E$ if, for each $u \in \mathbb{S}^{n-1}$, the quantity $MD(\varphi, x)(u)$ is continuous on $x \in E$.

Definition 3.5. Let $u_1, \ldots, u_k$ be a set of linearly independent vectors in $\mathbb{R}^n$, $1 < k \leq n$. Define a $k$-dimensional direction as $\text{span}\{u_1, \ldots, u_k\} \cap B(0, 1)$.

Definition 3.6. (See [15].) Let $E \subset \mathbb{R}^n$ be a measurable set and let $u_1, \ldots, u_k$, $k \leq n$, be a set of linearly independent vectors in $\mathbb{R}^n$. Put $L = \text{span}\{u_1, \ldots, u_k\} \cap B(0, 1)$. A point $x \in E$ is a $(k$-dimensional) density point in the direction $L$ if it is a density point of the set $(x + L) \cap E$, i.e.,
\[
\lim_{r \to 0} \frac{\mathcal{H}^k(B(x, r) \cap x + L)}{\mathcal{H}^k(B_k(x, r))} = 1,
\]
where $B_k(x, r)$ is a $k$-dimensional ball.

In the case of $k = 1$, we take $L = \text{span}\{u_1\} \cap B(0, 1)$ and say that $x$ is a linear density point in the direction $u_1$.

In what follows, we denote by $\mathcal{L}$ the countable dense set on $\mathbb{S}^{n-1}$ of $k$-dimensional directions. Henceforth, we assume that every point $x \in E$ is a $k$-dimensional density point in every direction $L_\nu \in \mathcal{L}$, $\nu \in \mathbb{N}$ [8].

Lemma 3.7. (See [15].) Let $E \subset \mathbb{R}^n$ be a measurable set and let $(X, d_X)$ be a metric space. Then the preimage of a set of $\mathcal{H}^k$-measure zero under a Lipschitz mapping $\varphi : E \to X$ with $\text{dim ker}(MD(\varphi, x)) = n - k$ almost everywhere, is the set of $\mathcal{H}^n$-measure zero.

Lemma 3.8. Suppose that $(Y, d_Y)$ and $(X, d_X)$ are metric spaces, $\varphi \in \text{Lip}(Y, X)$, and $E \subset Y$ is such that $\mathcal{H}^n(E) = 0$. Then the mapping $\varphi$ meets the following equality:
\[
\int_Y d\mathcal{H}^k(z) \int_{\varphi^{-1}(z) \cap E} d\mathcal{H}^{n-k}(u) = 0.
\]

Proof. It is known [5,8] that
\[
\int_Y \mathcal{H}^{n-k}(\varphi^{-1}(z) \cap E) d\mathcal{H}^k(z) \leq (\text{Lip}(\varphi))^k \frac{\omega_{n-k} \omega_k}{\omega_n} \mathcal{H}^n(E). \quad (3.1)
\]
Involving the fact that $\mathcal{H}^n(E) = 0$ and estimating the exterior integrals, we obtain
\[
0 \leq \int_Y d\mathcal{H}^k(z) \int_{\varphi^{-1}(z) \cap E} d\mathcal{H}^{n-k}(u) = \int_Y \mathcal{H}^{n-k}(\varphi^{-1}(z) \cap E) d\mathcal{H}^k(z) \leq 0.
\]
Hence,
\[
\int_Y d\mathcal{H}^k(z) \int_{\varphi^{-1}(z) \cap E} d\mathcal{H}^{n-k}(u) = 0. \quad \Box
\]
Lemma 3.9. Assume that $E \subset \mathbb{R}^n$ is a measurable set, $(\mathbb{X}, d_{\mathbb{X}})$ is a metric space, and $\varphi \in \text{Lip}(E, \mathbb{X})$. Suppose that there exists a measurable set $Z' \subset Z$ such that the equality
\[ \int_{\varphi^{-1}(z) \cap (Z \setminus Z')} dH^{n-k}(u) = 0 \]
holds for all $z \in \mathbb{X}$ and $\mathcal{H}^k(\varphi(Z')) = 0$. Then
\[ \int_{\mathbb{X}} d\mathcal{H}^k(z) \int_{\varphi^{-1}(z) \cap Z} dH^{n-k}(u) = 0. \]

Proof. Put
\[ \mathcal{E}_Z(z) = \int_{\varphi^{-1}(z) \cap Z} dH^{n-k}(u). \]
Then
\[ 0 \leq \int_{\mathbb{X}} \mathcal{E}_Z(z) d\mathcal{H}^k(z) = \int_{\mathbb{X}} d\mathcal{H}^k(z) \int_{\varphi^{-1}(z) \cap Z} dH^{n-k}(u) \]
\[ \leq \int_{\mathbb{X}} d\mathcal{H}^k(z) \int_{\varphi^{-1}(z) \cap Z'} dH^{n-k}(u) + \int_{\mathbb{X}} d\mathcal{H}^k(z) \int_{\varphi^{-1}(z) \cap (Z \setminus Z')} dH^{n-k}(u) = 0, \]
and, consequently,
\[ \int_{\mathbb{X}} d\mathcal{H}^k(z) \int_{\varphi^{-1}(z) \cap Z} dH^{n-k}(u) = 0. \]

Lemma 3.10. Assume that $E \subset \mathbb{R}^n$ is a measurable set, $(\mathbb{X}, d_{\mathbb{X}})$ is a metric space, and $\varphi \in \text{Lip}(E, \mathbb{X})$. Suppose that the coarea formula
\[ 0 = \int_Z J_k(MD(\varphi, x)) \, dx = \int_{\mathbb{X}} d\mathcal{H}^k(z) \int_{\varphi^{-1}(z) \cap Z} dH^{n-k}(u) \]
holds for the degeneration set $Z$. Then there exists a measurable set $Z' \subset Z$ such that the equality
\[ \int_{\varphi^{-1}(z) \cap (Z \setminus Z')} dH^{n-k}(u) = 0 \]
holds for all $z \in \mathbb{X}$ and $\mathcal{H}^k(\varphi(Z')) = 0$. 
Proof. Put
\[ \Xi_Z(z) = \int_{\varphi^{-1}(z) \cap Z} d\mathcal{H}^{n-k}(u) \]
and \( A = \{ \Xi_Z(z) \neq 0 \} \). Then \( \mathcal{H}^k(A) = 0 \). By Theorem 3.2 there exists a Borel set \( B \supset A \) such that \( \mathcal{H}^k(B) = \mathcal{H}^k(A) = 0 \). Hence, \( \mathcal{X} \setminus B \) and \( \varphi^{-1}(\mathcal{X} \setminus B) \) are also Borel sets. Put \( Z_1 = Z \cap \varphi^{-1}(\mathcal{X} \setminus B) \) and \( Z' = Z \setminus Z_1 \). In this case, \( \varphi(Z_1) \subset \{ \Xi_Z(z) = 0 \} \), \( \varphi(Z') \subset B \), and \( \mathcal{H}^k(\varphi(Z')) = 0 \). The sets \( \varphi(Z') \) and \( \varphi(Z_1) \) are disjoint. Indeed, \( \varphi(Z') \subset B \), and \( \varphi(Z_1) \subset \mathcal{X} \setminus B \). Therefore, every \( z \in \varphi(Z) \) belongs either to \( \varphi(Z') \) or to \( \varphi(Z_1) \). Hence, if \( z \in \varphi(Z_1) \) then \( \varphi^{-1}(z) \cap Z' = \emptyset \) and
\[ 0 = \Xi_Z(z) = \int_{\varphi^{-1}(z) \cap Z} d\mathcal{H}^{n-k}(u) = \int_{\varphi^{-1}(z) \cap Z_1} d\mathcal{H}^{n-k}(u). \]
If \( z \in \mathcal{X} \setminus \varphi(Z_1) \) then \( \varphi^{-1}(z) \cap Z_1 = \emptyset \) and
\[ \int_{\varphi^{-1}(z) \cap Z_1} d\mathcal{H}^{n-k}(u) = 0. \]
\[ \square \]

Remark 3.11. Using Lemmas 3.9 and 3.10, we see that the set \( Z \) can be decomposed into two disjoint measurable sets: one of them does not influence the integral on the right-hand side of the coarea formula and the \( \mathcal{H}^k \)-measure of the image of the second equals zero if and only if the coarea formula holds for this set.

Proposition 3.12. Thus, sets of the \( k \)-dimensional measure zero in the image and the \( n \)-dimensional measure zero in the preimage do not influence the values of integrals.

4. The coarea formula and the structure of the image

The goal of this section is to prove the following result.

Theorem 4.1. (See [15].) Assume that \( E \subset \mathbb{R}^n \) is a measurable set, \( (\mathcal{X}, d_{\mathcal{X}}) \) is a metric space, and \( \varphi \in \text{Lip}(E, \mathcal{X}) \) is such that \( \dim \ker(MD(\varphi, x)) \geq n - k \) almost everywhere. Suppose that the coarea formula holds for every measurable subset \( A \subset E \). Then there exists a set \( \Sigma \subset E \) such that \( \mathcal{H}^n(\Sigma) = 0 \) and \( \varphi(E \setminus (Z \cup \Sigma)) \) is an \( \mathcal{H}^k \)-rectifiable set.

Note that, the detailed proof of Theorem 4.1 is given in [15]. Here we give a scheme of its proof. The main idea is to split \( E \) up to a set of arbitrary small measure into measurable subsets so that the image of each subset is equal to the one of some \( k \)-dimensional measurable set in \( \mathbb{R}^k \), i.e., it is \( \mathcal{H}^k \)-rectifiable. To obtain the existence of such partition, we use Lemmas 4.6, 4.7 and 4.11. To prove Lemma 4.6, we need Lemmas 4.2, 4.4 and 4.5. Lemma 4.10 is used in the proof of Lemma 4.11.

Lemma 4.2. Let \( E \subset \mathbb{R}^n \) be a measurable set and let \( (\mathcal{X}, d_{\mathcal{X}}) \) be a metric space. If \( \varphi: E \to \mathcal{X} \) is a Lipschitz mapping with \( \dim \ker(MD(\varphi, x)) = n - k \) almost everywhere, then the preimage of a measurable set is measurable.
Proof. Let \( G \subset \mathbb{X} \) be a measurable set. Then \( G \) can be represented as \( G = G_B \setminus S \), where \( \mathcal{H}^k(S) = 0 \) and \( G_B \) is a Borel set. As \( \varphi \) is a Lipschitz mapping and, consequently, is continuous, the preimage \( \varphi^{-1}(G_B) \) is also a Borel set. By Lemma 3.7, we have \( \mathcal{H}^n(\varphi^{-1}(S)) = 0 \). Thus, the preimage of a measurable set is measurable. \( \square \)

Corollary 4.3. From the proofs of Lemmas 3.7 and 4.2, we see that, for every set \( A \subset \mathbb{X} \) with \( \mathcal{H}^k(A) = 0 \), there exists a Borel set \( B \supset A \) of \( \mathcal{H}^k \)-measure zero whose preimage can be represented as the union of a measurable subset of \( Z \) and a set of \( \mathcal{H}^n \)-measure zero.

Lemma 4.4. Suppose that \( E \) is a measurable set in \( \mathbb{R}^n \), \( \mathbb{X} \) is a metric space, and a mapping \( \varphi \in \text{Lip}(E, \mathbb{X}) \) is such that \( \dim \ker(MD(\varphi, x)) \geq n - k \) almost everywhere. Then there exists \( c > 0 \) such that \( MD(\varphi, x)(u) \geq c \) for all \( x \in E \) and \( u \in S^{n-1} \cap (\ker(MD(\varphi, x)))^\perp \).

Proof. By item II and the arguments of item IV of Theorem 2.3, we conclude that, for every point \( x \in E \), there exists \( r(x) > 0 \) such that \( MD(\varphi, y)(u) \geq c(x) > 0 \) for all \( y \in Q(x, r(x)) \cap E \) and \( u \in (\ker(MD(\varphi, x)))^\perp \cap S^{n-1} \). Assume without loss of generality that \( r(x) < R < \infty \) for all \( x \in E \). Every \( u \in (\ker(MD(\varphi, x)))^\perp \), \( |u| = 1 \), can be represented uniquely as \( u = \alpha_1 v_1 + \alpha_2 v_2 \), where \( 0 \leq \alpha_1, \alpha_2 \leq 1 \), \( v_1 \in (\ker(MD(\varphi, y)))^\perp \), \( v_2 \in \ker(MD(\varphi, y)) \), \( |v_1| = |v_2| = 1 \). Moreover, for each unit \( v_1 \in (\ker(MD(\varphi, y)))^\perp \), there exist corresponding unit vectors \( u \in (\ker(MD(\varphi, x)))^\perp \) and \( v_2 \in \ker(MD(\varphi, y)) \), and the relations

\[
c(x) \leq MD(\varphi, y)(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 MD(\varphi, y)(v_1)
\]

and \( MD(\varphi, y)(v) \geq c(x) \) hold for all \( y \in Q(x, r(x)) \cap E \) and \( v \in (\ker(MD(\varphi, y)))^\perp \). The open cubes \( \{Q(x, r(x))\} \) form an open covering of the compact set \( E \). Choose a finite subcovering \( Q(x_1, r(x_1)), \ldots, Q(x_K, r(x_K)) \) of this covering and put \( c = \min_{1 \leq i \leq K} c(x_i) \). The lemma is proved. \( \square \)

Lemma 4.5. Suppose that \( E \subset \mathbb{R}^n \) is a measurable set, \( (\mathbb{X}, d_\mathbb{X}) \) is a metric space, and \( \varphi \in \text{Lip}(E, \mathbb{X}) \) is such that \( \dim \ker(MD(\varphi, x)) \geq n - k \) almost everywhere. For each \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( \ker(MD(\varphi, y)) \cap S^{n-k-1} \) is a subset of the \( \varepsilon \)-neighborhood of the set \( \ker(MD(\varphi, x)) \cap S^{n-k-1} \) if \( |x - y| < \delta \), \( x, y \in E \). By another words, if \( x, y \in E \) and \( |x - y| < \delta \), then for each \( u \in \ker(MD(\varphi, y)) \) there exists \( v \in \ker(MD(\varphi, x)) \) such that \( |u - v| < \varepsilon \).

Proof. By Lemma 4.4, \( c \leq MD(\varphi, x)(u) \leq \text{Lip}(\varphi) \) for all \( u \in (\ker(MD(\varphi, x)))^\perp \). Next, every \( u \in S^{n-1} \) can be represented uniquely as \( u = \alpha_1 v_1 + \alpha_2 v_2 \), where \( 0 \leq \alpha_1, \alpha_2 \leq 1 \), \( v_1 \in (\ker(MD(\varphi, x)))^\perp \), \( v_2 \in \ker(MD(\varphi, x)) \), \( |v_1| = |v_2| = 1 \). Fix \( \varepsilon \in (0, 1) \). Let \( \delta > 0 \) be such that, for all \( x, y \in E \), \( |x - y| < \delta \), \( u \in S^{n-1} \), we have

\[
|MD(\varphi, x)(u) - MD(\varphi, y)(u)| < \frac{c_\varepsilon}{12}.
\]

On the one hand, from the equality \( MD(\varphi, x)(u) = \alpha_1 MD(\varphi, x)(v_1) \) it follows that \( MD(\varphi, y)(u) < \frac{c_\varepsilon}{6} \) if \( \alpha_1 < \frac{c_\varepsilon}{12 \text{Lip}(\varphi)} \). On the other hand, \( MD(\varphi, y)(u) > \frac{c_\varepsilon}{3} \) if \( \alpha_1 > \frac{c_\varepsilon}{2} \). Therefore, \( S^{n-1} \ni u = \alpha_1 v_1 + \alpha_2 v_2 \) can belong to \( \ker(MD(\varphi, y)) \) only if \( \alpha_1 \leq \frac{c_\varepsilon}{2} \), from which the existence of \( \delta > 0 \) follows. \( \square \)
Lemma 4.6. Suppose that $E \subset \mathbb{R}^n$ is a measurable set, $(\mathcal{X}, d_{\mathcal{X}})$ is a metric space, and $\varphi \in \text{Lip}(E, \mathcal{X})$ is such that $\dim \ker(M\text{D}(\varphi, x)) = n - k$ almost everywhere. Assume that the coarea formula
\[
\int_A J_k(M\text{D}(\varphi, x)) \, dx = \int_{\mathcal{X}} d\mathcal{H}^k(z) \int_{\varphi^{-1}(z) \cap A} d\mathcal{H}^{n-k}(u)
\] holds for any measurable subset $A \subset E$. Then, for each $E \subset E$, there exists $m_0 \in \mathbb{N}$ such that the function
\[
\Phi_m(x) = \frac{m^{n-k}}{2^{n-k}} \int_{\varphi^{-1}(\varphi(x)) \cap Q(x, 1/m) \cap E} d\mathcal{H}^{n-k}(u)
\] is measurable on $E$ for every $m \geq m_0$, $m \in \mathbb{N}$. Here the faces of the cube $Q(x, 1/m)$ are either parallel or orthogonal to $\ker(M\text{D}(\varphi, x))$.

Proof. I. From the validity of the coarea formula (4.1) it follows that the function
\[
\Xi_A(z) = \int_{\mathcal{X}} d\mathcal{H}^k(z) \int_{\varphi^{-1}(z) \cap A} d\mathcal{H}^{n-k}(u)
\] is measurable on $\mathcal{X}$ for every measurable subset $A \subset E$. The function $\Theta_A(x) = \Xi_A(\varphi(x))$ is also measurable, since $\varphi$ is a Lipschitz mapping, $\dim \ker(M\text{D}(\varphi, x)) = n - k$, and, hence, Lemma 4.2 implies the measurability of the preimage $\varphi^{-1}(G)$ for every measurable $G \subset \mathcal{X}$.

Fix $E \subset E$. Applying arguments similar to those in the proof of the Preimage-of-a-Point Theorem (see 1.2), we can find a sequence $\{D_i\}_{i \in \mathbb{N}}$ of compact subsets of $E$ meeting the conditions of Theorem 2.3. From now on, without loss of generality, we investigate the measurability of $\Phi_m(x)$ on a fixed set $D = D_i$, $D \supset E$, instead of $E$. For $\varepsilon < \frac{\varepsilon}{2}$, where $c$ is the constant of Lemma 4.4, consider $r(\varepsilon)$ such that $|M\text{D}(\varphi, x)(u) - M\text{D}(\varphi, y)(u)| < \varepsilon$ for all $u \in \mathbb{S}^{n-1}$, $x, y \in D$, $|x - y| < r(\varepsilon)$.

Since $D$ is compact, there exists $r_0 \in (0, r(\varepsilon))$ such that every cube of radius not exceeding $r_0$ lies in one of the cubes in the finite covering constructed in Lemma 4.4.

Fix $m_0 \in \mathbb{N}$ such that
\[
\frac{1}{m_0} < \frac{r_0}{3 \text{diam}(Q_1)}
\]
where $Q_1$ is the unit cube in $\mathbb{R}^n$. Below we suppose that $m \geq m_0$. Then every cube of radius $rac{3 \text{diam}(Q_1)}{2m}$ belongs to one of the cubes in the finite covering of Lemma 4.4.

II. Show that, for each $m \in \mathbb{N}$, $m \geq m_0$, there exists a sequence of measurable functions converging to $\Phi_m$ almost everywhere.

Fix $m \in \mathbb{N}$, $m \geq m_0$, and cover $D$ with a finite collection of cubes $Q_1(x_1), \ldots, Q_M(x_M)$ of radii $\frac{1}{m}$ with disjoint interiors. Then, by what was said at the beginning of the proof, the function
\[
\Phi_m^{Q_j}(y) = \frac{m^{n-k}}{2^{n-k}} \int_{\varphi^{-1}(\varphi(y)) \cap Q_j \cap E} d\mathcal{H}^{n-k}(u)
\] is measurable for every fixed $j = 1, \ldots, M$. 
Fix \( j \in \mathbb{N} \). Divide \( Q_j \) into a finite number of open cubes \( Q_{j,l} \) of radii \( \frac{1}{qm} \) for some \( q \in \mathbb{N} \) with disjoint interiors. Choose \( y_{j,l} \in Q_{j,l} \cap \mathbb{D} \) if \( Q_{j,l} \cap \mathbb{D} \neq \emptyset \). Consider measurable functions

\[
\Psi_{q,m}(y) = \frac{m^{n-k}}{2^{n-k}} \int_{\varphi^{-1}(\varphi(y)) \cap Q(y_{j,l}, \frac{1}{m})} d\mathcal{H}^{n-k}(u) \quad \text{if} \quad y \in Q_{j,l}, \quad y \in Q_{j,l} \cap \mathbb{E},
\]

\( q \in \mathbb{N}, \ m \geq m_0 \). Here the faces of \( Q(y_{j,l}, 1/m) \) are either parallel or orthogonal to \( \ker(MD(\varphi, y_{j,l})) \). By choice of \( m \in \mathbb{N} \), all the \( Q_{j,l} \)'s belong to one of the cubes of the finite covering constructed in Lemma 4.4. Then the functions \( \Phi_m(y) \) and \( \Psi_{q,m}(y) \) are finite, and

\[
|\Psi_{q,m}(y) - \Phi_m(y)| \leq C(n-k, q) \cdot (\frac{1}{\alpha(y)})^{n-k},
\]

where \( \alpha(y) \) is defined in item \( V \) of Theorem 2.3 and depends only on the cube of the finite covering constructed in Lemma 4.4 that contains \( y \), and on the level set \( \varphi^{-1}(\varphi(y)) \). Moreover, we have

\[
C(n-k, q) = O\left(\mathcal{H}^{n-k}\left(Q_{n-k}(0, \frac{1}{m}) \Delta Q_{n-k}\left(\frac{2\sqrt{n}}{qm}, \frac{1}{m}\right)\right)\right) \to 0 \quad \text{as} \quad q \to \infty
\]
in view of the continuity of \( MD(\varphi, x) \) and Lemma 4.5. Thus, \( |\Psi_{q,m}(y) - \Phi_m(y)| \to 0 \) as \( q \to \infty \) for almost all \( y \in \mathbb{D} \), and, consequently, \( \Phi_m(x) \) is measurable. \( \square \)

Lemma 4.7 shows that the level sets can be made close enough to \((n-k)\)-dimensional plains.

**Lemma 4.7.** Suppose that \( E \subset \mathbb{R}^n \) is a measurable set, \( (\mathbb{X}, d_\mathbb{X}) \) is a metric space, and \( \varphi \in \operatorname{Lip}(E, \mathbb{X}) \) is such that \( \dim \ker(MD(\varphi, x)) \geq n - k \) almost everywhere. Then

\[
\left\{|\Pr(\ker(MD(\varphi, x)))^\perp (y - v) : \varphi(y) = \varphi(v), \ y, v \in Q(x, r)\right\} = o(|y - v|),
\]

where \( y, v \in Q(x, r) \cap \mathbb{E}, o(\cdot) \) depends on \( r > 0 \) small enough, and it is uniform in \( x \in \mathbb{E} \). Here the faces of the cube \( Q(x, r) \) are either parallel or orthogonal to \( \ker(MD(\varphi, x)) \).

**Proof.** Show that if \( y, v \to x \), where \( y, v, x \in \mathbb{E} \) and \( \varphi(y) = \varphi(v) = \varphi(x) \), then the quantity \( |y - v| \) on this level set is uniformly close to the distance between their orthogonal projections onto \( \ker(MD(\varphi, x)) \).

This is equivalent to the following: for each \( \varepsilon \in (0, 1) \), there exists \( r > 0 \) such that

\[
|\frac{|y - v|}{|\varphi(y) - \varphi(v)|} - 1 + \varepsilon| < 1 + \varepsilon \quad \text{for all} \quad x \in \mathbb{E} \quad \text{and} \quad y, v \in \varphi^{-1}(\varphi(x)) \cap Q(x, r) \cap \mathbb{E},
\]

where \( \varphi \) is the mapping of Theorem 2.3. Assume the contrary. Consider \( \varepsilon \in (0, 1), \ r > 0 \), for which (2.1) holds for \( \frac{c\sqrt{\varepsilon}}{4} \), where \( c > 0 \) is the constant of Lemma 4.4, and \( x \in \mathbb{E}, \ y, v \in \varphi^{-1}(\varphi(x)) \cap Q(x, r) \cap \mathbb{E} \) such that

\[
\frac{|y - v|}{|\varphi(y) - \varphi(v)|} \geq 1 + \varepsilon.
\]

Put \( k = \Pr(\ker(MD(\varphi, x)))^\perp (y - v) \) and \( m = \Pr(\ker(MD(\varphi, x)))(y - v) \). Since the mapping \( \Pr(\ker(MD(\varphi, x))) \) is linear, \( m = \pi(y) - \pi(v) \). Therefore, from the hypothesis and (2.1), we infer

\[
|y - v|^2 = |k|^2 + |m|^2 \leq |k|^2 + \frac{|y - v|^2}{(1 + \varepsilon)^2},
\]

\[
|k|^2 \geq |y - v|^2 \left(1 - \frac{1}{(1 + \varepsilon)^2}\right) \geq |y - v|^2 \frac{2\varepsilon}{(1 + \varepsilon)^2},
\]

\[
MD(\varphi, x)(k) = MD(\varphi, x)(y - v) \leq \frac{c\sqrt{\varepsilon}}{4} |y - v|.
\]

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It follows that \( MD(\varphi, x)(\frac{k}{|k|}) \leq c \cdot \frac{1 + \varepsilon}{4\sqrt{2}} < c \), which contradicts the fact that \( MD(\varphi, x)(u) = 0 \) is separated from 0.

**Corollary 4.8.** Suppose that \( E \subset \mathbb{R}^n \) is a measurable set, \((X, d_X)\) is a metric space, and \( \varphi \in \text{Lip}(E, X) \) is such that \( \dim \ker(MD(\varphi, x)) \geq n - k \) almost everywhere. Then

\[
\sup \{|Pr_{(MD(\varphi, x))}(y - v)| : \varphi(y) = \varphi(v), y, v \in Q(x, r)\} = o(r),
\]

where \( y, v \in Q(x, r) \cap E, r \to 0 \), and \( o(\cdot) \) is uniform in \( x \in E \). Here the faces of the cube \( Q(x, r) \) are either parallel or orthogonal to \( \ker(MD(\varphi, x)) \).

**Theorem 4.9.** (See [8].) Let \( A \subset \mathbb{R}^n \) be a \( \mathcal{H}^m \)-rectifiable set of finite \( \mathcal{H}^m \)-measure. Then

\[
\lim_{r \to 0} \frac{\mathcal{H}^m(B(x, r) \cap A)}{\mathcal{H}^m(B_m(x, r))} = 1 \tag{4.2}
\]

for \( \mathcal{H}^m \)-almost all \( x \in A \), where \( B_m(x, r) \) is the \( m \)-dimensional ball of radius \( r \) centered at \( x \).

**Lemma 4.10.** The relation

\[
\lim_{r \to 0} \frac{\mathcal{H}^{n-k}(Q(x, r) \cap \varphi^{-1}(z) \cap E)}{\mathcal{H}^{n-k}(Q_{n-k}(x, r))} = 1
\]

holds on each level set \( \varphi^{-1}(z) \cap E \) for \( \mathcal{H}^{n-k} \)-almost \( x \in E \cap \varphi^{-1}(z) \), where \( Q(x, r) \) is a cube whose faces are either parallel or orthogonal to \( \ker(MD(\varphi, x)) \), and \( Q_{n-k}(x, r) \) is an \((n - k)\)-dimensional cube of radius \( r \) with center at \( x \).

**Proof.** In view of Theorem 2.3, the set \( \varphi^{-1}(z) \cap E \) satisfies the conditions of Theorem 4.9. From now on, in this proof, we will understand \( \varphi^{-1}(z) \) as a subset of \( E \). Consider the level set \( \varphi^{-1}(z) \) and a point \( x \in \varphi^{-1}(z) \) such that (4.2) holds, and fix \( \varepsilon > 0 \). Applying Lemma 4.7 and (4.2), consider \( r > 0 \) such that

\[
(1 + \varepsilon)|\pi(y) - \pi(v)| \geq |y - v| \geq |\pi(y) - \pi(v)|. \tag{4.3}
\]

for \( y, v \in \varphi^{-1}(z) \cap Q(x, r) \cap E \), where \( \pi \) is the mapping of Theorem 2.3, and

\[
\left| \frac{\mathcal{H}^{n-k}(Q(x, r) \cap \varphi^{-1}(z))}{\mathcal{H}^{n-k}(Q_{n-k}(x, r))} - 1 \right| < \varepsilon. \tag{4.4}
\]

Let \( Q(x, r/\sqrt{n}) \) be an \( n \)-dimensional cube of radius \( r/\sqrt{n} \) whose faces are parallel or orthogonal to \( \ker(MD(\varphi, x)) \). We have \( Q(x, r/\sqrt{n}) \subset B(x, r) \). Estimate the quantity

\[
\frac{\mathcal{H}^{n-k}(Q(x, r/\sqrt{n}) \cap \varphi^{-1}(z))}{\mathcal{H}^{n-k}(Q_{n-k}(x, r/\sqrt{n}))}.
\]

Put \( \Delta = B(x, r) \setminus Q(x, r/\sqrt{n}) \) and \( \Delta_{n-k} = (B(x, r) \setminus Q(x, r/\sqrt{n})) \cap (x + \ker(MD(\varphi, x))) \). Using (4.3), we infer

\[
\mathcal{H}^{n-k}(\varphi^{-1}(z) \cap \Delta) \leq (1 + \varepsilon)^{n-k} \mathcal{H}^{n-k}(\Delta_{n-k}). \tag{4.5}
\]
Furthermore,
\[
\mathcal{H}^{n-k}(Q(x, r/\sqrt{n}) \cap \varphi^{-1}(z)) = \mathcal{H}^{n-k}(B(x, r) \cap \varphi^{-1}(z)) - \mathcal{H}^{n-k}(\Delta \cap \varphi^{-1}(z)).
\]

Hence, from (4.4), (4.3) and (4.5) we have
\[
\mathcal{H}^{n-k}(Q(x, r/\sqrt{n}) \cap \varphi^{-1}(z)) \geq (1 - \varepsilon)\mathcal{H}^{n-k}(B_{n-k}(x, r)) - (1 + \varepsilon)n^{-k}\mathcal{H}^{n-k}(\Delta_{n-k})
\]
and
\[
\mathcal{H}^{n-k}(Q(x, r/\sqrt{n}) \cap \varphi^{-1}(z)) \leq (1 + \varepsilon)n^{-k}\mathcal{H}^{n-k}(Q_{n-k}(x, r/\sqrt{n})) + C_1\varepsilon \cdot \mathcal{H}^{n-k}(B_{n-k}(x, r)),
\]
where \(\varepsilon \to 0\) as \(r \to 0\). Letting \(r\) tend to 0, the lemma follows.

\section*{Lemma 4.11.} Suppose that \(E \subset \mathbb{R}^n\) is a measurable set, \((X, d_X)\) is a metric space, and \(\varphi \in \text{Lip}(E, X)\) is such that \(\text{dim ker}(MD(\varphi, x)) \geq n - k\) almost everywhere. Assume that the coarea formula holds for every measurable subset \(A \subset E\). Then, for each \(\varepsilon > 0\), there exists a measurable set \(\Sigma_E \subset E\) such that \(\mathcal{H}^n(\Sigma_E) < \varepsilon\) and
\[
\frac{1}{2^{n-k}n^{-k}} \int_{\varphi^{-1}(z) \cap Q(x, r) \cap E} d\mathcal{H}^{n-k}(u) = 1 + o(1)
\]
for every \(x \in E \setminus \Sigma_E\) as \(r \to 0\) and \(o(1)\) is uniform on \(E \setminus \Sigma_E\). Here the faces of the cube \(Q(x, r)\) are either parallel or orthogonal to \(\text{ker}(MD(\varphi, x))\).

\section*{Proof.} Consider the measurable functions
\[
\Phi_m(x) = \left(\frac{m}{2}\right)^{n-k}\mathcal{H}^{n-k}\left(\varphi^{-1}(\varphi(x)) \cap Q\left(x, \frac{1}{m}\right)\right), \quad m \in \mathbb{N}.
\]
Then the functions \(\Phi_0(x) = \lim_{m \to \infty} \Phi_m(x)\) and \(\Phi^0(x) = \lim_{m \to \infty} \Phi_m(x)\) are also measurable. Hence, the sets
\[
D = \{x \in E: \Phi_0(x) = \Phi^0(x)\} = \{x \in E: \exists \lim_{m \to \infty} \Phi_m(x) = \Phi(x)\}
\]
and \(\{x \in D: \Phi(x) = 1\}\) are measurable. Since \(\varphi^{-1}(z) \cap E\) is \(\mathcal{H}^{n-k}\)-rectifiable set and \(\mathcal{H}^{n-k}(\varphi^{-1}(z) \cap E) < \infty\), it follows that, by Lemma 4.10, \(\mathcal{H}^{n-k}(\varphi^{-1}(z) \setminus D) = 0\) for all \(z \in X\). Hence, \(\mathcal{H}^n(E \setminus D) = 0\) by the coarea formula, and, \(\Phi_m(x) \to 1\) on \(E\) almost everywhere. Using Egorov's theorem, remove some set \(\Sigma_E\) of measure not exceeding \(\varepsilon\) such that the functions \(\Phi_m\)
converge uniformly to the unity on the complement of $\Sigma_E$. Thus, for each $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that $|\Phi_m(x) - 1| < \varepsilon$ for all $m \in \mathbb{N}$, $m \geq m_0$, and $x \in E \setminus \Sigma_E$.

Hence,

$$\frac{1}{2^{n-k} \rho^{n-k}} \int_{\varphi^{-1}(\varphi(x)) \cap Q(x,r) \cap E} d\mathcal{H}^{n-k}(u) = 1 + o(1),$$

as $r \to 0$, $x \in E \setminus \Sigma_E$, where $o(1)$ is uniform on $E \setminus \Sigma_E$. \(\square\)

**Remark 4.12.** Consider a point $x \in E$, the level set $\varphi^{-1}(\varphi(x)) \cap E$, and a cube $Q(x,r)$ with faces parallel or orthogonal to $\ker(MD(\varphi,x))$. Notice that $\varphi^{-1}(\varphi(x)) \cap E \cap Q(x,r)$ is a subset of an $o(r)$-neighborhood of the set $(x + \ker(MD(\varphi,x))) \cap Q(x,r)$. Consider a cube $Q(y,r)$, where the point $y$ is obtained from the point $x$ by adding a vector of $(\ker(MD(\varphi,x)))^\perp$. If $|x - y| < r - |o(r)|$ then, by Lemma 4.7, we have $\varphi^{-1}(\varphi(x)) \cap Q(x,r) \cap E = \varphi^{-1}(\varphi(x)) \cap Q(y,r) \cap E$.

5. The $m$-coarea factor and the structure of the image

In this section, we consider item 5 of Theorem 1.3. In proofs of the main results of this section, we use Lemmas 5.1–5.3.

**Lemma 5.1.** (See Besicovitch’s lemma [10].) Let $A$ be a bounded set in $\mathbb{R}^n$. Suppose that, for each $x \in A$, there exists a closed cube $Q(x)$ of a radius not exceeding $R$ ($0 < R < \infty$) with center at $x$. Then there exists an at most countable sequence $\{Q_k\} \subset \{Q(x)\}_{x \in A}$ such that

1. this sequence covers $A$, i.e., $A \subset \bigcup_k Q_k$;
2. every point in $\mathbb{R}^n$ is contained in at most $M(n)$ cubes of the sequence $\{Q_k\}$, i.e., $\sum_k \chi_{Q_k}(y) \leq M(n)$ for all $y \in \mathbb{R}^n$, where $M(n)$ depends only on $n$;
3. the sequence $\{Q_k\}$ can be decomposed into $\xi(n)$ families of disjoint cubes, where $\xi(n)$ depends only on $n$.

**Lemma 5.2.** (See [17].) Let $\| \cdot \|$ be a norm on $\mathbb{R}^n$. Then

$$\mathcal{H}_n(\| \cdot \|)(A) = J(\| \cdot \|)\mathcal{H}^n(A)$$

for all measurable sets $A \subset \mathbb{R}^n$, where

$$J(\| \cdot \|) = \omega_n n \left( \int_{\mathbb{R}^{n-1}} \| u \|^{-n} d\mathcal{H}^{n-1}(u) \right)^{-1}.$$

In the following lemma, $E_\mathcal{U} \subset E \subset \mathbb{R}^n$ stands for the set consisting of linear density points of $E$ in all directions $u \in \mathcal{U}$. We remark [8] that $\mathcal{H}^n(E_\mathcal{U}) = \mathcal{H}^n(E)$.

**Lemma 5.3** (The Tangent Mapping Lemma [13]). Assume that $E \subset \mathbb{R}^n$ and $\mathcal{C} \subset \mathcal{C}$ are measurable sets, $(\mathbb{X}, d_\mathbb{X})$ is a metric space, and $\varphi \in \text{Lip}(E, \mathbb{X})$. Suppose that $MD(\varphi, x)(u) \neq 0$ for all $x \in E_\mathcal{U} \cap \mathcal{C}$, $MD(\varphi, x)(u)$ is continuous on $E_\mathcal{U} \cap \mathcal{C}$, and the “difference quotients”
\[ d_{\mathbb{H}}(\varphi(x + ru), \varphi(x))/r \text{ converge to the } m \text{-differential } MD(\varphi, x)(u) \text{ for all } u \in \mathbb{S}^{n-1} \text{ uniformly on } \mathcal{E}. \]

Then the relation
\[ |d_{\mathbb{H}}(\varphi(y), \varphi(z)) - MD(\varphi, x)(y - z)| = o(MD(\varphi, x)(y - z)) \quad (5.1) \]

holds for all \( x, y, z \in E \cup \mathcal{E} \) such that \( y, z \in B(x, r) \). If \( K \subset E \cup \mathcal{E} \) is a compact set, then the \( o(1) \) is uniform on \( K \).

**Theorem 5.4.** Let \( E \subset \mathbb{R}^n \) be a measurable set, let \( (\mathbb{H}, d_{\mathbb{H}}) \) be a metric space, and let \( \varphi \in \text{Lip}(E, \mathbb{H}) \). Suppose that the equality
\[ J_k(MD(\varphi, x)) = \lim_{r \to 0} \frac{\mathcal{H}^k(\varphi(B(x, r)))}{\mathcal{H}^k(B_k(x, r))}, \quad (5.2) \]

where \( B_k \) is a \( k \)-dimensional ball, holds on a compact set \( K \subset E \subset \mathbb{R}^n \) such that every \( x \in K \) is a \( k \)-dimensional density point of \( E \) in all directions \( \mathbb{L}_v \in \mathcal{L} \) and \( \text{dim ker}(MD(\varphi, x)) = n - k \). Then \( \varphi(K) \) is an \( \mathcal{H}^k \)-rectifiable metric space.

**Proof.** Since \( \text{dim ker}(MD(\varphi, x)) = n - k \), it follows that \( J_k(MD(\varphi, x)) \neq 0 \). First, fix \( \varepsilon_0 > 0 \). Equality (5.2) implies that, for each \( x \in K \), there exist \( r(x, \varepsilon_0) > 0 \) and \( v = v(x, \varepsilon_0) \in \mathbb{N} \) such that
\[ \frac{\mathcal{H}^k(\varphi(B(x, r)))}{\mathcal{H}^k(B_k(x, r))} < 1 + \varepsilon_0 \]
at \( r \leq r(x, \varepsilon_0) \), where \( B_{k,v} \) is a \( k \)-dimensional ball lying in \( x + \mathbb{L}_v \) and \( \mathbb{L}_v \) is close to \( \ker(MD(\varphi, x)) \). Hence,
\[ \mathcal{H}^k(\varphi(B(x, r))) < (1 + \varepsilon_0)\mathcal{H}^k(\varphi(B_{k,v}(x, r))). \]

The open balls \( \{B(x, r): x \in K, 0 < r \leq r(x, \varepsilon_0) < \text{diam}(K)\} \) form an open covering of the compact set \( K \). Note that, by choice of the balls, we have \( C = \sup_{x,r}\{\mathcal{H}^k(\varphi(B(x, r)))\} < \infty \).

Choose a finite system \( B_0 = \{B(x_i, r_i)\}_{i=1}^L \) such that \( K \subset \bigcup_{i=1}^L B(x_i, r_i) \). Then \( \varphi(K) \subset \bigcup_{i=1}^L \varphi(B(x_i, r_i)) \),
\[ \mathcal{H}^k(\varphi(K)) \leq \sum_{i=1}^L \mathcal{H}^k(\varphi(B(x_i, r_i))) < (1 + \varepsilon_0)\sum_{i=1}^L \mathcal{H}^k(\varphi(B_{k,v_i}(x_i, r_i))) < \infty, \]

where \( v_i \) corresponds to \( x_i \), and
\[ \mathcal{H}^k(\varphi(K) \setminus \bigcup_{i=1}^L \varphi(B_{k,v_i}(x_i, r_i))) \leq \mathcal{H}^k\left(\bigcup_{i=1}^L \varphi(B(x_i, r_i)) \setminus \bigcup_{i=1}^L \varphi(B_{k,v_i}(x_i, r_i))\right) \]
\[ \leq \mathcal{H}^k\left(\bigcup_{i=1}^L \varphi(B(x_i, r_i)) \setminus \varphi(B_{k,v_i}(x_i, r_i))\right). \]
\[ \leq \sum_{l=1}^{L} \mathcal{H}^k(\varphi(B(x_l, r_l)) \setminus \varphi(B_{k,v_l}(x_l, r_l))) \]

\[ = \sum_{l=1}^{L} \mathcal{H}^k(\varphi(B(x_l, r_l))) - \mathcal{H}^k(\varphi(B_{k,v_l}(x_l, r_l))) \]

\[ < \varepsilon_0 \sum_{l=1}^{L} \mathcal{H}^k(\varphi(B_{k,v_l}(x_l, r_l))) \]

\[ \leq \varepsilon_0 \sum_{l=1}^{L} \mathcal{H}^k(\varphi(B(x_l, r_l))). \]

Next, at \( \varepsilon < \varepsilon_0, \varepsilon \to 0, \) we may assume without loss of generality, that we choose from \( \{B(x, r) : x \in K, 0 < r \leq r(x, \varepsilon) < \text{diam}(K)\} \) only the balls which are contained in \( \bigcup B_0. \) From here we have

\[
\mathcal{H}^k \left( \varphi(K) \setminus \bigcup_{l=1}^{L(\varepsilon)} \varphi(B_{k,v_l}(x_l, r_l)) \right) \leq \varepsilon \sum_{l=1}^{L(\varepsilon)} \mathcal{H}^k(\varphi(B(x_l, r_l)))
\]

\[ \leq \varepsilon \cdot M(n) \sum_{B(x_l, r_l) \in B_0} \mathcal{H}^k(\varphi(B(x_l, r_l))) \to 0 \quad \text{at} \quad \varepsilon \to 0, \]

where the constant \( M(n) \) of Lemma 5.1 depends only on the dimension. The theorem follows.

\[ \square \]

**Theorem 5.5.** Assume that \( E \subset \mathbb{R}^n \) is a measurable set, \( (\mathcal{X}, d_{\mathcal{X}}) \) is a metric space, \( \varphi \in \text{Lip}(E, \mathcal{X}), \) and \( \text{dimker}(MD(\varphi, x)) = n - k \) almost everywhere. Suppose that, on each intersection \( E \cap B(0, s), s \in \mathbb{R}, \) for each \( \varepsilon > 0, \) there exists a measurable set \( \Sigma_\varepsilon \subset E \) such that \( \mathcal{H}^n(\Sigma_\varepsilon) < \varepsilon \) and the equality

\[
\mathcal{J}_k(MD(\varphi, x)) = \lim_{r \to 0} \frac{\mathcal{H}^k(\varphi(B(x, r) \setminus \Sigma_\varepsilon))}{\mathcal{H}^k(B_k(x, r))}
\]

holds everywhere on the compact set \( E \cap B(0, s) \setminus \Sigma_\varepsilon. \) Then there exists a set \( \Sigma \subset E, \) \( \mathcal{H}^n(\Sigma) = 0, \) such that \( \varphi(E \setminus \Sigma) \) is an \( \mathcal{H}^k \)-rectifiable metric space.

Setting \( s_m = m, \varepsilon_m = \frac{1}{m}, K_m = E \cap B(0, m) \setminus \Sigma_m, \Sigma = \bigcap_{m \in \mathbb{N}} \Sigma_m \) and representing \( E \) as \( E = \bigcup_{m \in \mathbb{N}} (E \cap B(0, m)), \) from Theorem 5.4 with \( E \setminus \Sigma_\varepsilon \) instead of \( E, \) we obtain Theorem 5.5.

**Theorem 5.6.** Assume that \( E \subset \mathbb{R}^n \) is a measurable set, \( (\mathcal{X}, d_{\mathcal{X}}) \) is a metric space and \( \varphi \in \text{Lip}(E, \mathcal{X}). \) Suppose that for each intersection \( E \cap B(0, s), s \in \mathbb{R}, \) and each \( \varepsilon > 0, \) there exists a measurable set \( \Sigma_\varepsilon \subset E \cap B(0, s) \) and a collection of compact sets \( \{K_i\}_{i \in \mathbb{N}} \) such that \( \mathcal{H}^n(\Sigma_\varepsilon) < \varepsilon, (E \cap B(0, s)) \setminus (\Sigma_\varepsilon \cup Z) \subset \bigcup_{i \in \mathbb{N}} K_i, \) and the equality

\[
\lim_{v \to \infty} \lim_{r \to 0} \frac{\mathcal{H}^k(\varphi|_{K_i}(B(x, r) \setminus (Z \cup \Sigma_\varepsilon)))}{\mathcal{H}^k(\varphi|_{K_i}(B_{k,v}(x, r) \setminus (Z \cup \Sigma_\varepsilon)))} = 1.
\]
where \( B_{k,\nu}(x, r) \) is a \( k \)-dimensional ball of radius \( r \) in \( x + \mathbb{L}_\nu \subset \mathbb{R}^n \), \( \mathbb{L}_\nu \to (\ker(MD(\varphi, x)))^\perp \), holds almost everywhere on \( (E \cap B(0, s) \cap K_i) \setminus (\Sigma_g \cup Z) \) for all \( i \in \mathbb{N} \). Then, \( \mathcal{X} \) is a union of an \( \mathcal{H}^k \)-rectifiable metric space, \( \varphi(Z) \) and an image of a set of measure zero.

**Proof.** Fix \( \varepsilon > 0 \). Put \( E = E \setminus Z \), \( E_j = K_j \cap E \setminus E_i \) and \( \varphi_i = \varphi|_{E_j} \). Denote by \( \Sigma_{E_i} \) the subset of \( K_i \) of measure zero on which (5.3) does not hold and on which \( J_k(MD(\varphi_i, x)) \neq \lim_{r \to 0} \frac{\mathcal{H}^k(\varphi_i(B_k(x, r)))}{\mathcal{H}^k(B_k(x, r))} \). Next, choose a sequence \( \{E_{ij}\}_{j \in \mathbb{N}} \) of compact sets \( E_{ij} \subset E_{i} \setminus \Sigma_{E_i} \) with \( \mathcal{H}^0((E_i \setminus \Sigma_{E_i}) \setminus E_{ij}) < \frac{1}{j}, j \in \mathbb{N} \). Thus, we have

\[
J_k(MD(\varphi_i, x)) = \lim_{r \to 0} \frac{\mathcal{H}^k(\varphi_i(B_k(x, r)))}{\mathcal{H}^k(B_k(x, r))}
\]

everywhere on \( E_{ij} \cap B(0, s) \) for all \( j \in \mathbb{N} \). From here and from Theorems 5.4 and 5.5 we obtain the existence of a set \( \Sigma \subset E \) of \( \mathcal{H}^n \)-measure zero and the \( \mathcal{H}^k \)-rectifiability of \( \varphi(E \setminus (\Sigma \cup Z)) \).

**Lemma 5.7.** Assume that \( E \subset \mathbb{R}^n \) and \( A \subset \mathbb{R}^k \) are measurable sets, \( K \subset E \) is a compact set, and \( \mathcal{X} \) is a metric space such that \( \mathcal{X} = \alpha(A) \), where \( \alpha : A \to \mathcal{X} \) is a bi-Lipschitz continuously \( m \)-differentiable mapping on \( A \). Suppose also that \( \varphi \in \text{Lip}(E, \mathcal{X}) \) is a continuously \( m \)-differentiable mapping on \( E \) and \( \alpha^{-1} \circ \varphi \in \text{Lip}(E, A) \) is a continuously differentiable mapping. Assume that every point \( x \in K \) is a \( k \)-dimensional density point of \( E \) in every \( \mathbb{L}_\nu \in \mathcal{L} \). Then

\[
\lim_{r \to 0} \frac{\mathcal{H}^k((\alpha^{-1} \circ \varphi)(B(x, r) \cap E))}{\mathcal{H}^k(B_k(x, r))} = J_k(\alpha^{-1} \circ \varphi, x)
\]

for all \( x \in K \).

**Proof.** Consider \( x \in K \) and \( \mathbb{L}_\nu \) close to \( (\ker(D(\alpha^{-1} \circ \varphi)(x)))^\perp \). Put \( B_{k,\nu}(r) = B(x, r) \cap (x + \mathbb{L}_\nu) \). Note that, \( J_k(\alpha^{-1} \circ \varphi, x) = \lim_{r \to 0} J_k(\alpha^{-1} \circ \varphi|_{x + \mathbb{L}_\nu}, x) \) for all \( x \in K \).

Using the Kirsbraun Extension Theorem, extend the mapping \( \alpha^{-1} \circ \varphi \) to a \( C^1 \)-mapping defined on \( \mathbb{R}^n \). Denote this extension by \( \psi \). The mapping \( \alpha^{-1} \circ \varphi|_{B_k,\nu(r)} \) is bi-Lipschitz if \( r > 0 \) is small enough. Since the partial derivatives of \( \psi \) coincide \( \mathcal{H}^k \)-almost everywhere on \( B_{k,\nu}(r) \cap E \) with those of \( \alpha^{-1} \circ \varphi \) and these partial derivatives are uniformly continuous on the compact set \( B_{k,\nu}(r) \) if \( r > 0 \) is sufficiently small, the restriction \( \psi|_{B_{k,\nu}(r)} \) is bi-Lipschitz. Hence, \( MD(\psi|_{B_{k,\nu}(r)}, x) \) is a norm on \( \mathbb{R}^k \). Moreover, for \( y \in K \) we have \( J_k(\psi|_{x + \mathbb{L}_\nu}, y) = J_k(\alpha^{-1} \circ \varphi, y) + \delta(y, v) \), where \( \delta(y, v) \to 0 \) as \( v \to \infty \). In particular, \( J_k(\psi|_{x + \mathbb{L}_\nu}, x) = J_k(\alpha^{-1} \circ \varphi, x) + \delta(x, v) \).

Recall that \( \psi \) is defined on \( \mathbb{R}^n \). From Lemma 5.2 it follows that

\[
\mathcal{H}^0_{MD(\psi|_{x + \mathbb{L}_\nu}, x)}(D) = \mathcal{J}(MD(\psi|_{x + \mathbb{L}_\nu}, x)) \mathcal{H}^k(D)
\]

for every measurable \( D \subset B_{k,\nu}(r) \). By Lemma 5.3, we have

\[
(1 - o(1))^k \mathcal{H}^k_{MD(\psi|_{x + \mathbb{L}_\nu}, x)}(D) \leq \mathcal{H}^k(\psi(D)) \leq (1 + o(1))^k \mathcal{H}^k_{MD(\psi|_{x + \mathbb{L}_\nu}, x)}(D)
\]

(here the Hausdorff measure \( \mathcal{H}^k_{MD(\psi|_{x + \mathbb{L}_\nu}, x)} \) is considered with respect to the metric defined by the norm \( MD(\psi|_{x + \mathbb{L}_\nu}, x) \) on \( \mathbb{R}^k \)). Hence,
\[
\frac{(1 - o(1))^k \mathcal{H}^k_{MD(\psi|x+L_v,x)}(B_k,v(r) \cap E)}{(1 + o(1))^k \mathcal{H}^k_{MD(\psi|x+L_v,x)}(B_k,v(r))} \leq \frac{\mathcal{H}^k(\psi(B_k,v(r) \cap E)}{\mathcal{H}^k(\psi(B_k,v(r)))} \leq \frac{(1 + o(1))^k \mathcal{H}^k_{MD(\psi|x+L_v,x)}(B_k,v(r) \cap E)}{(1 - o(1))^k \mathcal{H}^k_{MD(\psi|x+L_v,x)}(B_k,v(r))},
\]

where \( o(r) \to 0 \) as \( r \to 0 \). Therefore,

\[
\lim_{r \to 0} \frac{\mathcal{H}^k(\psi(B_k,v(r) \cap E)}{\mathcal{H}^k(\psi(B_k,v(r)))} = \lim_{r \to 0} \frac{\mathcal{H}^k_{MD(\psi|x+L_v,x)}(B_k,v(r) \cap E)}{\mathcal{H}^k_{MD(\psi|x+L_v,x)}(B_k,v(r))} = \lim_{r \to 0} \frac{\mathcal{J}(MD(\psi|x+L_v,x))\mathcal{H}^k(B_k,v(r) \cap E)}{\mathcal{J}(MD(\psi|x+L_v,x))\mathcal{H}^k(B_k,v(r))} = 1 \quad (5.4)
\]

since \( x \in K \) is a density point of \( E \) in the \( k \)-dimensional direction \( \mathbb{L}_v \). Thus,

\[
\mathcal{J}_k(\alpha^{-1} \circ \psi|_{x+L_v,x}) = \lim_{r \to 0} \frac{\mathcal{H}^k(\psi(B_k,v(r) \cap E)}{\mathcal{H}^k(B_k(x,r))} = \lim_{r \to 0} \frac{\mathcal{H}^k(\psi(B_k,v(r)))}{\mathcal{H}^k(B_k(x,r))}.
\]

It is left to prove that \( \mathcal{H}^k(\psi(B_k,v(r))) \) can be replaced by \( \mathcal{H}^k(\psi(B(x,r))) \). Next, the image of the boundary of a \( k \)-dimensional sphere under \( \psi \) is the boundary of the image. Show that \( \partial(\psi(B(x,r))) \) is a subset of \( (o(r) + r\delta(x,v)) \)-neighborhood of \( \partial(\psi(B_k,v)) \).

Note that, \( \psi(x + ru) = \psi(x) + D\psi(x)(ru) + o(r) \), where \( u \in \mathbb{S}^{n-1} \). From here it follows that \( \partial(\psi(B_k,v(x,r))) \) is a subset of \( o(r) \)-neighborhood of \( \psi(x) + \partial(D\psi(x)(B_k,v(0,r))) \), and \( \psi(x) + \partial(D\psi(x)(B_k,v(0,r))) \) is a subset of \( o(r) \)-neighborhood of \( \partial(\psi(B_k,v(x,r))) \). The same is true for \( B(x,r) \). Thus, it is sufficient to show that \( \partial(D\psi(x)(B(0,r))) \) is a subset of \( r\delta(x,v) \)-neighborhood of \( \partial(D\psi(x)(B_k,v(0,r))) \). Here \( o(r) \to 0 \) as \( r \to 0 \) and \( \delta(x,v) \to 0 \) as \( v \to \infty \).

Since every unit vector \( u \in \mathbb{S}^{n-1} \) can be uniquely represented as \( u = \alpha_1 v_1 + \alpha_2 v_2 \), where \( 0 \leq \alpha_1, \alpha_2 \leq 1 \), \( v_1 \in (\ker(MD(\psi,x)))^\perp \), \( v_2 \in \ker(MD(\psi,x)) \), \( |v_1| = |v_2| = 1 \), it follows that

\[
D\psi(x)(u) = D\psi(x)(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 D\psi(x)(v_1).
\]

Thus, we have that, for each \( u \in \mathbb{S}^{n-1} \), there exists \( v_1 \in (\ker(MD(\psi,x)))^\perp \cap \mathbb{S}^{n-1} \) such that \( D\psi(x)(u) = \alpha_1 D\psi(x)(v_1) \), \( \alpha_1 \leq 1 \). Consequently, \( \delta(D\psi(x)(B(0,r))) = \delta(D\psi(x)(B(0,r) \cap (\ker(MD(\psi,x)))^\perp)) \). In particular, every unit vector \( u \in \mathbb{L}_v \) can be uniquely represented this way. Note that, \( \alpha_2 \to 0 \) and \( \alpha_1 \to 1 \) as \( v \to \infty \), and

\[
|D\psi(x)(u) - D\psi(x)(v_1)| = |D\psi(x)(\alpha_1 v_1) - D\psi(x)(v_1)| \leq \text{Lip}(\psi)(1 - \alpha_1) \to 0
\]

as \( v \to \infty \).

Therefore, for each \( u \in \mathbb{L}_v \), there exists \( v_1 \in (\ker(MD(\psi,x)))^\perp \cap \mathbb{S}^{n-1} \) such that

\[
|D\psi(x)(u) - D\psi(x)(v_1)| \leq \delta(x,v),
\]

where \( \delta(x,v) \to 0 \) as \( v \to \infty \).
From here we have that \( \partial(\psi(B(x, r))) \) is a subset of \( (\rho(r) + r\delta(x, v)) \)-neighborhood of \( \partial(\psi(B_k, v(x, r))) \). Note that \( \mathcal{H}^k(\partial B_k(v(x, r))) = 0 \) and \( \mathcal{H}^{k-1}(\partial B_k(v(x, r))) < \infty \). Consequently, \( \mathcal{H}^k(\psi(B(x, r)) \setminus \psi(B_k, v(x, r))) = o(r^k) + r^k\delta(x, v) \) as \( r \to 0 \). Passage to the limit as \( v \to \infty \) yields the lemma:

\[
\mathcal{J}_k(\alpha^{-1} \circ \varphi, x) = \lim_{v \to \infty} \mathcal{J}_k(\alpha^{-1} \circ \varphi|_{x + \mathbb{Z}_v}, x) = \lim_{v \to \infty \sigma r \to 0} \frac{\mathcal{H}^k(\psi(B_k, v(r) \cap E)}{\mathcal{H}^k(B_k(x, r))}
\]

\[
= \lim_{r \to 0} \frac{\mathcal{H}^k(\psi(B(x, r) \cap E)}{\mathcal{H}^k(B_k(x, r))}
\]

in view of the inclusion

\[
(\alpha^{-1} \circ \varphi)(B_k, v(r))) = \psi(B_k, v(r) \cap E) \subset (\alpha^{-1} \circ \varphi)(B(x, r) \cap E) \subset \psi(B(x, r))
\]

and relation (5.4).

**Theorem 5.8.** Suppose that \( E \subset \mathbb{R}^n \) is a measurable set, \( \varphi \in \text{Lip}(E, \mathbb{X}) \), and \( \mathbb{X} \) is a union of an \( \mathcal{H}^k \)-rectifiable metric space and an image of a set \( \Sigma \) with \( \mathcal{H}^n(\Sigma) = 0 \). Then, for each intersection \( E \cap B(0, s) \), \( s \in \mathbb{R} \), and each \( \varepsilon > 0 \), there exists a measurable set \( \Sigma_\varepsilon \subset E \cap B(0, s) \) and a collection of compact sets \( \{K_i\}_{i \in \mathbb{N}} \) such that \( \mathcal{H}^n(\Sigma_\varepsilon) < \varepsilon \), \( (E \cap B(0, s)) \setminus (\Sigma_\varepsilon \cup Z) \subset \bigcup_{i \in \mathbb{N}} K_i \), and the equality

\[
\lim_{v \to \infty \sigma r \to 0} \frac{\mathcal{H}^k(\varphi|_{k}(B_k, v(x, r) \setminus (Z \cup \Sigma_\varepsilon)))}{\mathcal{H}^k(B_k(x, r) \setminus (Z \cup \Sigma_\varepsilon)))} = 1,
\]

where \( B_k, v(x, r) \) is a \( k \)-dimensional ball of radius \( r \) in \( x + \mathbb{L}_v \subset \mathbb{R}^n \), \( \mathbb{L}_v \varepsilon \to (\text{ker}(\text{MD}(\varphi, x))) \setminus \), holds almost everywhere on \( (E \cap B(0, s) \cap K_i) \setminus (\Sigma_\varepsilon \cup Z) \) for all \( i \in \mathbb{N} \).

**Proof.** Fix \( s \in \mathbb{R} \). Below we assume without loss of generality that \( E \subset B(0, s) \). The proof of Theorem 1.2 and [15] imply that there exists a compact subset \( \mathcal{E} \subset E \) satisfying the conditions of Theorem 2.3 with \( \mathcal{H}^n((E \setminus Z) \setminus E) < \varepsilon \), in particular, \( \text{dim ker}(\text{MD}(\varphi, x)) \geq n - k \) almost everywhere. Put \( \Sigma_\varepsilon = (E \setminus Z) \setminus E \). By Lemma 3.7, we may assume that \( \Sigma \subset E \setminus E \). From the definition of an \( \mathcal{H}^k \)-rectifiable metric space and Lemma 3.7 it follows that we can suppose that \( \mathbb{X} = \alpha(A) \), where \( A \subset \mathbb{R}^k \) is a compact set, and \( \alpha : A \to \mathbb{X} \) is a continuously \( m \)-differentiable bi-Lipschitz mapping. Moreover, we may assume that \( \mathcal{E} \) is such that \( \alpha^{-1} \circ \varphi \) is a continuously differentiable mapping, and \( \mathcal{E} = \varphi^{-1} \circ \alpha(A) \). Choose a sequence of \( k \)-dimensional directions \( \{\mathbb{L}_v\}_{v \in \mathbb{N}} \) converging to \( (\text{ker}(\text{MD}(\varphi, x))) \setminus \) as \( v \to \infty \). Note that, almost every \( x \in \mathcal{E} \) is a \( k \)-dimensional density point of \( E \) in the direction \( \mathbb{L}_v \). Take such a point \( x \in \mathcal{E} \). Put \( V_v = \varphi(B_k, v(x, r) \cap E) \), where \( B_k, v(x, r) \) is a \( k \)-dimensional ball of radius \( r \) in \( x + \mathbb{L}_v \subset \mathbb{R}^n \), and \( U = \varphi(B(x, r) \cap E) \). We see that, \( U = \alpha(\alpha^{-1}(U)) \), \( \text{MD}(\alpha, y) \) is a norm on \( \mathbb{R}^k \), and by Lemma 5.2

\[
\mathcal{H}^k_{\text{MD}(\alpha, y)}(\alpha^{-1}(U)) = \mathcal{J}(\text{MD}(\alpha, y))\mathcal{H}^k(\alpha^{-1}(U)).
\]  

Consider the set \( A_{\mathcal{U}} \) (see the definition in the beginning of the section). Note that [8], \( \mathcal{H}^k(A \setminus A_{\mathcal{U}}) = 0 \). Since \( \alpha \) possesses the \( \mathcal{N} \)-property, then the set \( A \setminus A_{\mathcal{U}} \) does not influence the result of (5.7) (see below), and we may assume without loss of generality that \( \alpha^{-1}(U) \subset A_{\mathcal{U}} \) and \( \alpha^{-1}(V_v) \subset A_{\mathcal{U}} \) for all \( v \in \mathbb{N} \).
Thus, in view of the Tangent Mapping Lemma 5.3,
\[(1 - o(1)) \frac{\mathcal{H}^k_{MD(\alpha,Y)}(\alpha^{-1}(U))}{H^k(U)} \leq \mathcal{H}^k(U) \leq (1 + o(1)) \frac{\mathcal{H}^k_{MD(\alpha,Y)}(\alpha^{-1}(U))}{H^k(U)}.
\]

Similarly, we obtain the same result for \(V\):
\[(1 + o(1)) \frac{\mathcal{H}^k_{MD(\alpha,Y)}(\alpha^{-1}(V))}{H^k(V)} \geq \mathcal{H}^k(V) \geq (1 - o(1)) \frac{\mathcal{H}^k_{MD(\alpha,Y)}(\alpha^{-1}(V))}{H^k(V)}.
\]

Hence,
\[(1 - o(1)) \frac{\mathcal{H}^k_{MD(\alpha,Y)}(\alpha^{-1}(U))}{H^k(U)} \leq \mathcal{H}^k(U) \leq (1 + o(1)) \frac{\mathcal{H}^k_{MD(\alpha,Y)}(\alpha^{-1}(U))}{H^k(U)} \leq (1 - o(1)) \frac{\mathcal{H}^k_{MD(\alpha,Y)}(\alpha^{-1}(V))}{H^k(V)} \leq (1 + o(1)) \frac{\mathcal{H}^k_{MD(\alpha,Y)}(\alpha^{-1}(V))}{H^k(V)}.
\]

By (5.5), Lemma 5.7, the fact that
\[J_k(MD(\alpha^{-1} \circ \varphi, x)) = \lim_{\nu \to \infty} J_{k,\nu}(MD(\alpha^{-1} \circ \varphi, x)),\]
where
\[J_{k,\nu}(MD(\alpha^{-1} \circ \varphi, x)) \overset{\text{def}}{=} \omega_k \left( \int_{\mathbb{R}^{n-k} \cap E} [MD(\alpha^{-1} \circ \varphi)(u)]^{-k} d\mathcal{H}^{n-k}(u) \right)^{-1},\]

and the relation
\[J_{k,\nu}(MD(\alpha^{-1} \circ \varphi, x)) = \lim_{r \to 0} \frac{\mathcal{H}^k((\alpha^{-1} \circ \varphi)(B_{k,\nu}(x, r) \cap E))}{\mathcal{H}^k(B_k(x, r))},\]

we have
\[\lim_{\nu \to \infty} \lim_{r \to 0} \frac{\mathcal{H}^k(\alpha^{-1}(V_\nu))}{\mathcal{H}^k(\alpha^{-1}(U))} = 1.\]

Since \(o(1) \to 0\) as \(r \to 0\), we have
\[\lim_{\nu \to \infty} \lim_{r \to 0} \frac{\mathcal{H}^k(\varphi(B_{k,\nu}(x, r) \cap E))}{\mathcal{H}^k(B(x, r) \cap E)} = \lim_{\nu \to \infty} \lim_{r \to 0} \frac{\mathcal{H}^k_{MD(\alpha,Y)}(\alpha^{-1}(V_\nu))}{\mathcal{H}^k_{MD(\alpha,Y)}(\alpha^{-1}(U))} = \lim_{\nu \to \infty} \lim_{r \to 0} \frac{\mathcal{H}^k(\alpha^{-1}(V_\nu))}{\mathcal{H}^k(\alpha^{-1}(U))} = 1.
\]

In the general case, when \(\mathcal{H}^k(\mathbb{R} \setminus \alpha_j(A_j)) = 0\), we obtain the existence of a collection of compact sets \(\{K_i\}_{i \in \mathbb{N}}\) by representing \(A_j\) as a union of a countable collection of “regular” parts (i.e., possessing the properties described above) and a set of measure zero, and considering their preimages under the mappings \(\varphi^{-1} \circ \alpha_j\).

**Theorem 5.9.** (See [12,15].) Suppose that \(E\) is a measurable set in \(\mathbb{R}^n\), \((\mathbb{X}, d_\mathbb{X})\) is an \(\mathcal{H}^k\)-rectifiable metric space, \(n \geq k\), and \(\varphi \in \text{Lip}(E, \mathbb{X})\). Then the formula
\[
\int_E f(x) J_k(MD(\varphi, x)) \, dx = \int_{\mathbb{X}} d\mathcal{H}^k(z) \int_{\varphi^{-1}(z)} f(u) \, d\mathcal{H}^{n-k}(u) \quad (5.8)
\]
holds for every function \( f : E \to \mathbb{E} \) (where \( \mathbb{E} \) is an arbitrary Banach space) such that \( f(x)J_k(MD(\varphi(x))) \) is integrable.

**Corollary 5.10.** The coarea formula (5.8) is valid for any set \( A \subset E \) such that \( A \setminus Z \) is measurable.

**Proof of Theorem 1.3.** (The case of \( \mathbb{Y} = E \subset \mathbb{R}^n \).) Observe that \( 1 \Leftrightarrow 2 \) was established in Remark 1.6; the implication \( 1 \Rightarrow 4 \) follows from Lemmas 3.7–3.10, Corollary 4.3, and Theorem 4.1; \( 3 \Leftrightarrow 4 \) follows from Lemma 3.7; \( 4 \Rightarrow 1 \) follows from Lemmas 3.8, 3.9, Theorem 5.9, and Corollary 5.10; \( 4 \Leftrightarrow 5 \) is a consequence of Lemmas 3.9, 3.10, and Theorems 5.6 and 5.8.

Thus, we have proved the following equivalences:

\[
1 \Leftrightarrow 2, \quad 1 \Leftrightarrow 4, \quad 3 \Leftrightarrow 4, \quad 4 \Leftrightarrow 5. \quad \square
\]

### 6. Coarea formula as a countably additive set function

Suppose that \( E \subset \mathbb{R}^n \) is a measurable set, \((\mathbb{X}, d_\mathbb{X})\) is a metric space, and \( \varphi \in \text{Lip}(E, \mathbb{X}) \) is such that \( \dim \ker(MD(\varphi, x)) \geq n - k \) almost everywhere.

Consider some set \( E \subset E \) and the set function \( \Phi \) defined on Borel sets in \( \mathbb{R}^n \) as follows:

\[
A \mapsto \Phi(A) = \int_{\mathbb{X}} dH^k(z) \int_{\varphi^{-1}(z) \cap A \cap E} dH^{n-k}(u). \quad (6.1)
\]

It follows from Lemma 3.8 that \( \Phi \) is absolutely continuous. Show that \( \Phi \) is countably additive. Indeed, let \( A_1 \) and \( A_2 \) be disjoint Borel sets in \( \mathbb{R}^n \). Then

\[
\Phi(A_1) + \Phi(A_2) = \int_{\mathbb{X}} dH^k(z) \int_{\varphi^{-1}(z) \cap A_1 \cap E} dH^{n-k}(u) + \int_{\mathbb{X}} dH^k(z) \int_{\varphi^{-1}(z) \cap A_2 \cap E} dH^{n-k}(u)
\]

\[
= \int_{\mathbb{X}} dH^k(z) \left[ \int_{\varphi^{-1}(z) \cap A_1 \cap E} dH^{n-k}(u) + \int_{\varphi^{-1}(z) \cap A_2 \cap E} dH^{n-k}(u) \right]
\]

\[
= \int_{\mathbb{X}} dH^k(z) \int_{\varphi^{-1}(z) \cap (A_1 \cup A_2) \cap E} dH^{n-k}(u) = \Phi(A_1 \cup A_2). \quad (6.2)
\]

For a disjoint collection \( \{A_m\}_{m \in \mathbb{N}} \) of Borel sets, the equality \( \Phi(\bigcup_{m \in \mathbb{N}} A_m) = \sum_{m \in \mathbb{N}} \Phi(A_m) \) follows from (6.2) and B. Levi’s theorem.

Hence (see, for example [30]), \( \Phi \) is differentiable almost everywhere, i.e., the limit

\[
\lim_{\delta \to 0, Q_\delta \exists x} \frac{\Phi(Q_\delta)}{H^n(Q_\delta)} = \Phi'(x), \quad (6.3)
\]

where \( Q_\delta \) is an arbitrary cube of radius \( \delta \), exists almost everywhere.
Theorem 6.1. Let $E \subset \mathbb{R}^n$ be a measurable set, let $(\mathbb{X}, d_{\mathbb{X}})$ be a metric space, and let $\varphi \in \text{Lip}(E, \mathbb{X})$ be such that $\dim \ker(MD(\varphi, x)) \geq n - k$ almost everywhere. Denote by $D_E$ the set on which the functions $\Phi_m$, 

$$\Phi_m(x) = \frac{m^{n-k}}{2^{n-k}} \int_{\varphi^{-1}(\varphi(x)) \cap Q(x, 1/m) \cap E} d\mathcal{H}^{n-k}(u), \quad x \in E,$$

converge to unity. Then the coarea formula holds for every measurable set $A \subset E \setminus Z$ if and only if $\mathcal{H}^n(E \setminus D_E) = 0$ for every $E \subset E$.

Proof. Assume that $\mathcal{H}^n(E \setminus D_E) = 0$ for every $E \subset E$. By Lemma 3.8, for proving the validity of the coarea formula, it suffices to consider the case of a compact set $A$. It is known that if $A \subset \mathbb{R}^n$ is measurable and $\mathcal{H}^n(A) < \infty$ then the function $\Xi_A(z) = \int_{\varphi^{-1}(\varphi(z)) \cap A} d\mathcal{H}^{n-k}(u)$ is $\mathcal{H}^k$-measurable on $\mathbb{X}$ (see [8, 2.10.26]). Then, repeating the arguments of Lemma 4.6, we conclude that the functions $\Phi_m(x), x \in E$, are measurable for all $m \in \mathbb{N}$ greater than some $m_0 \in \mathbb{N}$. Assume that $D_E \subset E$ is the set on which $\Phi_m \to 1$ (it is measurable, see Lemma 4.11), $D_l \subset D_E$ is a compact set on which $\Phi_m \Rightarrow 1$ such that $\mathcal{H}^n(D_E \setminus D_l) < \frac{1}{l}$, $A \subset D_l$ is an arbitrary compact set, and $Q(x, r)$ is an $n$-dimensional cube meeting the conditions of Theorem 4.1. We can suppose without loss of generality that $D_l \subset D_{l+1}$ for all $l \in \mathbb{N}$.

Let us show that $J_k(MD(\varphi, x)) = \Phi'(x)$ for $x \in D_l$.

Repeating the arguments of item II in the proof of Theorem 6.22 in [15], we infer

$$\varphi(\mathcal{Q}(t, 0) \cap E) = \varphi(\mathcal{Q}_{k,v}(x, r)) \cup \varphi(\mathcal{Q}_e)$$

for almost all $x \in D_l$ at $r < r_0(x, \varepsilon)$, where $\mathcal{Q}_{k,v}(x, r) \subset (x + \mathbb{L}_v) \cap \overline{Q(x, r)} \cap D_l$ is compact, the $k$-dimensional direction $\mathbb{L}_v \in \mathcal{L}$ is sufficiently close to $(\ker(MD(\varphi, x)))^\perp$, $\mathcal{Q}_{k,v}(x, r)$ has $k$-dimensional density 1 at the point $x$, the values $\Phi_m(y)$ are uniformly close to the unity for all $y \in \mathcal{Q}_{k,v}(x, r)$, and $\mathcal{H}^n(\mathcal{Q}_e) < \varepsilon \mathcal{H}^n(\mathcal{Q}(x, r))$, where $\varepsilon \to 0$ as $r \to 0$ and $v \to \infty$. Moreover,

$$\int_{\varphi(\mathcal{Q}_{k,v}(x, r))} d\mathcal{H}^k(z) \int_{\varphi^{-1}(\varphi(z)) \cap \overline{\mathcal{Q}_{k,v}(x, r)} \cap E} d\mathcal{H}^{n-k}(u) \leq \int_{\varphi(\mathcal{Q}(x, r) \cap E)} d\mathcal{H}^k(z) \int_{\varphi^{-1}(\varphi(z)) \cap \overline{Q(x, r)} \cap E} d\mathcal{H}^{n-k}(u) \leq \int_{\varphi(\mathcal{Q}_{k,v}(x, r))} d\mathcal{H}^k(z) \int_{\varphi^{-1}(\varphi(z)) \cap \overline{\mathcal{Q}_{k,v}(x, r)} \cap E} d\mathcal{H}^{n-k}(u) + \int_{\varphi(\mathcal{Q}_e)} d\mathcal{H}^k(z) \int_{\varphi^{-1}(\varphi(z)) \cap \overline{\mathcal{Q}_e}} d\mathcal{H}^{n-k}(u),$$

and, by Lemma 3.8,
\[
\frac{1}{2^{n} r^{n}} \int_{\varphi^{-1}(z) \cap Q_{\varepsilon}} dH^{k}(z) \int_{\varphi^{-1}(z) \cap Q_{\varepsilon}} dH^{n-k}(u) \leq \frac{(\text{Lip}(\varphi))^{k} \omega_{n-k} \omega_{k}}{\omega_{n} 2^{n} r^{n}} H^{n}(Q_{\varepsilon}) \leq \varepsilon \frac{(\text{Lip}(\varphi))^{k} \omega_{n-k} \omega_{k}}{\omega_{n} 2^{n} r^{n}} H^{n}(Q(x, r)) \rightarrow 0
\]
as \( r \rightarrow 0 \) and \( \nu \rightarrow \infty \). Hence,

\[
\Phi'(x) = \lim_{r \rightarrow 0} \frac{1}{2^{n} r^{n}} \int_{\varphi^{-1}(z) \cap (Q_{k, v}(x, r))} dH^{k}(z) \int_{\varphi^{-1}(z) \cap (Q_{k, v}(x, r))} dH^{n-k}(u)
\]

where \( Q_{k, v}^{n} = \varphi^{-1}(z) \cap (\varphi^{-1}(\varphi(Q_{k, v}(x, r)))) \cap Q(x, r) \cap E \). Furthermore, the last integral can be represented as

\[
\frac{1}{2^{k} r^{k}} \int_{\varphi^{-1}(z) \cap (Q_{k, v}(x, r))} dH^{k}(z) \frac{1}{2^{n-k} r^{n-k}} \int_{Q_{k, v}^{n}} dH^{n-k}(u)
\]

Since \( \frac{\mathcal{H}^{k}(\varphi(Q_{k, v}(x, r)))}{2^{k} r^{k}} \leq (\text{Lip}(\varphi))^{k} \) and \( o(1) \rightarrow 0 \) as \( r \rightarrow 0 \) and \( \nu \rightarrow \infty \), we have

\[
\lim_{\nu \rightarrow \infty} \lim_{r \rightarrow 0} \frac{1}{2^{k} r^{k}} \int_{\varphi^{-1}(z) \cap (Q_{k, v}(x, r))} dH^{k}(z) \frac{1}{2^{n-k} r^{n-k}} \int_{Q_{k, v}^{n}} dH^{n-k}(u) = \lim_{\nu \rightarrow \infty} J_{k,v}(MD(\varphi, x)).
\]

Since \( MD(\varphi, x)(u) \in \text{Lip}(S^{n-1}) \) for all \( x \in E \), it follows that \( J_{k,v}(MD(\varphi, x)) \rightarrow J_{k}(MD(\varphi, x)) \) as \( \nu \rightarrow \infty \).

Thus, the equality \( J_{k}(MD(\varphi, x)) = \Phi'(x) \) holds at almost all points \( x \in D_{l} \) of differentiability of the set function \( \Phi(A) \), and the coarea formula holds for every measurable set \( A \subset D_{l} \).

If \( A \subset D_{E} \) then, using the countable additivity and the absolute continuity of \( \Phi \) [30], we obtain

\[
\int_{A} J_{k}(MD(\varphi, x)) \, dx = \lim_{m \rightarrow \infty} \int_{A \cap D_{m}} J_{k}(MD(\varphi, x)) \, dx
\]
\[
\lim_{m \to \infty} \int_X d\mathcal{H}^k(z) \int_{\varphi^{-1}(z) \cap A \cap D_m} d\mathcal{H}^{n-k}(u) = \int_X d\mathcal{H}^k(z) \int_{\varphi^{-1}(z) \cap \bigcup_{m \in \mathbb{N}} D_m} d\mathcal{H}^{n-k}(u) = \int_X d\mathcal{H}^k(z) \int_{\varphi^{-1}(z) \cap A} d\mathcal{H}^{n-k}(u),
\]

i.e., in this case, the coarea formula also holds. Moreover, by the definition of \( D_E \), we have \( \mathcal{H}^{n-k}(\varphi^{-1}(z) \cap E \setminus D_E) = 0 \) for all \( z \in X \). Representing \( E \) as a set of measure zero, and applying B. Levi's theorem, we see that, the coarea formula holds for every measurable \( A \subset E \setminus Z \), if \( \mathcal{H}^n(E \setminus D_E) = 0 \) for every \( E \subset E \).

The necessity of this condition is obvious from the coarea formula. \( \square \)

**Proposition 6.2.** The \( m \)-coarea factor can be considered as the derivative \( \Phi'(x) \) of the absolutely continuous countably additive set function \( \Phi \) defined on Borel sets in (6.1). Then, for every measurable set \( A \subset E \setminus Z \), the coarea formula can be represented as follows:

\[
\int_X d\mathcal{H}^k(z) \int_{\varphi^{-1}(z) \cap A} d\mathcal{H}^{n-k}(u) = \Phi(A) = \int_A \Phi'(x) dx = \int_A J_k(MD(\varphi, x)) dx.
\]

**Theorem 6.3.** Assume that \( E \subset \mathbb{R}^n \) is a measurable set, \( (X, d_X) \) is a metric space, the mapping \( \varphi \in \text{Lip}(E, X) \) is such that \( \dim \ker(MD(\varphi, x)) \leq n-k \) almost everywhere, and \( \varphi(E) \subset X \) is \( \mathcal{H}^k \)-\( \sigma \)-finite. Then the coarea formula holds.

**Proof.** I. Prove that \( \dim \ker(MD(\varphi, x)) = n-k \) almost everywhere. Indeed, we have

\[
\mathcal{H}^{k+1}(\varphi(E)) = 0.
\]

By Corollary 4.3, \( J_{k+1}(MD(\varphi, x)) = 0 \) almost everywhere. It means that

\[
\dim \ker(MD(\varphi, x)) \geq n-k
\]

almost everywhere, and, by hypothesis, we have \( \dim \ker(MD(\varphi, x)) = n-k \) almost everywhere.

II. Denote by

\[
\Sigma_E = \{ x \in E : \Phi_m(x) \not\to 1 \} = E \setminus D_E.
\]

Recall that \( \mathcal{H}^{n-k}(\Sigma_E \cap \varphi^{-1}(z)) = 0 \) for all \( z \in X \). Fix \( x \in \Sigma_E \) and a cube \( Q(x, r) \) whose faces are parallel or orthogonal to \( \ker(MD(\varphi, x)) \). We may assume without loss of generality that \( x = 0 \) and \( \ker(MD(\varphi, 0)) = \text{span}\{e_{k+1}, \ldots, e_n\} \). Define a mapping \( \psi : E \cap Q(0, r) \to X \times \mathbb{R}^{n-k} \) and a metric \( d \) on \( \psi(E \cap Q(0, r)) \) as follows:
Indeed, suppose the contrary that for any \( r > S \) of Lemma 4.4, that (2.1) holds with the constant 0 Lipschitz with respect to the metric \( d \). It is easy to see that

\[
d((z, w), (t, v)) = \max\{d_{\mathcal{F}}(z, t), |w - v|_{\infty}\}.
\]

Let us show that, there exists \( \gamma > 0 \) such that \( d((\psi(y), \psi(w)), (\psi(y), \psi(w))) \geq \gamma |y - w| \).

1. The case \( \varphi(y) = \varphi(w) \). The continuity of the \( m \)-differential implies that, if \( \varphi(y) = \varphi(w) \) then there exists a constant \( \alpha(r) \in [0, 1] \), such that

\[
|y^{n-k} - w^{n-k}| \geq (1 - \alpha(r))|y - w|.
\]

Indeed, suppose the contrary that for any \( r > 0 \) and \( \alpha \in [0, 1] \) there exist \( y \in Q(0, r) \cap E \) and \( w, v \in \varphi^{-1}(\varphi(y)) \cap E \cap Q(0, r) \) such that

\[
|\text{Pr}_{\ker(MD(y))}(w - v)| < (1 - \alpha)|w - v|.
\]

On the one hand, it means that \( \frac{w - v}{|w - v|} \) is close to \((\ker(MD(\varphi, 0))) / \mathbb{S}^{n-1}\).

On the other hand, since \( w, v \in \varphi^{-1}(\varphi(y)) \), we have that \( \frac{w - v}{|w - v|} \) is close to \((\ker(MD(\varphi, y))) / \mathbb{S}^{n-1}\). By Lemma 4.5, for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( \ker(MD(\varphi, y)) / \mathbb{S}^{n-1} \) is a subset of an \( \varepsilon \)-neighborhood of \((\ker(MD(\varphi, y))) / \mathbb{S}^{n-1} \) if \( |y - w| < \delta \). Note that, the same is true for \( (\ker(MD(\varphi, y))) / \mathbb{S}^{n-1} \) and \((\ker(MD(\varphi, w))) / \mathbb{S}^{n-1} \).

Fix \( \varepsilon > 0 \), such that \( 3\varepsilon \)-neighborhoods of \((\ker(MD(\varphi, w))) / \mathbb{S}^{n-1} \) and \((\ker(MD(\varphi, w))) / \mathbb{S}^{n-1} \) do not intersect for each \( w \). Take \( r < \delta \) such that for all \( w, v \in Q(0, r) \) with \( \varphi(w) = \varphi(v) \), we have \( \frac{w - v}{|w - v|} \) belong to \( \varepsilon \)-neighborhood of \((\ker(MD(\varphi, y))) / \mathbb{S}^{n-1} \) for \( y \in Q(0, r) \) with \( \varphi(y) = \varphi(w) \) (the existence of such \( r > 0 \) follows from Lemma 4.7). Finally, take \( \alpha \in [0, 1] \) such that if (6.5) holds for \( w, v \in Q(0, r) \), then \( \frac{w - v}{|w - v|} \) belong to \( \varepsilon \)-neighborhood of

\[
(\ker(MD(\varphi, 0))) / \mathbb{S}^{n-1}.
\]

For the chosen \( r \) and \( \alpha \), consider \( w_{r,\alpha} \) and \( v_{r,\alpha} \) such that (6.5) holds. Denote by \( \xi \) the vector \( \frac{w_{r,\alpha} - v_{r,\alpha}}{|w_{r,\alpha} - v_{r,\alpha}|} \). By the choice of \( \alpha \), we have that, there exists \( \xi_0 \in (\ker(MD(\varphi, 0))) / \mathbb{S}^{n-1} \) with \( |\xi - \xi_0| < \varepsilon \). By the choice of \( \varepsilon > 0 \), we have that, there exists \( \xi_1 \in \ker(MD(\varphi, y)) / \mathbb{S}^{n-1} \) with \( |\xi - \xi_1| < \varepsilon \). Consequently, since \( r < \delta \), for \( \xi \) there exists \( \xi_2 \in \ker(MD(\varphi, 0)) \) with \( |\xi_1 - \xi_2| < \varepsilon \). From here we have that \( |\xi_0 - \xi_2| < 3\varepsilon \), and we obtain the contradiction. Thus, there exist \( r > 0 \) and \( \alpha(r) \in [0, 1] \) such that (6.4) holds.

2. The case \( \varphi(y) \neq \varphi(w) \). We may assume without loss of generality, that \( r > 0 \) is such that \( MD(\varphi, y)(u) \geq \frac{c}{2} \) for \( y \in Q(0, r) \cap E \) and \( u \in (\ker(MD(\varphi, 0))) / \mathbb{S}^{n-1} \), where \( c \) is the constant of Lemma 4.4, that (2.1) holds with the constant \( 0 < 2\beta < \min\{\frac{c}{2K}, \frac{1}{K}\} \), where \( |y - w| \leq K(|y^k - w^k| + |y^{n-k} - w^{n-k}|) \) for all \( y, w \in \mathbb{R}^n \), instead of \( \varepsilon \), and that \( MD(\varphi, y)(v) \leq \beta |v| \) for all \( v \in \ker(MD(\varphi, 0)) \).
From here we have

\[ d_X(\varphi(y), \varphi(w)) = MD(\varphi, y)(y - w) + o(|y - w|) \]
\[ = MD(\varphi, y)(y^k - w^k) + o(|y - w|) \geq \frac{c}{2}|y^k - w^k| - \beta|y - w|, \]

and

\[ \rho(\psi(y), \psi(w)) \overset{\text{def}}{=} d_X(\varphi(y), \varphi(w)) + |y^{n-k} - w^{n-k}| \]
\[ \geq \frac{c}{2}|y^k - w^k| - \beta|y - w| + |y^{n-k} - w^{n-k}| \]
\[ \geq \frac{c}{2}|y^k - w^k| - K\beta(|y^k - w^k| + |y^{n-k} - w^{n-k}|) + |y^{n-k} - w^{n-k}| \]
\[ = |y^k - w^k| \left( \frac{c}{2} - K\beta \right) + |y^{n-k} - w^{n-k}| (1 - K\beta) \]
\[ \geq K\theta(|y^k - w^k| + |y^{n-k} - w^{n-k}|) \geq \theta|y - w|, \]

where \( \theta < \frac{\min(\frac{1}{2} - K\beta, 1 - K\beta)}{K} \), \( \theta > 0 \). Thus, in view of obvious inequality

\[ \rho((z, w), (t, v)) \leq 2d((z, w), (t, v))^2, \]

we have for \( \varphi(y) \neq \varphi(w) \),

\[ d(\psi(y), \psi(w)) \geq \frac{\theta}{2}|y - w|. \]

Put \( \gamma = \min\{1 - \alpha(r), \frac{\theta}{2}\} \). Then, \( \gamma > 0 \) and \( d(\psi(y), \psi(w)) \geq \gamma|y - w| \). Thus, \( \psi \) is bi-Lipschitz.

**III.** Note that, we may assume that \( \psi \) is defined on \( \Sigma_E \cap Q(0, r) \).

Repeating arguments of the proof of [8, 2.10.45] for our metric \( d \), the set \( \mathbb{X} \times Q(0, R) \supset \psi(\Sigma_E \cap Q(0, r)) \), and Hausdorff measure \( \mathcal{H}^{n-k}_d \) on \( \mathbb{R}^{n-k} \), such that \( \mathcal{H}^{n-k}_{|\cdot|_{\infty}}(Q(0, R)) = \omega_{n-k} R^{n-k} \), we deduce that \( \mathcal{H}^{n-k}_{d}(\mathbb{X} \times Q(0, R)) = \mathcal{H}^{k}(\mathbb{X}) \mathcal{H}^{n-k}_{|\cdot|_{\infty}}(Q(0, R)) \cdot \omega_{n-k}. \) Consequently, in view of [8, 2.10.25], for \( C = \frac{\omega_{n-k}}{\omega_{\varphi^{-1}(z)}} \) we have

\[ \mathcal{H}^{n-k}_{d}(\psi(\Sigma_E \cap Q(0, r))) \leq C \int_{\psi(\Sigma_E \cap Q(0, r))} d\mathcal{H}^{k}(z) \int_{\{y^{n-k}: (z, y^{n-k}) \in \psi(\Sigma_E \cap Q(0, r))\}} d\mathcal{H}^{n-k}(u). \]

The last integral equals zero, since \( \psi \) is a bi-Lipschitz mapping,

\[ \{y^{n-k}: (z, y^{n-k}) \in \psi(\Sigma_E \cap Q(0, r))\} = \text{Pr}_{\ker(MD(\varphi, 0))}(\Sigma_E \cap \varphi^{-1}(z) \cap Q(0, r)), \]

where the projection mapping \( \text{Pr}_{\ker(MD(\varphi, 0))} \) is non-degenerate on level sets (see (6.4)), and, consequently,

\[ \mathcal{H}^{n-k}(\Sigma_E \cap \varphi^{-1}(z) \cap Q(0, r)) = \mathcal{H}^{n-k}(\{y^{n-k}: (z, y^{n-k}) \in \psi(\Sigma_E \cap Q(0, r))\}) = 0. \]
But $\mathcal{H}^2_0(\psi(\Sigma E \cap Q(0,r))) = 0$ if and only if $\mathcal{H}^n(\Sigma E \cap Q(0,r)) = 0$ since $\psi$ is bi-Lipschitz. Hence, $\mathcal{H}^2(\Sigma E) = 0$.

By Theorem 6.1, the coarea formula holds. \hfill \Box

**Corollary 6.4.** Suppose that, under the conditions of Theorem 6.1, $\dim \ker(MD(\varphi, x)) \leq n - k$ almost everywhere and $\varphi(E) \subset \mathbb{X}$ is $\mathcal{H}^k$-finite. Then assertions 1, 3--5 in the Coarea Formula Validity Criterion hold.

### 7. Generalizations and applications

The next lemma is used in the proof of the area formula for mappings of level sets (see below).

**Lemma 7.1.** Let $A \subset \mathbb{R}^l$ be a measurable set, and $x \in A$ be a density point of $A$. Then, $x$ is a linear density point of $A$ in $\mathcal{H}^{l-1}$-almost all $u \in \mathbb{S}^{l-1}$.

**Proof.** For each $q \in \mathbb{N}$, consider functions $\psi_q(u) = q|(x, x + \frac{1}{q}u) \cap A|: \mathbb{S}^{l-1} \to \mathbb{R}$, where $| \cdot |$ denotes 1-dimensional Lebesgue measure. By Fubini’s theorem, the value $|(x, x + \frac{1}{q}u) \cap A|$ is defined for $\mathcal{H}^{l-1}$-almost all $u \in \mathbb{S}^{l-1}$. Moreover, each function $\psi_q$ is integrable, and, consequently, is measurable in view of Fubini’s theorem.

From here it follows that the function $\psi(u) = \lim_{q \to \infty} \psi_q(u)$ is measurable.

Denote the set of directions in which $x$ is not a linear density point, by $\Omega$. Note that, $\Omega$ is measurable, since $\Omega = \{u \in \mathbb{S}^{l-1}: \psi(u) < 1\}$.

Assume that $\mathcal{H}^{l-1}(\Omega) > 0$. Then, there exists $\alpha > 0$ and a measurable set $\Omega' \subset \Omega$ of positive measure, such that $\psi(u) < 1 - 2\alpha$ on $u \in \Omega'$.

We may assume by Egorov’s theorem, that $\psi_q(u) \Rightarrow \psi(u)$ for $u \in \Omega'$. Then, there exists $q_0 \in \mathbb{N}$ with $\psi_q(u) < 1 - \alpha$ for $q > q_0$ and $u \in \Omega'$.

Since $\Omega'$ is measurable, we have

\[
(1 - o(1))\omega_l r^l = \mathcal{H}^l(A \cap B(x, r)) = \int_{\mathbb{S}^{l-1}} \frac{r^{l-1}}{l} |(x, x + ru) \cap A| d\mathcal{H}^{l-1}(u)
\]

\[
= \int_{\mathbb{S}^{l-1} \setminus \Omega'} \frac{r^{l-1}}{l} |(x, x + ru) \cap A| d\mathcal{H}^{l-1}(u) + \int_{\Omega'} \frac{r^{l-1}}{l} |(x, x + ru) \cap A| d\mathcal{H}^{l-1}(u)
\]

\[
< (1 - o(1))r^l \left( \omega_l - \frac{\mathcal{H}^{l-1}(\Omega')}{l} \right) + (1 - \alpha)r^l \frac{\mathcal{H}^{l-1}(\Omega')}{l} < (1 - o(1))\omega_l r^l,
\]

where $r = \frac{1}{q} < \frac{1}{q_0}$ and $o(1) \to 0$ as $q \to \infty$. Letting $q$ tend to infinity, we obtain the contradiction. The theorem follows. \hfill \Box

Let $(\mathbb{Y}, d_Y)$ be an $\mathcal{H}^n$-rectifiable metric space. Denote the Lipschitz mappings from the definition of the $\mathcal{H}^n$-rectifiable metric space by $\beta_j : B_j \to \mathbb{Y}$, $j \in \mathbb{N}$, where $B_j \subset \mathbb{R}^n$ are measurable, and $\mathcal{H}^n(\mathbb{Y} \setminus \bigcup_{j \in \mathbb{N}} \beta_j(B_j)) = 0$.

**Definition 7.2.** (See [15].) Let $\{A_i\}_{i \in \mathbb{N}}$ be a countable family of sets. We say that $\{A_i\}_{i \in \mathbb{N}}$ is a partition of $A$ if the following conditions hold:

1. $A_i \cap A_j = \emptyset$ if $i \neq j$,
2. $A = \bigcup_{i \in \mathbb{N}} A_i$. 

Lemma 7.3. (See [15].) Suppose that \((Y, d_Y)\) is an \(H^n\)-rectifiable metric space, \((X, d_X)\) is an \(H^k\)-rectifiable metric space, \(n \geq k\), and \(\varphi \in \text{Lip}(Y, X)\). Then there exist a set of \(H^n\)-measure zero \(Y_0 \subset Y\), measurable sets \(D_j \subset B_j\), \(j \in \mathbb{N}\), and partitions into measurable sets \(Y \setminus Y_0 = \bigcup_{j, v \in \mathbb{N}} Y_{j,v}\), \(B_j \setminus D_j = \bigcup_{v \in \mathbb{N}} B_{j,v}\) such that \(Y_{j,v} = \beta_j(B_{j,v})\) for some \(j \in \mathbb{N}\) and each \(\beta_j\) is bi-Lipschitz, \(m\)-differentiable everywhere on \(B_{j,v}\), and \(\text{MD}(\beta_j, y)(u) \neq 0\) for all \(y \in B_{j,v}\) and \(u \in S^{n-1}\), \(j, v \in \mathbb{N}\).

Remark 7.4. A similar assertion is formulated in [17].

To simplify the notations, assume that \(Y \setminus Y_0 = \bigcup_{m \in \mathbb{N}} Y_m\), and \(Y_m = \beta_m(B_m)\), \(m \in \mathbb{N}\). In what follows, we denote by \(M\) the countable dense set on \(S^{n-1}\) of \((n-k)\)-dimensional directions \(\{M_\mu\}_{\mu \in \mathbb{N}}\).

Theorem 7.5 (The area formula for mappings of level sets). Suppose that \(z \in X\), \(\dim \ker(MD(\varphi \circ \beta_m, y)) = n - k\), \(\varphi \circ \beta_m\) is bi-Lipschitz, \(m\)-differentiable everywhere on \(B_m\), and \(\text{MD}(\beta_m, y)(u) \neq 0\) for all \(y \in B_m\) and \(u \in S^{n-1}\).

\[
\int_{(\varphi \circ \beta_m)^{-1}(z) \cap B_m \setminus B_{m,0}} J_{n-k}(MD(\beta_m, y)) \, dH^{n-k}(y) = \int_{(\varphi \circ \beta_m)^{-1}(z) \cap Y_m \setminus \beta_m(B_{m,0})} \, dH^{n-k}(x) \quad (7.1)
\]

holds. Here

\[
J_{n-k}(MD(\beta_m, y)) = \omega_{n-k}(n - k) \left( \int_{S^{n-k-1}} [MD(\beta_m, y)(u)]^{-(n-k)} dH^{n-k-1}(u) \right)^{-1}, \quad (7.2)
\]

and \(S^{n-k-1}\) is the \((n-k-1)\)-dimensional unit sphere of \(\ker(MD(\varphi \circ \beta_m, y))\).

Proof. Any \(B_m \setminus B_{m,0}\), \(H^n(B_{m,0}) = 0\), is the union of at most countable family of compact sets, such that the conditions of Theorem 2.3 hold on every compact set. Moreover, we may assume that each point of \(B_m \setminus B_{m,0}\) is a \((n-k)\)-dimensional density point of \(B_m\) in each direction \(M \in M\). Thus, we have obtained the existence of \(B_{m,0}\).

Therefore, suppose that \(B_m\) enjoys the above conditions.

Fix a level set \(S\) of \(\varphi \circ \beta_m\). By Theorem 2.3, it is \(H^{n-k}\)-rectifiable. Since \(\beta_m\) is Lipschitz, and, therefore, it possesses \(\mathcal{N}\)-property, we may assume that (4.2) holds in each \(y \in S\).

Define a set function \(\Phi\) on Borel subsets of \(S\) as

\[
\Phi(A) = H^{n-k}(\beta_m(A \cap S)).
\]

Note that, it is countably additive and absolutely continuous. Moreover, the measure \(H^{n-k}\) on \(S\) is doubling locally. Thus, to prove the area formula (7.1), it suffices to show that \(\Phi'(y) = \)
\(J_{n-k}(MD(\beta_m, y))\) in each \(y \in S\) [30]. By definition of the derivative of the set function, we have

\[
\Phi'(y) = \lim_{r \to 0} \frac{\mathcal{H}_{n-k}(\beta_m(B(y, r) \cap S))}{\mathcal{H}_{n-k}(B(y, r) \cap S)} = \lim_{r \to 0} \frac{\mathcal{H}_{n-k}(\beta_m(B(y, r) \cap S))}{\omega_{n-k}r^{n-k}}
\]

in view of (4.2). Consider a sequence \(M_\mu \to \ker(MD(\varphi \circ \beta_m, y))\) as \(\mu \to \infty\). For each \(\mu \in \mathbb{N}\), construct the projection mapping \(\pi_{\mu,y} : S \to y + M_\mu\) along \((\ker(MD(\varphi \circ \beta_m, y)))\). By construction, \(\pi_{\mu,y}(y) = y\).

Denote the set \(\pi_{\mu,y}(B(y, r) \cap S)\) by \(A_\mu(y, r)\). Then, \(\beta_m(B(y, r) \cap S) = \beta_m \circ \pi_{\mu,y}^{-1}(A_\mu(y, r))\). Fix \(r_0 > 0\) and suppose that \(r < r_0\). Then, in view of (4.2) and the properties of \(\pi_{\mu,y}\) following from Lemma 4.7, we have

\[
\mathcal{H}_{n-k}(A_\mu(y, r)) = (1 + o(1))\mathcal{H}_{n-k}(A_\mu(y, r_0) \cap B(y, r)) = (1 + o(1))\omega_{n-k}r^{n-k},
\]

where \(o(1) \to 0\) as \(r \to 0\) and \(\mu \to \infty\). Thus, since \(\beta_m\) is Lipschitz, we infer

\[
\Phi'(y) = \lim_{\mu \to \infty} \lim_{r \to 0} \frac{\mathcal{H}_{n-k}(\beta_m \circ \pi_{\mu,y}^{-1}(A_\mu(y, r_0) \cap B(y, r)))}{\mathcal{H}_{n-k}(A_\mu(y, r_0) \cap B(y, r))} = \lim_{\mu \to \infty} J_{n-k}(MD(\beta_m \circ \pi_{\mu,y}^{-1}, y))
\]

since \(y\) is a density point of \(A_\mu(y, r_0)\). (Here \(J_{n-k}(MD(\beta_m \circ \pi_{\mu,y}^{-1}, y))\) is the Jacobian of \(\beta_m \circ \pi_{\mu,y}^{-1}\).)

It is left to prove that \(MD(\beta_m \circ \pi_{\mu,y}^{-1}, y)(u) = (1 + o(1))MD(\beta_m, y)(u)\) for each \(u \in S^{n-1} \cap M_\mu\), where \(o(1) \to 0\) as \(\mu \to \infty\).

Put \(D_\mu(y, r) = A_\mu(y, r) \cap B_m\). By above assumptions, \(y\) is a density point of \(D_\mu(y, r)\) for each \(r \in (0, R)\), where \(R > 0\) is small enough. By Lemma 7.1, it is a linear density point in almost all directions in \(S^{n-1} \cap M_\mu\). Then, there exists a countable dense set \(U_\mu\) on \(S^{n-1} \cap M_\mu\) such that \(y\) is a linear density point in each direction \(u \in U_\mu\). Consequently, for each \(u \in U_\mu\) we have

\[
MD(\beta_m \circ \pi_{\mu,y}^{-1}, y)(u) = \lim_{r \to 0} \frac{d_Y(\beta_m(y), \beta_m \circ \pi_{\mu,y}^{-1}(y + ru))}{r} = \lim_{r \to 0} \frac{(1 + o(1))d_Y(\beta_m(y), \beta_m(y + ru))}{r},
\]

where \(r \in (0, R)\) is such that \(y + ru\) belong to \(\pi_{\mu,y}^{-1}(D_\mu(y, R))\) in the first relation and \(D_\mu(y, r_0)\) in the second relation, respectively; and \(o(1) \to 0\) as \(r \to 0\) and \(\mu \to \infty\). Here the convergence of \(o(1)\) to zero depending on \(\mu\) is uniform in \(u \in S^{n-1}\). Thus, for \(u \in \ker(MD(\varphi \circ \beta_m, y))\), we have

\[
MD(\beta_m, y)(u) = \lim_{\mu \to \infty} MD(\beta_m \circ \pi_{\mu,y}^{-1}, y)(u),
\]

where \(u_\mu \to u, u_\mu \in U_\mu\), as \(\mu \to \infty\). From here we obtain \(\Phi'(y) = J_{n-k}(MD(\beta_m, y))\) in those \(y\), in which we have (4.2), i.e., in \(\mathcal{H}_{n-k}\)-almost all \(y \in S\).
The area formula follows from result of [30]:

\[
\int_{S \cap A} \mathcal{J}_{n-k}(MD(\beta_m, y)) \, d\mathcal{H}^{n-k}(y) = \int_{S \cap A} \Phi'(y) \, d\mathcal{H}^{n-k}(y)
= \Phi(A) = \int_{\beta_m(A \cap S)} d\mathcal{H}^{n-k}(y)
\]

for each measurable \( A \subset S \). \( \square \)

**Theorem 7.6.** (The coarea formula [12,15].) Suppose that \((\mathcal{Y}, d_\mathcal{Y})\) is \(\mathcal{H}^n\)-rectifiable metric space, \((\mathcal{X}, d_\mathcal{X})\) is \(\mathcal{H}^k\)-rectifiable metric space, \(n \geq k\), and \(\varphi \in \text{Lip}(\mathcal{Y}, \mathcal{X})\). Then, the formula

\[
\int_{\mathcal{Y}} f(x) \mathcal{J}_k(\varphi, x) \, d\mathcal{H}^n(x) = \int_{\mathcal{X}} d\mathcal{H}^k(z) \int_{\varphi^{-1}(z)} f(u) \, d\mathcal{H}^{n-k}(u),
\]

where

\[
\mathcal{J}_k(\varphi, x) = \frac{\mathcal{J}_k(MD(\varphi \circ \beta_m, \beta_m^{-1}(x))) \mathcal{J}_{n-k}(MD(\beta_m, \beta_m^{-1}(x)))}{\mathcal{J}(MD(\beta_m, \beta_m^{-1}(x)))}
\]

on \(\mathcal{Y}_m\), holds for every function \(f: \mathcal{Y} \to \mathcal{E}\) (where \(\mathcal{E}\) is an arbitrary Banach space) such that \(f(x)\mathcal{J}_k(\varphi, x)\) is integrable.

**Proof of Theorems 1.2–1.4.** I. Applying the Preimage-of-a-Point Theorem to each of the mappings \(\varphi \circ \beta_m, m \in \mathbb{N}\), we conclude that there exist sets \(\Sigma_m \subset B_m\), \(\mathcal{H}^n(\Sigma_m) = 0\), such that \((\varphi \circ \beta_m)^{-1}(z) \cap B_m \setminus (Z_m \cup \Sigma_m)\) is a subset of the union of an at most countable collection of the images of measurable sets \(D_{l}^{z} \subset \mathbb{R}^{n-k}\) under Lipschitz mappings \(\psi_{m,l}^{z}\), \(l \in \mathbb{N}\); in particular, this set is \(\mathcal{H}^{n-k}\)-rectifiable in \(\mathbb{R}^{n}\) for all \(z \in \mathcal{X}\). Denote \(\Sigma = \bigcup_{m \in \mathbb{N}} \beta_m(\Sigma_m)\). Since each of the mappings \(\beta_m, m \in \mathbb{N}\), possesses the \(\mathcal{N}\)-property, we have \(\mathcal{H}^n(\Sigma) = 0\). Furthermore, for every mapping \(\psi_{m,l}^{z}: D_{l}^{z} \to (\varphi \circ \beta_m)^{-1}(z) \cap B_m \setminus (Z_m \cup \Sigma_m)\), \(D_{l}^{z} \subset \mathbb{R}^{n-k}\), consider the compositions \(\beta_m \circ \psi_{m,l}^{z}: D_{l}^{z} \to \varphi^{-1}(z) \setminus (Z \cup \Sigma), l \in \mathbb{N}\), where \(Z = \bigcup_{m \in \mathbb{N}} \beta_m(Z_m)\). Obviously, \(\beta_m \circ \psi_{m,l}^{z} \in \text{Lip}(D_{l}^{z})\). Putting \(\Sigma_{\mathcal{Y}} = \Sigma \cup \Sigma_0\), we see that the set \(\varphi^{-1}(z) \setminus (Z \cup \Sigma_{\mathcal{Y}})\) is an \(\mathcal{H}^{n-k}\)-rectifiable metric space for every \(z \in \mathcal{X}\), or, more exactly, is a subset of the union of an at most countable collection of images of measurable sets in \(\mathbb{R}^{n-k}\) under Lipschitz mappings.

II. The proof of the equivalence \(1 \iff 2\) is carried out as was indicated in 1.6.

\(1 \implies 3\). In view of the equivalence \(1 \iff 2\), it suffices to consider the case of \(f(x) = \chi_A(x)\), where \(A \setminus Z \subset \mathcal{Y}\) is measurable. Suppose that the coarea formula (7.3) holds for every such a set \(A \subset \mathcal{Y}\). Since \(\beta_m\) is a bi-Lipschitz mapping, the preimage \(A_m = \beta_m^{-1}(A \cap \mathcal{Y}_m \setminus Z)\) is also measurable, and the function \(\mathcal{J}_{n-k}(MD(\beta_m, \beta_m^{-1}(x)))^{-1}\) is bounded and measurable. Then, applying the coarea formula to the function \(f(x) = \mathcal{J}_{n-k}(MD(\beta_m, \beta_m^{-1}(x)))^{-1}\), the change-of-variable formula to the mapping \(\beta_m: A_m \to A \cap \mathcal{Y}_m \setminus Z\) and Theorem 7.5, we have

\[
\int_{A_m} \mathcal{J}_k(MD(\varphi \circ \beta_m), x) \, dx = \int_{\mathcal{X}} d\mathcal{H}^k(z) \int_{(\varphi \circ \beta_m)^{-1}(z) \cap A_m} d\mathcal{H}^{n-k}(u).
\]
Since $\beta_m$ is bi-Lipschitz, we may assume that $A_m$ is an arbitrary measurable subset of $B_m$. Then $\varphi \circ \beta_m(B_m) = \varphi(\Sigma_m) \cup X_{Z_m} \cup X_{0,m}$, $m \in \mathbb{N}$, where $X_{\Sigma_m}$, $X_{Z_m}$, and $X_{0,m}$ correspond to the sets $X_{\Sigma}$, $X_{Z}$, and $X_{0}$ in Theorem 1.3. Put $\Sigma = \bigcup_{m \in \mathbb{N}} \beta_m(\Sigma_m) \cup \Sigma_0$, $X_{\Sigma} = \bigcup_{m \in \mathbb{N}} X_{\Sigma_m} \cup \varphi(\Sigma)$, $X_{Z} = \varphi(Z) = \bigcup_{m \in \mathbb{N}} X_{Z_m}$, and $X_{0} = \bigcup_{m \in \mathbb{N}} X_{0,m}$.

The proof of the assertion on the properties of $X_Z$ follows the scheme of 3.10 with obvious changes.

The implication $3 \Rightarrow 1$ follows from Lemma 3.8, the proof of Lemma 3.9, and Theorem 7.6.

To prove the equivalence $3 \Leftrightarrow 4$, it suffices to consider the mappings $\varphi \circ \beta_m$, $m \in \mathbb{N}$, and apply the proof of Lemma 3.7; and Lemma 3.8 and Corollary 4.3.

The proof of the equivalence of the assertion about the set $Z$ in 3 and item 5(b) follows the schemes of 3.9 and 3.10 (see 1.5) with obvious changes.

III. Applying arguments similar to those in proof of Lemma 6.3, we see that $\dim \ker(MD(\varphi \circ \beta_m, x)) \leq n - k$ almost everywhere, $m \in \mathbb{N}$, and the coarea formula holds for each of the mappings $\varphi \circ \beta_m$. Consider the set $\Sigma_m$ and a mapping $\beta_m \in \text{Lip}(B_m, \Sigma_m)$, where $B_m \subset \mathbb{R}^n$.

By the area formula, the equivalence 1 $\Leftrightarrow$ 2, and Theorems 6.3 and 7.5, we have

$$\int_{\Sigma_m} f(x) J_k(\varphi, x) d\mathcal{H}^n(x)$$

$$= \int_{B_m} (f \circ \beta_m)(y) J_k(MD(\varphi \circ \beta_m, y)) J_{n-k}(MD(\beta_m, y)) d\mathcal{H}^n(y)$$

$$= \int_X d\mathcal{H}^k(z) \int_{(\varphi \circ \beta_m)^{-1}(z)} (f \circ \beta_m)(v) J_{n-k}(MD(\beta_m, v)) d\mathcal{H}^{n-k}(v)$$

$$= \int_X d\mathcal{H}^k(z) \int_{\varphi^{-1}(z)} f(u) d\mathcal{H}^{n-k}(u).$$

Applying a standard argument, we deduce the coarea formula (7.3) for $\varphi \in \text{Lip}(\Sigma, X)$.

Corollary 7.7. Suppose that $\dim \ker(MD(\varphi \circ \beta_m, x)) \leq n - k$ almost everywhere, $m \in \mathbb{N}$, and $\varphi(\Sigma) \subset X$ is $\mathcal{H}^k$-$\sigma$-finite. Then items 2–4, 5(b) of the Coarea Formula Validity Criterion hold.

Definition 7.8. Let $\Sigma$ be an $\mathcal{H}^n$-rectifiable metric space and let $(X, d_X)$ be an arbitrary metric space. We say that a mapping $\varphi : \Sigma \to X$ belongs to the class $\Phi(\Sigma, X)$ if there exists an at most countable disjoint collection of measurable sets $\{A_l\}$ such that, for all $l \in \mathbb{N}$, the mapping $\varphi|_{A_l}$ is Lipschitz on $A_l$ and

$$\mathcal{H}^n\left(\Sigma \setminus \bigcup_{l \in \mathbb{N}} A_l\right) = 0.$$

Put $\Sigma_\varphi = \Sigma \setminus \bigcup_{l \in \mathbb{N}} A_l$.

Definition 7.9. (See [16].) Suppose that $E \subset \mathbb{R}^n$ is a measurable set, $(X, d_X)$ is a metric space, and $\varphi : E \to (X, d_X)$. The mapping $\varphi$ is approximately metrically differentiable, or approximately
**m-differentiable** at a point \( x \in E \) if there exists a seminorm \( L_x \) on \( \mathbb{R}^n \) such that

\[
\lim_{y \to x} \frac{L_x(x - y) - d_X(\varphi(x), \varphi(y))}{|x - y|} = 0,
\]
i.e., the set

\[
A_\varepsilon = \left\{ y \in E : \left| \frac{L_x(x - y) - d_X(\varphi(x), \varphi(y))}{|x - y|} \right| < \varepsilon \right\}
\]
has density 1 at \( x \) for every \( \varepsilon > 0 \). The seminorm \( L_x \) is called the *approximate metric differential*, or the *approximate m-differential* of the mapping \( \varphi \) at the point \( x \) and is denoted by \( MD_{ap}(\varphi, x) \).

**Theorem 7.10.** Let \( \mathcal{Y} \) be an \( \mathcal{H}^n \)-rectifiable metric space and let \((\mathbb{X}, d_\mathbb{X})\) be an arbitrary metric space. Then, for every mapping \( \varphi \in \Phi(\mathcal{Y}, \mathbb{X}) \), Theorem 1.2, items 1–4, 5(b) of Theorem 1.3, and Theorem 1.4 hold but the coarea formula takes the following form:

\[
\int_{\mathcal{Y}} \mathcal{J}_k(\varphi, x) \, d\mathcal{H}^n(x) = \int_{\mathbb{X}} d\mathcal{H}^k(z) \int_{\varphi^{-1}(z) \setminus \Sigma \varphi} d\mathcal{H}^{n-k}(u).
\]

In the case, where \( \mathcal{Y} = E \subset \mathbb{R}^n \) is a measurable set, all the five statements of Theorem 1.3 hold but the coarea formula is of the form

\[
\int_{A} \mathcal{J}_k(MD_{ap}(\varphi, x)) \, d\mathcal{H}^n(x) = \int_{\mathbb{X}} d\mathcal{H}^k(z) \int_{\varphi^{-1}(z) \setminus \Sigma \varphi} d\mathcal{H}^{n-k}(u).
\]

**Proof.** It suffices to consider the restrictions \( \varphi|_{A_l}, l \in \mathbb{N} \), and apply Theorems 1.2–1.4 and standard arguments.

**Remark 7.11.** In [26,27], it was proved that, given a domain \( \Omega \subset \mathbb{R}^n \), continuous quasi-monotone mappings of the Sobolev class \( W^{1,p}_{n,loc}(\Omega, \mathbb{X}) \), continuous mappings of the Sobolev class \( W^{1,q}_{q,loc}(\Omega, \mathbb{X}), q > n \), with values in an arbitrary metric space, and mappings of the Sobolev class \( W^{1,q}_q(\Omega; \mathbb{X}), q \geq 1 \), with values in separable metric space [25], belong to \( \Phi(\Omega, \mathbb{X}) \). Furthermore, in [4,29], it was proved that mappings of class \( BV(\Omega, \mathbb{X}) \) [2] also belong to \( \Phi(\Omega, \mathbb{X}) \). Hence, Theorem 7.10 holds for all these classes of mappings.

We remark that, for a mapping \( \varphi : \mathbb{R}^n \to \mathbb{R}^k, k \leq n, [11,21–23] \) deal with a subtler characteristic of the geometry of level sets of Sobolev mappings than the Hausdorff measure \( \mathcal{H}^{n-k} \).

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