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A Cayley–Hamilton trace identity for 2×2 matrices over Lie-solvable rings ☆,☆☆,☆☆☆

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ABSTRACT

First we construct an algebra satisfying the polynomial identity $[[x, y], [u, v]] = 0$, but none of the stronger identities $[x, y][u, v] = 0$ and $[[x, y], z] = 0$. Then we exhibit a Cayley–Hamilton trace identity for 2×2 matrices with entries in a ring R satisfying $[[x, y], [x, z]] = 0$ and $\frac{1}{2} \in R$.

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1. Introduction

The Cayley–Hamilton theorem and the corresponding trace identity play a fundamental role in proving classical results about the polynomial and trace identities of the $n \times n$ matrix algebra $\mathrm{M}_n(K)$ over a field K (see [2,3]). In case of $\mathrm{char}(K) = 0$, Kemer's pioneering work (see [5]) on the T-ideals of associative algebras revealed the importance of the identities satisfied by the $n \times n$ matrices over the Grassmann (exterior) algebra

$$E = K\langle v_1, v_2, \dots, v_r, \dots \mid v_i v_j + v_j v_i = 0 \text{ for all } 1 \leqslant i \leqslant j\rangle$$

generated by the infinite sequence of anticommutative indeterminates $(v_i)_{i \geqslant 1}$.

For $n \times n$ matrices over a Lie-nilpotent ring R satisfying the polynomial identity

$$[[[\dots[[x_1, x_2], x_3], \dots], x_m], x_{m+1}] = 0$$

(with $[x, y] = xy - yx$), a Cayley–Hamilton identity of degree n^m (with left- or right-sided scalar coefficients) was found in [6]. Since E is Lie-nilpotent with $m = 2$, the above mentioned Cayley–Hamilton identity for a matrix $A \in \mathrm{M}_n(E)$ is of degree n^2.

In [1] Domokos presented a slightly modified version of this identity in which the coefficients are invariant under the conjugation action of $\mathrm{GL}_n(K)$. For a matrix $A \in \mathrm{M}_2(E)$ he obtained the trace identity

$$\begin{aligned} &A^4 - 2\mathrm{tr}(A)A^3 + \Big(2\mathrm{tr}^2(A) - \mathrm{tr}(A^2)\Big)A^2 + \Big(\frac{1}{2}\mathrm{tr}(A)\mathrm{tr}(A^2) + \frac{1}{2}\mathrm{tr}(A^2)\mathrm{tr}(A) - \mathrm{tr}^3(A)\Big)A \\ &\quad + \frac{1}{4}\Big(\mathrm{tr}^4(A) + \mathrm{tr}^2(A^2) - \frac{5}{2}\mathrm{tr}^2(A)\mathrm{tr}(A^2) + \frac{1}{2}\mathrm{tr}(A^2)\mathrm{tr}^2(A) \\ &\quad - 2\mathrm{tr}(A^3)\mathrm{tr}(A) + 2\mathrm{tr}(A)\mathrm{tr}(A^3)\Big)I = 0, \end{aligned}$$

where I is the identity matrix and $\mathrm{tr}(A)$ denotes the sum of the diagonal entries of A. A similar identity with right coefficients also holds for A. Here E can be replaced by any ring R which is Lie-nilpotent of index 2.

The identity $[x, y][x, z] = 0$ is a consequence of Lie-nilpotency of index 2 (see [4]), as is obviously $[[x, y], [x, z]] = 0$. The first aim of the present paper is to provide an example of an algebra satisfying $[[x, y], [u, v]] = 0$, but neither $[x, y][u, v] = 0$ nor $[[x, y], z] = 0$. Since the above mentioned trace identity cannot be used for matrices over such an algebra, our second purpose is to exhibit a new trace identity of the same kind (of degree 4 in A) for a matrix A in $\mathrm{M}_2(R)$, where R is any ring satisfying the identity

$$[[x, y], [x, z]] = 0$$

and $\frac{1}{2} \in R$. We note that a ring satisfying $[[x, y], [u, v]] = 0$ is called Lie-solvable of index 2.

From now onward R and S are rings with 1. In Section 2 we consider the ring $\mathrm{U}_3^*(R)$ of upper triangular 3×3 matrices with equal diagonal entries over R. First we observe that $\mathrm{U}_3^*(R)$ is never commutative. We prove that if R is commutative then the algebra $\mathrm{U}_3^*(R)$ satisfies the identities $[x, y][u, v] = 0$ and $[[x, y], z] = 0$. However, for a non-commutative R we show that the ring $\mathrm{U}_3^*(R)$ never satisfies any of the identities $[x, y][u, v] = 0$ and $[[x, y], z] = 0$.

The main result in Section 2 states that if S satisfies the identities $[x, y][u, v] = 0$ and $[[x, y], z] = 0$, then the matrix ring $\mathrm{U}_3^*(S)$ is Lie-solvable of index 2. It follows that if R is commutative, then $\mathrm{U}_3^*(\mathrm{U}_3^*(R))$ is an example of an algebra satisfying $[[x, y], [u, v]] = 0$, but neither $[x, y][u, v] = 0$ nor $[[x, y], z] = 0$.

Section 3 is entirely devoted to the construction of our Cayley–Hamilton trace identity.

2. A particular Lie-solvable matrix algebra

Since

$$E_{1,2}, E_{2,3} \in \mathrm{U}_3^*(R) = \left\{ \begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} \mid a, b, c, d \in R \right\}$$

and $E_{1,2}E_{2,3} = E_{1,3} \neq 0 = E_{2,3}E_{1,2}$, the ring $U_3^*(R)$ is never commutative. Any element of $U_3^*(R)$ can be written as $aI + X$, where X is strictly upper triangular. We note that $XYZ = 0$ for strictly upper triangular 3×3 matrices. If R is commutative, then aI is central in $U_3^*(R)$ (of course, also in $M_3(R)$), $[aI + X, bI + Y] = [X, Y]$ for all $a, b \in R$ and so $U_3^*(R)$ satisfies all polynomial identities in which each summand is a product of certain (possibly iterated) commutators. For example,

$$[x, y][u, v] = 0 \quad \text{and} \quad [[x, y], z] = 0$$

are typical such identities for $U_3^*(R)$. If R is non-commutative, say $[r, s] \neq 0$ for some $r, s \in R$, then for $x = rI$, $y = sE_{1,2}$, $u = E_{2,2}$, $v = z = E_{2,3}$ in $U_3^*(R)$ we have

$$[x, y][u, v] = [[x, y], z] = [r, s]E_{1,3} \neq 0.$$

Theorem 2.1. *If S satisfies $[x, y][u, v] = 0$ and $[[x, y], z] = 0$, then $U_3^*(S)$ satisfies $[[x, y], [u, v]] = 0$.*

Proof. Using the matrices

$$x = \begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} e & f & g \\ 0 & e & h \\ 0 & 0 & e \end{bmatrix}$$

in $U_3^*(S)$, a straightforward calculation gives that

$$[x, y] = \begin{bmatrix} [a, e] & [a, f] + [b, e] & [a, g] + [c, e] + (bh - fd) \\ 0 & [a, e] & [a, h] + [d, e] \\ 0 & 0 & [a, e] \end{bmatrix} = [a, e]I + C + \alpha E_{1,3},$$

where $\alpha = bh - fd$ and C is a strictly upper triangular matrix with entries in $[S, S]$ (the additive subgroup of S generated by all commutators). Now $[[a, e], s] = 0$ for all $s \in S$, hence $[a, e]I$ is central in $U_3^*(S)$ (also in $M_3(S)$). Thus we have

$$[[x, y], [u, v]] = [[a, e]I + C + \alpha E_{1,3}, [a', e']I + C' + \alpha' E_{1,3}] = [C + \alpha E_{1,3}, C' + \alpha' E_{1,3}] = 0$$

because of $(C + \alpha E_{1,3})(C' + \alpha' E_{1,3}) = (C' + \alpha' E_{1,3})(C + \alpha E_{1,3}) = 0$. Indeed, $CC' = C'C = 0$ is a consequence of $C, C' \in M_3([S, S])$ and of $[x, y][u, v] = 0$ in S, and $CE_{1,3} = E_{1,3}C = C'E_{1,3} = E_{1,3}C' = 0$ follows from the fact that C and C' are strictly upper triangular. □

Corollary 2.2. *If R is commutative, then the algebra $U_3^*(U_3^*(R))$ satisfies $[[x, y], [u, v]] = 0$, but neither $[x, y][u, v] = 0$ nor $[[x, y], z] = 0$.*

3. Matrices with commutator entries

The following can be considered as the "real" 2×2 Cayley–Hamilton trace identity.

Proposition 3.1. *If $\frac{1}{2} \in R$ and $A = [a_{ij}] \in M_2(R)$, then*

$$A^2 - \operatorname{tr}(A)A + \frac{1}{2}(\operatorname{tr}^2(A) - \operatorname{tr}(A^2))I = \begin{bmatrix} \frac{1}{2}[a_{11}, a_{22}] + \frac{1}{2}[a_{12}, a_{21}] & [a_{12}, a_{22}] \\ [a_{21}, a_{11}] & -\frac{1}{2}[a_{11}, a_{22}] - \frac{1}{2}[a_{12}, a_{21}] \end{bmatrix}.$$

Proof. A straightforward computation suffices. □

Corollary 3.2. *If $\frac{1}{2} \in R$ and $B = [b_{ij}] \in M_2(R)$ with $\operatorname{tr}(B) = 0$, then*

$$B^2 - \frac{1}{2}\operatorname{tr}(B^2)I = \begin{bmatrix} \frac{1}{2}[b_{12}, b_{21}] & -[b_{12}, b_{11}] \\ [b_{21}, b_{11}] & -\frac{1}{2}[b_{12}, b_{21}] \end{bmatrix}.$$

Proof. Since $b_{22} = -b_{11}$, we have $[b_{11}, b_{22}] = 0$ and $[b_{12}, b_{22}] = -[b_{12}, b_{11}]$. Thus the formula in Proposition 3.1 immediately gives the identity for B. $\square$

Theorem 3.3. *If $\frac{1}{2} \in R$ and R satisfies $[[x, y], [x, z]] = 0$, then*

$$\left(C^2 - \frac{1}{2}\mathrm{tr}(C^2)I\right)^2 - \frac{1}{2}\mathrm{tr}\left(\left(C^2 - \frac{1}{2}\mathrm{tr}(C^2)I\right)^2\right)I = 0$$

for all $C \in \mathrm{M}_2(R)$ with $\mathrm{tr}(C) = 0$.

Proof. Take $C = [c_{ij}]$. In view of Corollary 3.2 we have

$$C^2 - \frac{1}{2}\mathrm{tr}(C^2)I = \begin{bmatrix} \frac{1}{2}[c_{12}, c_{21}] & -[c_{12}, c_{11}] \\ [c_{21}, c_{11}] & -\frac{1}{2}[c_{12}, c_{21}] \end{bmatrix}.$$

Since $\mathrm{tr}(C^2 - \frac{1}{2}\mathrm{tr}(C^2)I) = 0$, the repeated application of Corollary 3.2 to $B = C^2 - \frac{1}{2}\mathrm{tr}(C^2)I$ gives that

$$\left(C^2 - \frac{1}{2}\mathrm{tr}(C^2)I\right)^2 - \frac{1}{2}\mathrm{tr}\left((C^2 - \frac{1}{2}\mathrm{tr}(C^2)I)^2\right)I$$
$$= \frac{1}{2}\begin{bmatrix} -[[c_{12}, c_{11}], [c_{21}, c_{11}]] & [[c_{12}, c_{11}], [c_{12}, c_{21}]] \\ [[c_{21}, c_{11}], [c_{12}, c_{21}]] & [[c_{12}, c_{11}], [c_{21}, c_{11}]] \end{bmatrix}.$$

Now we have

$$[[c_{12}, c_{11}], [c_{21}, c_{11}]] = [[c_{11}, c_{12}], [c_{11}, c_{21}]]$$

and

$$[[c_{21}, c_{11}], [c_{12}, c_{21}]] = -[[c_{21}, c_{11}], [c_{21}, c_{12}]].$$

Thus each entry of the above 2×2 matrix is of the form $\pm[[x, y], [x, z]] = 0$ and the desired identity follows. $\square$

In Corollaries 3.4 and 3.5 we assume that $\frac{1}{2} \in R$ and R satisfies $[[x, y], [x, z]] = 0$.

Corollary 3.4. *If $C \in \mathrm{M}_2(R)$ with $\mathrm{tr}(C) = \mathrm{tr}(C^2) = \mathrm{tr}(C^4) = 0$, then $C^4 = 0$.*

Proof. Expanding the left hand side of the identity in Theorem 3.3, we get

$$C^4 - \frac{1}{2}\mathrm{tr}(C^2)C^2 - \frac{1}{2}C^2\mathrm{tr}(C^2) + \frac{1}{2}\left(\mathrm{tr}^2(C^2) - \mathrm{tr}(C^4)\right)I = 0,$$

whose all terms but C^4 contain a factor $\mathrm{tr}(C^2)$ or $\mathrm{tr}(C^4)$. $\square$

Corollary 3.5. *If $\frac{1}{2} \in R$ and R is a ring satisfying $[[x, y], [x, z]] = 0$, then for all $A \in \mathrm{M}_2(R)$ we have*

$$A^4 - \frac{1}{2}A^2\mathrm{tr}(A)A - \frac{1}{2}A\mathrm{tr}(A)A^2 - \frac{1}{2}A^3\mathrm{tr}(A) - \frac{1}{2}\mathrm{tr}(A)A^3 + \frac{1}{2}A^2\mathrm{tr}^2(A) + \frac{1}{2}\mathrm{tr}^2(A)A^2$$
$$- \frac{1}{2}A^2\mathrm{tr}(A^2) - \frac{1}{2}\mathrm{tr}(A^2)A^2 + \frac{1}{4}A\mathrm{tr}(A)A\mathrm{tr}(A) + \frac{1}{4}\mathrm{tr}(A)A\mathrm{tr}(A)A$$

$$
\begin{aligned}
&+\frac{1}{4}\mathrm{tr}(A)A^2\mathrm{tr}(A)+\frac{1}{4}A\mathrm{tr}^2(A)A-\frac{1}{4}\mathrm{tr}(A)A\mathrm{tr}^2(A)-\frac{1}{4}\mathrm{tr}^2(A)A\mathrm{tr}(A)\\
&+\frac{1}{4}\mathrm{tr}(A)A\mathrm{tr}(A^2)+\frac{1}{4}\mathrm{tr}(A^2)A\mathrm{tr}(A)-\frac{1}{4}A\mathrm{tr}^3(A)-\frac{1}{4}\mathrm{tr}^3(A)A\\
&+\frac{1}{4}A\mathrm{tr}(A)\mathrm{tr}(A^2)+\frac{1}{4}\mathrm{tr}(A^2)\mathrm{tr}(A)A-\frac{1}{2}\mathrm{tr}^2(A)\mathrm{tr}(A^2)I-\frac{1}{2}\mathrm{tr}(A^2)\mathrm{tr}^2(A)I\\
&+\frac{1}{2}\mathrm{tr}^2(A^2)I+\frac{1}{4}\mathrm{tr}\left(A^2\mathrm{tr}(A)A\right)I+\frac{1}{4}\mathrm{tr}(A\mathrm{tr}(A)A^2)I+\frac{1}{4}\mathrm{tr}(A^3)\mathrm{tr}(A)I+\frac{1}{4}\mathrm{tr}(A)\mathrm{tr}(A^3)I\\
&-\frac{1}{8}\mathrm{tr}(A)\mathrm{tr}(A\mathrm{tr}(A)A)I-\frac{1}{8}\mathrm{tr}(A\mathrm{tr}(A)A)\mathrm{tr}(A)I-\frac{1}{8}\mathrm{tr}(A\mathrm{tr}^2(A)A)I-\frac{1}{8}\mathrm{tr}(A)\mathrm{tr}(A^2)\mathrm{tr}(A)I\\
&+\frac{1}{2}\mathrm{tr}^4(A)I-\frac{1}{2}\mathrm{tr}(A^4)I=0.
\end{aligned}
$$

Proof. Apply Theorem 3.3 for $C = A - \frac{1}{2}\mathrm{tr}(A)I$; using linearity of $\mathrm{tr}(-)$, we get the identity above. $\square$

We note that the trace identity in Corollary 3.5 is different from the trace identity given by Domokos [1] in the following respect: in the latter in each term a power of A is multiplied from the left by a trace expression, whereas in our identity terms like $A^2\mathrm{tr}(A)A$ appear.

Throughout this section we have used the identity $[[x, y], [x, z]] = 0$. The referee pointed out that this identity implies the "seemingly stronger" identity $[[x, y], [u, v]] = 0$ of Lie solvability, which plays an important role in Section 2.

Starting with a matrix $C \in \mathrm{M}_2(R)$ such that $\mathrm{tr}(C) = 0$, define the sequence $(C_k)_{k\geqslant 0}$ by the following recursion: $C_0 = C$ and

$$C_{k+1} = C_k^2 - \frac{1}{2}\mathrm{tr}(C_k^2)I.$$

Clearly, $\mathrm{tr}(C_k) = 0$ for all $k \geqslant 0$ and C_k is a trace polynomial expression of C. In view of Corollary 3.2, the entries of C_1 are of the form $[x_1, x_2]$. The repeated application of Corollary 3.2 (as it can be seen in the proof of Theorem 3.3) and a straightforward induction show that the (four) entries of C_k are all of the form $[x_1, x_2, \ldots, x_{2^k}]_{\mathrm{solv}}$, where $[x_1, x_2]_{\mathrm{solv}} = [x_1, x_2]$ and for $i \geqslant 1$ we take the Lie brackets as

$$[x_1, x_2, \ldots, x_{2^{i+1}}]_{\mathrm{solv}} = [[x_1, x_2, \ldots, x_{2^i}]_{\mathrm{solv}}, [x_{2^i+1}, x_{2^i+2}, \ldots, x_{2^i+2^i}]_{\mathrm{solv}}].$$

If R satisfies the general identity

$$[x_1, x_2, \ldots, x_{2^k}]_{\mathrm{solv}} = 0$$

of Lie solvability, then $C_k = 0$, whence we can derive a trace identity for C. Thus the substitution $C = A - \frac{1}{2}\mathrm{tr}(A)I$ gives a trace identity for an arbitrary $A \in \mathrm{M}_2(R)$.

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References

[1] M. Domokos, Cayley–Hamilton theorem for 2×2 matrices over the Grassmann algebra, J. Pure Appl. Algebra 133 (1998) 69–81.
[2] V. Drensky, Free Algebras and PI-Algebras, Springer-Verlag, 2000.
[3] V. Drensky, E. Formanek, Polynomial Identity Rings, Birkhäuser-Verlag, 2004.
[4] S.A. Jennings, On rings whose associated Lie rings are nilpotent, Bull. Amer. Math. Soc. 53 (1947) 593–597.
[5] A.R. Kemer, Ideals of Identities of Associative Algebras, AMS, Providence, Rhode Island, 1991.
[6] J. Szigeti, New determinants and the Cayley–Hamilton theorem for matrices over Lie nilpotent rings, Proc. Amer. Math. Soc. 125 (8) (1997) 2245–2254.