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Sufficiency and Duality in Nonsmooth Multiobjective Optimization Involving Generalized (F, α, ρ, d)-Type I Functions

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Abstract—In this paper, new classes of generalized (F, α, ρ, d)-Type I functions are introduced for a nonsmooth multiobjective programming problem. Based upon these generalized functions, sufficient optimality conditions are established. Weak, strong, and strict converse duality theorems are also derived for Wolfe and Mond-Weir type multiobjective dual programs in order to relate the efficient and weak efficient solutions of primal and dual problems.

Keywords—Nonsmooth multiobjective programming, (F, α, ρ, d)-Type I function, Efficiency, Sufficiency, Duality.

1. INTRODUCTION

Hanson [1] introduced a class of functions by generalizing the difference vector $x - \bar{x}$ in the definition of a convex function to any vector function $\eta(x, \bar{x})$. These functions were named invex by Craven [2] and η-convex by Kaul and Kaur [3]. Hanson and Mond [4] defined two new classes of functions called Type I and Type II functions, which were further generalized to pseudo Type I and quasi Type I functions by Rueda and Hanson [5]. Zhao [6] established optimality conditions and duality in nonsmooth scalar programming problems assuming Clarke [7] generalized subgradients under Type I functions.

Kaul *et al.* [8] extended the concept of Type I functions from a single objective to a multiobjective programming problem by defining the Type I and its various generalizations. They investigated necessary and sufficient optimality conditions and derived Wolfe type and Mond-Weir type duality results. Suneja and Srivastava [9] introduced generalized d-Type I functions in terms of directional derivative for a multiobjective programming problem and discussed Wolfe type and Mond-Weir type duality results. In [10], Aghezzaf and Hachimi introduced classes of generalized Type I vector valued functions for a differentiable multiobjective programming problem and established duality results. Recently, Kuk and Tanino [11] derived optimality conditions

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and duality theorems for nonsmooth multiobjective programming problems involving generalized Type I vector valued functions.

Motivated by various concepts of generalized convexity, Liang *et al.* [12] introduced a unified formulation of generalized convexity called (F, α, ρ, d)-convexity and obtained some optimality conditions and duality results for nonlinear fractional programming problems. In [13], Hachimi and Aghezzaf extended the concept to (F, α, ρ, d)-Type I functions and obtained several sufficiency optimality conditions and established weak and strong duality theorems for mixed type duality. Chen [14] gave definitions for the generalized (F, ρ)-convex class about the Clark subgradient and obtained optimality and duality results for multiobjective fractional programming problems.

In this paper, we define new classes of functions called generalized (F, α, ρ, d)-Type I functions for a nonsmooth multiobjective programming problem and derive sufficient optimality conditions. We also obtain Wolfe type and Mond-Weir type duality results.

2. NOTATIONS AND PRELIMINARIES

The following convention of vectors in R^n will be followed throughout this paper: $x \geqq y \Leftrightarrow x_i \geqq y_i$, $i = 1, 2, \ldots, n$; $x \geq y \Leftrightarrow x \geqq y$, $x \neq y$; $x > y \Leftrightarrow x_i > y_i$, $i = 1, 2, \ldots, n$.

A function $f : R^n \mapsto R$ is said to be locally Lipschitz at a point $\bar{x} \in R^n$ if there exist scalar $K > 0$ and $\epsilon > 0$ such that

$$\left|f\left(x^1\right) - f\left(x^2\right)\right| \leqq K \left\|x^1 - x^2\right\|,$$

for all $x^1, x^2 \in \bar{x} + \epsilon B$, where $\bar{x} + \epsilon B$ is the open ball of radius ϵ about $\bar{x}$.

The Clarke generalized directional derivative [7] of a locally Lipschitz function f at $\bar{x}$ in the direction v, denoted by $f^\circ(\bar{x}; v)$, is defined as follows:

$$f^\circ\left(\bar{x}; v\right) = \limsup_{y \to \bar{x},\, t \to 0} \frac{f(y + tv) - f(y)}{t}.$$

The Clarke generalized gradient [7] of f at $\bar{x}$, denoted by $\partial^c f(\bar{x})$, is defined as

$$\partial^c f(\bar{x}) = \left\{\xi \mid f^\circ(\bar{x}; v) \geqq \xi v, \text{ for all } v \in R^n\right\}.$$

We consider the following nonlinear multiobjective programming problem:

$$\begin{aligned} &\text{Minimize } f(x) = (f_1(x), f_2(x), \ldots, f_k(x)), \\ &\text{subject to } x \in X_\circ = \{x \in X : g(x) \leqq 0\}, \end{aligned} \tag{MOP}$$

where $X \subseteq R^n$ is an open set and the functions $f = (f_1, f_2, \ldots, f_k) : X \mapsto R^k$ and $g = (g_1, g_2, \ldots, g_m) : X \mapsto R^m$ are locally Lipschitz on X.

Since the objectives in multiobjective problems generally conflict with one another, an optimal solution is chosen from the set of efficient/weak efficient solutions. Geoffrion [15] defined a restricted concept of efficiency, called proper efficiency.

DEFINITION 2.1. *A point $\bar{x} \in X_\circ$ is said to be a weak efficient (weak Pareto) solution of (MOP), if there exists no $x \in X_\circ$ such that*

$$f(x) < f(\bar{x}).$$

DEFINITION 2.2. *A point $\bar{x} \in X_\circ$ is said to be an efficient solution of (MOP), if there exists no $x \in X_\circ$ such that*

$$f(x) \leq f(\bar{x}).$$

DEFINITION 2.3. *An efficient solution $\bar{x}$ of (MOP) is said to be properly efficient if there exists a scalar $M > 0$ such that for each $r \in K$ and $x \in X_\circ$ satisfying $f_r(x) < f_r(\bar{x})$, we have*

$$f_r(\bar{x}) - f_r(x) \leqq M[f_j(x) - f_j(\bar{x})]$$

for at least one j satisfying $f_j(\bar{x}) < f_j(x)$.

Let $F : X \times X \times R^n \mapsto R$ be a sublinear functional in the third variable. Let $K = \{1, 2, \dots, k\}$, $M = \{1, 2, \dots, m\}$, and $d : X \times X \mapsto R$. Let $\alpha = (\alpha^1, \alpha^2) : X \times X \mapsto R_+ \setminus \{0\}$ and $\rho = (\rho^1, \rho^2)$ such that $\rho^1 = (\rho_1, \rho_2, \dots, \rho_k) \in R^k$, $\rho^2 = (\rho_{k+1}, \rho_{k+2}, \dots, \rho_{k+m}) \in R^m$, i.e., ρ^1 has k components corresponding to k components of f and ρ^2 has m components corresponding to m components of g. The number of components in ρ^1 and ρ^2 may vary depending upon the way the objective and constraint functions are involved in various hypotheses, e.g., the hypothesis may be on f, g, λf, or μg, etc. Also for $\bar{x} \in X_\circ$, $J(\bar{x}) = \{j : g_j(\bar{x}) = 0\}$ and g_J will denote the vector of active constraints at $\bar{x}$. For a vector function $f : X \mapsto R^k$, $\xi \in \partial^c f(\bar{x})$ means $\xi_i \in \partial^c f_i(\bar{x})$ for $i \in K$ and the symbol $F(x, \bar{x}; \xi)$ denotes the vector of components $F(x, \bar{x}; \xi_1), \dots, F(x, \bar{x}; \xi_k)$.

DEFINITION 2.4. *(f, g) is said to be (F, α, ρ, d)-Type I at $\bar{x}$ if for each $x \in X_\circ$*

$$f(x) - f(\bar{x}) \geqq F\left(x, \bar{x}; \alpha^1(x, \bar{x})\xi\right) + \rho^1 d^2(x, \bar{x}), \qquad \text{for all } \xi \in \partial^c f(\bar{x}),$$
$$-g(\bar{x}) \geqq F\left(x, \bar{x}; \alpha^2(x, \bar{x})\eta\right) + \rho^2 d^2(x, \bar{x}), \qquad \text{for all } \eta \in \partial^c g(\bar{x}).$$

If the first inequality is satisfied as

$$f(x) - f(\bar{x}) > F\left(x, \bar{x}; \alpha^1(x, \bar{x})\xi\right) + \rho^1 d^2(x, \bar{x}), \qquad \text{for all } \xi \in \partial^c f(\bar{x}),$$

then (f, g) is said to be semistrictly (F, α, ρ, d)-Type I at $\bar{x}$.

DEFINITION 2.5. *(f, g) is said to be pseudoquasi (F, α, ρ, d)-Type I at $\bar{x}$ if for each $x \in X_\circ$*

$$f(x) < f(\bar{x}) \Rightarrow F\left(x, \bar{x}; \alpha^1(x, \bar{x})\xi\right) + \rho^1 d^2(x, \bar{x}) < 0, \qquad \text{for all } \xi \in \partial^c f(\bar{x}),$$
$$-g(\bar{x}) \leqq 0 \Rightarrow F\left(x, \bar{x}; \alpha^2(x, \bar{x})\eta\right) + \rho^2 d^2(x, \bar{x}) \leqq 0, \qquad \text{for all } \eta \in \partial^c g(\bar{x}).$$

DEFINITION 2.6. *(f, g) is said to be strictly pseudoquasi (F, α, ρ, d)-Type I at $\bar{x}$ if for each $x \in X_\circ$*

$$f(x) \leqq f(\bar{x}) \Rightarrow F\left(x, \bar{x}; \alpha^1(x, \bar{x})\xi\right) + \rho^1 d^2(x, \bar{x}) < 0, \qquad \text{for all } \xi \in \partial^c f(\bar{x}),$$
$$-g(\bar{x}) \leqq 0 \Rightarrow F\left(x, \bar{x}; \alpha^2(x, \bar{x})\eta\right) + \rho^2 d^2(x, \bar{x}) \leqq 0, \qquad \text{for all } \eta \in \partial^c g(\bar{x}).$$

DEFINITION 2.7. *(f, g) is said to be weak strictly pseudoquasi (F, α, ρ, d)-Type I at $\bar{x}$ if for each $x \in X_\circ$*

$$f(x) \leq f(\bar{x}) \Rightarrow F\left(x, \bar{x}; \alpha^1(x, \bar{x})\xi\right) + \rho^1 d^2(x, \bar{x}) < 0, \qquad \text{for all } \xi \in \partial^c f(\bar{x}),$$
$$-g(\bar{x}) \leqq 0 \Rightarrow F\left(x, \bar{x}; \alpha^2(x, \bar{x})\eta\right) + \rho^2 d^2(x, \bar{x}) \leqq 0, \qquad \text{for all } \eta \in \partial^c g(\bar{x}).$$

DEFINITION 2.8. *(f,g) is said to be strong pseudoquasi (F,α,ρ,d)-Type I at $\bar{x}$ if for each $x \in X_\circ$*

$$\begin{aligned} f(x) \leq f(\bar{x}) &\Rightarrow F\left(x,\bar{x};\alpha^1(x,\bar{x})\xi\right) + \rho^1 d^2(x,\bar{x}) \leq 0, && \text{for all } \xi \in \partial^c f(\bar{x}), \\ -g(\bar{x}) \leqq 0 &\Rightarrow F\left(x,\bar{x};\alpha^2(x,\bar{x})\eta\right) + \rho^2 d^2(x,\bar{x}) \leqq 0, && \text{for all } \eta \in \partial^c g(\bar{x}). \end{aligned}$$

If the first inequality is satisfied as

$$f(x) < f(\bar{x}) \Rightarrow F\left(x,\bar{x};\alpha^1(x,\bar{x})\xi\right) + \rho^1 d^2(x,\bar{x}) \leq 0, \qquad \text{for all } \xi \in \partial^c f(\bar{x}),$$

then (f,g) is said to be weak pseudoquasi (F,α,ρ,d)-Type I at $\bar{x}$.

DEFINITION 2.9. *(f,g) is said to be quasi strictly pseudo (F,α,ρ,d)-Type I at $\bar{x}$ if for each $x \in X_\circ$*

$$\begin{aligned} f(x) \leqq f(\bar{x}) &\Rightarrow F\left(x,\bar{x};\alpha^1(x,\bar{x})\xi\right) + \rho^1 d^2(x,\bar{x}) \leqq 0, && \text{for all } \xi \in \partial^c f(\bar{x}), \\ -g(\bar{x}) \leqq 0 &\Rightarrow F\left(x,\bar{x};\alpha^2(x,\bar{x})\eta\right) + \rho^2 d^2(x,\bar{x}) < 0, && \text{for all } \eta \in \partial^c g(\bar{x}). \end{aligned}$$

DEFINITION 2.10. *(f,g) is said to be weak quasi strictly pseudo (F,α,ρ,d)-Type I at $\bar{x}$ if for each $x \in X_\circ$*

$$\begin{aligned} f(x) \leq f(\bar{x}) &\Rightarrow F\left(x,\bar{x};\alpha^1(x,\bar{x})\xi\right) + \rho^1 d^2(x,\bar{x}) \leqq 0, && \text{for all } \xi \in \partial^c f(\bar{x}), \\ -g(\bar{x}) \leqq 0 &\Rightarrow F\left(x,\bar{x};\alpha^2(x,\bar{x})\eta\right) + \rho^2 d^2(x,\bar{x}) < 0, && \text{for all } \eta \in \partial^c g(\bar{x}). \end{aligned}$$

3. SUFFICIENT OPTIMALITY CONDITIONS

In this section we establish some sufficient optimality conditions for the problem (MOP). The first theorem gives a properly efficient solution of (MOP). As we go on weakening the assumptions, we get a weaker conclusion of the efficient/weak efficient solution of (MOP).

THEOREM 3.1. *Suppose that there exists a feasible solution $\bar{x}$ for (MOP) and scalars $\bar{\lambda} > 0 \in R^k$, $\bar{\mu}_J \geqq 0$ such that*

(i) $0 \in \bar{\lambda}\partial^c f(\bar{x}) + \bar{\mu}_J \partial^c g_J(\bar{x})$,
(ii) *(f, g_J) is (F,α,ρ,d)-Type I at $\bar{x}$,*
(iii) $\bar{\lambda}\rho^1/\alpha^1(x,\bar{x}) + \bar{\mu}_J\rho^2/\alpha^2(x,\bar{x}) \geqq 0$.

Then $\bar{x}$ is a properly efficient solution for (MOP).

PROOF. Since Hypothesis (ii) holds, we have for all $x \in X_\circ$

$$\begin{aligned} f(x) - f(\bar{x}) &\geqq F\left(x,\bar{x};\alpha^1(x,\bar{x})\xi\right) + \rho^1 d^2(x,\bar{x}), && \text{for all } \xi \in \partial^c f(\bar{x}), \\ 0 = -g_J(\bar{x}) &\geqq F\left(x,\bar{x};\alpha^2(x,\bar{x})\eta\right) + \rho^2 d^2(x,\bar{x}), && \text{for all } \eta \in \partial^c g_J(\bar{x}). \end{aligned}$$

Using $\bar{\lambda} > 0$, $\bar{\mu}_J \geqq 0$ and $\alpha^1(x,\bar{x}) > 0$, $\alpha^2(x,\bar{x}) > 0$, we get

$$\begin{aligned} \frac{\bar{\lambda} f(x)}{\alpha^1(x,\bar{x})} - \frac{\bar{\lambda} f(\bar{x})}{\alpha^1(x,\bar{x})} &\geqq F\left(x,\bar{x};\bar{\lambda}\xi\right) + \frac{\bar{\lambda}\rho^1 d^2(x,\bar{x})}{\alpha^1(x,\bar{x})}, \\ 0 &\geqq F(x,\bar{x};\bar{\mu}_J\eta) + \frac{\bar{\mu}_J\rho^2 d^2(x,\bar{x})}{\alpha^2(x,\bar{x})}. \end{aligned}$$

By sublinearity of F, we have for all $\xi \in \partial^c f(\bar{x})$ and $\eta \in \partial^c g_J(\bar{x})$

$$\begin{aligned} F\left(x,\bar{x};\bar{\lambda}\xi + \bar{\mu}_J\eta\right) &\leqq F\left(x,\bar{x};\bar{\lambda}\xi\right) + F\left(x,\bar{x};\bar{\mu}_J\eta\right) \\ &\leqq \frac{\bar{\lambda} f(x)}{\alpha^1(x,\bar{x})} - \frac{\bar{\lambda} f(\bar{x})}{\alpha^1(x,\bar{x})} - \left(\frac{\bar{\lambda}\rho^1}{\alpha^1(x,\bar{x})} + \frac{\bar{\mu}_J\rho^2}{\alpha^2(x,\bar{x})}\right) d^2(x,\bar{x}) \\ &\leqq \frac{\bar{\lambda} f(x)}{\alpha^1(x,\bar{x})} - \frac{\bar{\lambda} f(\bar{x})}{\alpha^1(x,\bar{x})} \qquad \text{(using Hypothesis (iii))}. \end{aligned}$$

By Hypothesis (i), there exists some $\bar{\xi} \in \partial^c f(\bar{x})$ and $\bar{\eta} \in \partial^c g_J(\bar{x})$ such that

$$\bar{\lambda}\bar{\xi} + \bar{\mu}_J\bar{\eta} = 0, \qquad \text{which implies } F\left(x, \bar{x}; \bar{\lambda}\bar{\xi} + \bar{\mu}_J\bar{\eta}\right) = F(x, \bar{x}; 0) = 0.$$

So, the above inequality yields

$$\frac{\bar{\lambda}f(\bar{x})}{\alpha^1(x,\bar{x})} \leqq \frac{\bar{\lambda}f(x)}{\alpha^1(x,\bar{x})}.$$

As $\alpha^1(x, \bar{x}) > 0$, we get

$$\bar{\lambda}f(\bar{x}) \leqq \bar{\lambda}f(x).$$

Hence, by Theorem 1 in [15], $\bar{x}$ is a properly efficient solution for (MOP).

THEOREM 3.2. *Suppose that there exists a feasible solution $\bar{x}$ for (MOP) and vectors $\bar{\lambda} > 0 \in R^k$, $\bar{\mu}_J \geqq 0$ such that*

(i) $0 \in \bar{\lambda}\partial^c f(\bar{x}) + \bar{\mu}_J\partial^c g_J(\bar{x})$,
(ii) (f, g_J) *is strong pseudoquasi* (F, α, ρ, d)*-Type I at* $\bar{x}$,
(iii) $\bar{\lambda}\rho^1/\alpha^1(x,\bar{x}) + \bar{\mu}_J\rho^2/\alpha^2(x,\bar{x}) \geqq 0$.

Then $\bar{x}$ is an efficient solution for (MOP).

PROOF. Suppose that $\bar{x}$ is not an efficient solution for (MOP). Then there exists $x \in X_\circ$ such that

$$f(x) \leq f(\bar{x}).$$

Also, since $g_J(\bar{x}) = 0$, Hypothesis (ii) gives

$$F\left(x, \bar{x}; \alpha^1(x,\bar{x})\xi\right) + \rho^1 d^2(x,\bar{x}) \leq 0, \qquad \text{for all } \xi \in \partial^c f(\bar{x}),$$
$$F\left(x, \bar{x}; \alpha^2(x,\bar{x})\eta\right) + \rho^2 d^2(x,\bar{x}) \leqq 0, \qquad \text{for all } \eta \in \partial^c g_J(\bar{x}).$$

Multiplying the above inequalities by $\bar{\lambda}/\alpha^1(x,\bar{x})$ and $\bar{\mu}_J/\alpha^2(x,\bar{x})$, respectively, we have

$$\bar{\lambda}F(x,\bar{x};\xi) < -\frac{\bar{\lambda}\rho^1 d^2(x,\bar{x})}{\alpha^1(x,\bar{x})}, \qquad \text{and} \qquad \bar{\mu}_J F(x,\bar{x};\eta) \leqq -\frac{\bar{\mu}_J\rho^2 d^2(x,\bar{x})}{\alpha^2(x,\bar{x})}.$$

For all $\xi \in \partial^c f(\bar{x})$ and $\eta \in \partial^c g_J(\bar{x})$, sublinearity of F yields

$$\begin{aligned} F\left(x,\bar{x};\bar{\lambda}\xi + \bar{\mu}_J\eta\right) &\leqq F\left(x,\bar{x};\bar{\lambda}\xi\right) + F\left(x,\bar{x};\bar{\mu}_J\eta\right) \\ &< -\left(\frac{\bar{\lambda}\rho^1}{\alpha^1(x,\bar{x})} + \frac{\bar{\mu}_J\rho^2}{\alpha^2(x,\bar{x})}\right) d^2(x,\bar{x}) \\ &\leqq 0 \qquad \text{(using Hypothesis (iii))}, \end{aligned}$$

which contradicts $F(x,\bar{x};0) = 0$ as by Hypothesis (i) there exists some $\bar{\xi} \in \partial^c f(\bar{x})$, $\bar{\eta} \in \partial^c g_J(\bar{x})$ such that

$$\bar{\lambda}\bar{\xi} + \bar{\mu}_J\bar{\eta} = 0.$$

Hence, $\bar{x}$ is an efficient solution for (MOP).

THEOREM 3.3. *Suppose that there exists a feasible solution $\bar{x}$ for (MOP) and vectors $\bar{\lambda} \geq 0 \in R^k$, $\bar{\mu}_J \geqq 0$ such that*

(i) $0 \in \bar{\lambda}\partial^c f(\bar{x}) + \bar{\mu}_J\partial^c g_J(\bar{x})$,
(ii) (f, g_J) *is weak strictly pseudoquasi* (F, α, ρ, d)*-Type I at* $\bar{x}$,
(iii) $\bar{\lambda}\rho^1/\alpha^1(x,\bar{x}) + \bar{\mu}_J\rho^2/\alpha^2(x,\bar{x}) \geqq 0$.

Then $\bar{x}$ is an efficient solution for (MOP).

PROOF. Suppose that $\bar{x}$ is not an efficient solution for (MOP). Then there exists $x \in X_\circ$ such that

$$f(x) \leq f(\bar{x}).$$

Also, since $g_J(\bar{x}) = 0$, Hypothesis (ii) gives

$$\begin{aligned} &F\left(x, \bar{x}; \alpha^1(x,\bar{x})\xi\right) + \rho^1 d^2(x,\bar{x}) < 0, \qquad &&\text{for all } \xi \in \partial^c f(\bar{x}),\\ &F\left(x, \bar{x}; \alpha^2(x,\bar{x})\eta\right) + \rho^2 d^2(x,\bar{x}) \leqq 0, \qquad &&\text{for all } \eta \in \partial^c g_J(\bar{x}).\end{aligned}$$

The rest of the proof is similar to that of the previous theorem.

THEOREM 3.4. *Suppose that there exists a feasible solution $\bar{x}$ for (MOP) and vectors $\bar{\lambda} \geqq 0 \in R^k$, $\bar{\mu}_J \geq 0$ such that*

(i) $0 \in \bar{\lambda}\partial^c f(\bar{x}) + \bar{\mu}_J \partial^c g_J(\bar{x})$,
(ii) *(f, g_J) is weak quasi strictly pseudo (F, α, ρ, d)-Type I at $\bar{x}$,*
(iii) $\bar{\lambda}\rho^1/\alpha^1(x,\bar{x}) + \bar{\mu}_J \rho^2/\alpha^2(x,\bar{x}) \geqq 0$.

Then $\bar{x}$ is an efficient solution for (MOP).

PROOF. Suppose that $\bar{x}$ is not an efficient solution for (MOP). Then there exists $x \in X_\circ$ such that

$$f(x) \leq f(\bar{x}).$$

Also, since $g_J(\bar{x}) = 0$, Hypothesis (ii) gives

$$\begin{aligned} &F\left(x, \bar{x}; \alpha^1(x,\bar{x})\xi\right) + \rho^1 d^2(x,\bar{x}) \leqq 0, \qquad &&\text{for all } \xi \in \partial^c f(\bar{x}),\\ &F\left(x, \bar{x}; \alpha^2(x,\bar{x})\eta\right) + \rho^2 d^2(x,\bar{x}) < 0, \qquad &&\text{for all } \eta \in \partial^c g_J(\bar{x}).\end{aligned}$$

The rest of the proof is similar to that of Theorem 3.2.

THEOREM 3.5. *Suppose that there exists a feasible solution $\bar{x}$ for (MOP) and vectors $\bar{\lambda} > 0 \in R^k$, $\bar{\mu}_J \geqq 0$ such that*

(i) $0 \in \bar{\lambda}\partial^c f(\bar{x}) + \bar{\mu}_J \partial^c g_J(\bar{x})$,
(ii) *(f, g_J) is weak pseudoquasi (F, α, ρ, d)-Type I at $\bar{x}$,*
(iii) $\bar{\lambda}\rho^1/\alpha^1(x,\bar{x}) + \bar{\mu}_J \rho^2/\alpha^2(x,\bar{x}) \geqq 0$.

Then $\bar{x}$ is a weak efficient solution for (MOP).

PROOF. Suppose that $\bar{x}$ is not a weak efficient solution for (MOP). Then there exists $x \in X_\circ$ such that

$$f(x) < f(\bar{x}).$$

Also, we have

$$g_J(\bar{x}) = 0.$$

Hence, Hypothesis (ii) gives

$$\begin{aligned} &F\left(x, \bar{x}; \alpha^1(x,\bar{x})\xi\right) + \rho^1 d^2(x,\bar{x}) \leq 0, \qquad &&\text{for all } \xi \in \partial^c f(\bar{x}),\\ &F\left(x, \bar{x}; \alpha^2(x,\bar{x})\eta\right) + \rho^2 d^2(x,\bar{x}) \leqq 0, \qquad &&\text{for all } \eta \in \partial^c g_J(\bar{x}).\end{aligned}$$

The rest of the proof is similar to that of Theorem 3.2.

THEOREM 3.6. *Suppose that there exists a feasible solution $\bar{x}$ for (MOP) and vectors $\bar{\lambda} \geq 0 \in R^k$, $\bar{\mu}_J \geqq 0$ such that*

(i) $0 \in \bar{\lambda}\partial^c f(\bar{x}) + \bar{\mu}_J \partial^c g_J(\bar{x})$,
(ii) *(f, g_J) is pseudoquasi (F, α, ρ, d)-Type I at $\bar{x}$,*
(iii) $\bar{\lambda}\rho^1/\alpha^1(x,\bar{x}) + \bar{\mu}_J \rho^2/\alpha^2(x,\bar{x}) \geqq 0$.

Then $\bar{x}$ is a weak efficient solution for (MOP).

PROOF. Suppose that $\bar{x}$ is not an weak efficient solution for (MOP). Then there exists $x \in X_\circ$ such that

$$f(x) < f(\bar{x}).$$

Also, we have

$$g_J(\bar{x}) = 0.$$

Hence, Hypothesis (ii) gives

$$\begin{aligned} F\left(x,\bar{x};\alpha^1(x,\bar{x})\xi\right) + \rho^1 d^2(x,\bar{x}) &< 0, &\quad \text{for all } \xi \in \partial^c f(\bar{x}),\\ F\left(x,\bar{x};\alpha^2(x,\bar{x})\eta\right) + \rho^2 d^2(x,\bar{x}) &\leqq 0, &\quad \text{for all } \eta \in \partial^c g_J(\bar{x}). \end{aligned}$$

The rest of the proof is similar to that of Theorem 3.2.

4. WOLFE TYPE DUALITY

In this section, we consider the following Wolfe type dual for (MOP) and establish weak, strong, and strict converse duality theorems.

$$\text{Maximize } f(y) + \mu g(y)e, \tag{WD}$$
$$\text{subject to } y \in X,$$
$$0 \in \lambda\partial^c f(y) + \mu\partial^c g(y), \tag{1}$$
$$\lambda_i \geqq 0, \qquad i = 1,2,\dots,k, \tag{2}$$
$$\mu_j \geqq 0, \qquad j = 1,2,\dots,m, \tag{3}$$
$$\lambda e = 1, \tag{4}$$

where e is a k-dimensional vector whose all components are ones.

THEOREM 4.1. WEAK DUALITY. *Let x and (y,λ,μ) be feasible solutions of (MOP) and (WD), respectively, such that*

(i) *(f,g) is (F,α,ρ,d)-Type I at y with $\alpha^1(x,y) = \alpha^2(x,y)$.*
Also, if either
(a) *$\lambda > 0$ and $\lambda\rho^1 + \mu\rho^2 \geqq 0$, or*
(b) *$\lambda\rho^1 + \mu\rho^2 > 0$ holds,*

then the following cannot hold:

$$f(x) \leq f(y) + \mu g(y)e. \tag{5}$$

PROOF. Under Hypothesis (a): by Hypothesis (i), we have

$$\begin{aligned} f(x) - f(y) &\geqq F\left(x,y;\alpha^1(x,y)\xi\right) + \rho^1 d^2(x,y), &\quad \text{for all } \xi \in \partial^c f(y),\\ -g(y) &\geqq F\left(x,y;\alpha^1(x,y)\eta\right) + \rho^2 d^2(x,y), &\quad \text{for all } \eta \in \partial^c g(y). \end{aligned}$$

Since $\lambda > 0$, we get

$$\frac{\lambda f(x)}{\alpha^1(x,y)} - \frac{\lambda f(y)}{\alpha^1(x,y)} \geqq F(x,y;\lambda\xi) + \frac{\lambda\rho^1 d^2(x,y)}{\alpha^1(x,y)},$$
$$\frac{-\mu g(y)}{\alpha^1(x,y)} \geqq F(x,y;\mu\eta) + \frac{\mu\rho^2 d^2(x,y)}{\alpha^1(x,y)}.$$

Hence, the above inequalities with sublinearity of F give

$$\begin{aligned} F(x,y;\lambda\xi+\mu\eta) &\leqq F(x,y;\lambda\xi) + F(x,y;\mu\eta) \\ &\leqq \frac{\lambda f(x)}{\alpha^1(x,y)} - \frac{\lambda f(y)}{\alpha^1(x,y)} - \frac{\mu g(y)}{\alpha^1(x,y)} - \left(\frac{\lambda\rho^1}{\alpha^1(x,y)} + \frac{\mu\rho^2}{\alpha^1(x,y)}\right) d^2(x,y). \end{aligned} \tag{6}$$

Since $\alpha^1(x,y) > 0$, Hypothesis (a) reduces (6) to

$$F(x,y;\lambda\xi+\mu\eta) \leqq \frac{\lambda f(x)}{\alpha^1(x,y)} - \frac{\lambda f(y)}{\alpha^1(x,y)} - \frac{\mu g(y)}{\alpha^1(x,y)}. \tag{7}$$

Now suppose, to the contrary, that (5) holds, that is

$$f(x) \leq f(y) + \mu g(y)e.$$

Since, $\lambda > 0$, $\lambda e = 1$, and $\alpha^1(x,y) > 0$, the above inequality implies

$$\frac{\lambda f(x)}{\alpha^1(x,y)} < \frac{\lambda f(y)}{\alpha^1(x,y)} + \frac{\mu g(y)}{\alpha^1(x,y)}.$$

Hence, (7) yields

$$F(x,y;\lambda\xi+\mu\eta) < 0, \qquad \text{for all } \xi \in \partial^c f(y) \quad \text{and} \quad \eta \in \partial^c g(y),$$

which contradicts the dual constraint (1) as, for some $\bar{\xi} \in \partial^c f(y)$ and $\bar{\eta} \in \partial^c g(y)$,

$$\lambda\bar{\xi} + \mu\bar{\eta} = 0, \qquad \text{which implies } F\left(x,y;\lambda\bar{\xi}+\mu\bar{\eta}\right) = 0.$$

Hence, (5) cannot hold.

Under Hypothesis (b): using (2) and (4), inequality (5) yields

$$\frac{\lambda f(x)}{\alpha^1(x,y)} \leqq \frac{\lambda f(y)}{\alpha^1(x,y)} + \frac{\mu g(y)}{\alpha^1(x,y)}.$$

The above inequality along with Hypothesis (b) reduces (6) to

$$F(x,y;\lambda\xi+\mu\eta) < 0,$$

which contradicts $F(x,y;0) = 0$. Hence the result of the theorem holds.

The proofs of the following weak duality theorems are similar to Theorem 4.1 and hence are omitted.

THEOREM 4.2. WEAK DUALITY. *Let x and (y,λ,μ) be feasible solutions of (MOP) and (WD), respectively, such that*

(i) *(f,g) is semistrictly (F,α,ρ,d)-Type I at y,*
(ii) *$\alpha^1(x,y) = \alpha^2(x,y)$,*
(iii) *$\lambda\rho^1 + \mu\rho^2 \geqq 0$.*

Then the following cannot hold:

$$f(x) \leq f(y) + \mu g(y)e.$$

THEOREM 4.3. WEAK DUALITY. *Let x and (y, λ, μ) be feasible solutions of (MOP) and (WD), respectively, such that*

(i) (f, g) *is* (F, α, ρ, d)*-Type I at* y,
(ii) $\alpha^1(x, y) = \alpha^2(x, y)$,
(iii) $\lambda\rho^1 + \mu\rho^2 \geqq 0$.

Then the following cannot hold:

$$f(x) < f(y) + \mu g(y)e.$$

COROLLARY 4.4. *Let x° and $(y^\circ, \lambda^\circ, \mu^\circ)$ be feasible solutions for (MOP) and (WD), respectively, such that $f(x^\circ) = f(y^\circ) + \mu^\circ g(y^\circ)e$. If the weak duality holds between (MOP) and (WD) for all feasible solutions of the two problems, then x° is efficient for (MOP) and $(y^\circ, \lambda^\circ, \mu^\circ)$ is efficient for (WD).*

PROOF. Suppose that x° is not efficient for (MOP) then for some $x \in X_\circ$

$$f(x) \leq f(x^\circ).$$

Since $f(x^\circ) = f(y^\circ) + \mu^\circ g(y^\circ)e$, the above inequality can be written as

$$f(x) \leq f(y^\circ) + \mu^\circ g(y^\circ) e,$$

which contradicts the result of Theorem 4.1 as $(y^\circ, \lambda^\circ, \mu^\circ)$ is feasible for (WD) and x is feasible for (MOP). So, x° is efficient for (MOP). Similarly $(y^\circ, \lambda^\circ, \mu^\circ)$ is efficient for (WD).

DEFINITION 4.1. COTTLE CONSTRAINT QUALIFICATION. *Let f_i, $i \in K$, g_j, $j \in M$ be locally Lipschitz functions at a point $u \in X$. The problem (MOP) satisfies the Cottle constraint qualification at u, if either $g_j(u) < 0$ for all $j \in M$ or $0 \notin \operatorname{conv}\{\partial^c g_j(u) : g_j(u) = 0\}$, where $\operatorname{conv} S$ denotes the convex hull of the set S.*

THEOREM 4.5. KARUSH-KUHN-TUCKER NECESSARY CONDITIONS [16, p. 49]. *Assume that $\bar{u}$ is an efficient solution for (MOP) at which the Cottle constraint qualification is satisfied. Then, there exist multipliers $\bar{\lambda}_i \geqq 0$, $i \in K$, $\bar{\lambda}e = 1$, and $\bar{\mu} \in R^m$, such that*

$$0 \in \bar{\lambda}\partial^c f(\bar{u}) + \bar{\mu}\partial^c g(\bar{u}),$$
$$\bar{\mu} g(\bar{u}) = 0.$$

THEOREM 4.6. STRONG DUALITY. *Let $\bar{x}$ be an efficient solution for (MOP) at which the Cottle constraint qualification is satisfied. Then, there exist $\bar{\lambda} \in R^k$, $\bar{\mu} \in R^m$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is feasible for (WD) and the two objectives are equal. Furthermore, if weak duality holds between (MOP) and (WD) for all feasible solutions of the primal and dual problems, then $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is efficient for (WD).*

PROOF. Since $\bar{x}$ is an efficient solution for (MOP) and the Cottle constraint qualification is satisfied at $\bar{x}$, from Theorem 4.5, there exist $\bar{\lambda} \geqq 0 \in R^k$, $\bar{\lambda}e = 1$, $\bar{\mu} \geqq 0 \in R^m$ such that

$$0 \in \bar{\lambda}\partial^c f(\bar{x}) + \bar{\mu}\partial^c g(\bar{x}),$$
$$\bar{\mu} g(\bar{x}) = 0,$$

which gives that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is feasible for (WD) and the objectives are equal. Efficiency of $(\bar{x}, \bar{\lambda}, \bar{\mu})$ for (WD) follows from Corollary 4.4.

THEOREM 4.7. STRICT CONVERSE DUALITY. *Let $\bar{x}$ and $(\bar{y}, \bar{\lambda}, \bar{\mu})$ be feasible for (MOP) and (WD), respectively, such that*

(i) $\bar{\lambda} f(\bar{x}) \leqq \bar{\lambda} f(\bar{y}) + \bar{\mu} g(\bar{y})$,
(ii) (f, g) *is semistrictly* (F, α, ρ, d)*-Type I at* $\bar{y}$ *with* $\alpha^1(\bar{x}, \bar{y}) = \alpha^2(\bar{x}, \bar{y})$,
(iii) $\bar{\lambda}\rho^1 + \bar{\mu}\rho^2 \geqq 0$.

Then $\bar{x} = \bar{y}$.

PROOF. We suppose that $\bar{x} \neq \bar{y}$ and exhibit a contradiction. Since $(\bar{y}, \bar{\lambda}, \bar{\mu})$ is feasible for (WD), by the dual constraint (1), for some $\bar{\xi} \in \partial^c f(\bar{y})$ and $\bar{\eta} \in \partial^c g(\bar{y})$,

$$\bar{\lambda}\bar{\xi} + \bar{\mu}\bar{\eta} = 0,$$

which implies

$$F\left(\bar{x}, \bar{y}; \bar{\lambda}\bar{\xi} + \bar{\mu}\bar{\eta}\right) = 0. \tag{8}$$

Also, using Hypothesis (ii), we have

$$f(\bar{x}) - f(\bar{y}) > F\left(\bar{x}, \bar{y}; \alpha^1(\bar{x}, \bar{y})\xi\right) + \rho^1 d^2(\bar{x}, \bar{y}), \qquad \text{for all } \xi \in \partial^c f(\bar{y}),$$
$$-g(\bar{y}) \geqq F\left(\bar{x}, \bar{y}; \alpha^1(\bar{x}, \bar{y})\eta\right) + \rho^2 d^2(\bar{x}, \bar{y}), \qquad \text{for all } \eta \in \partial^c g(\bar{y}).$$

By the dual constraints (2)–(4) and $\alpha^1(\bar{x}, \bar{y}) > 0$, we obtain

$$\frac{\bar{\lambda} f(\bar{x})}{\alpha^1(\bar{x}, \bar{y})} - \frac{\bar{\lambda} f(\bar{y})}{\alpha^1(\bar{x}, \bar{y})} > \bar{\lambda} F(\bar{x}, \bar{y}; \xi) + \frac{\bar{\lambda}\rho^1 d^2(\bar{x}, \bar{y})}{\alpha^1(\bar{x}, \bar{y})},$$
$$\frac{-\bar{\mu} g(\bar{y})}{\alpha^1(\bar{x}, \bar{y})} \geqq \bar{\mu} F(\bar{x}, \bar{y}; \eta) + \frac{\bar{\mu}\rho^2 d^2(\bar{x}, \bar{y})}{\alpha^1(\bar{x}, \bar{y})}.$$

Adding the above inequalities, we get

$$\begin{aligned}\bar{\lambda} F(\bar{x}, \bar{y}; \xi) + \bar{\mu} F(\bar{x}, \bar{y}; \eta) &< \frac{1}{\alpha^1(\bar{x}, \bar{y})} \left(\bar{\lambda} f(\bar{x}) - \bar{\lambda} f(\bar{y}) - \bar{\mu} g(\bar{y}) - \left\{\bar{\lambda}\rho^1 + \bar{\mu}\rho^2\right\} d^2(\bar{x}, \bar{y})\right) \\ &\leqq \frac{1}{\alpha^1(\bar{x}, \bar{y})} \left(\bar{\lambda} f(\bar{x}) - \bar{\lambda} f(\bar{y}) - \bar{\mu} g(\bar{y})\right) \qquad \text{(using Hypothesis (iii))}.\end{aligned}$$

By sublinearity of F, we have for all $\xi \in \partial^c f(\bar{y})$ and $\eta \in \partial^c g(\bar{y})$

$$\begin{aligned}F\left(\bar{x}, \bar{y}; \bar{\lambda}\xi + \bar{\mu}\eta\right) &\leqq \bar{\lambda} F(\bar{x}, \bar{y}; \xi) + \bar{\mu} F(\bar{x}, \bar{y}; \eta) \\ &< \frac{1}{\alpha^1(\bar{x}, \bar{y})} \left(\bar{\lambda} f(\bar{x}) - \bar{\lambda} f(\bar{y}) - \bar{\mu} g(\bar{y})\right).\end{aligned}$$

Since $\alpha^1(\bar{x}, \bar{y}) > 0$, (8) reduces the above inequality to

$$\bar{\lambda} f(\bar{x}) > \bar{\lambda} f(\bar{y}) + \bar{\mu} g(\bar{y}),$$

which contradicts Hypothesis (i). Hence $\bar{x} = \bar{y}$.

5. MOND-WEIR TYPE DUALITY

In this section, we consider the following Mond-Weir type dual for (MOP) and establish weak, strong, and strict converse duality theorems. In this section f and g are assumed to be regular locally Lipschitz functions on X.

$$\text{Maximize } f(y), \tag{MOD}$$
$$\text{subject to } y \in X,$$
$$0 \in \lambda \partial^c f(y) + \mu \partial^c g(y), \tag{9}$$
$$\mu g(y) \geqq 0, \tag{10}$$
$$\lambda_i \geqq 0, \qquad i = 1, 2, \ldots, k, \quad \lambda e = 1, \tag{11}$$
$$\mu_j \geqq 0, \qquad j = 1, 2, \ldots, m. \tag{12}$$

To prove the weak duality theorem, we shall need the following result.

LEMMA 5.1. *(See [17, pp. 144–147].) Let the functions $f_i : R^n \mapsto R$, $i = 1, \dots, k$, be regular locally Lipschitz at a point $x^* \in R^n$. Then, for weights $w_i \geqq 0 \in R$ we have*

$$\partial^c \left(\sum_{i=1}^{k} w_i f_i \right) (x^*) = \sum_{i=1}^{k} w_i \partial^c f_i(x^*).$$

THEOREM 5.1. WEAK DUALITY. *Let x and (y, λ, μ) be feasible solutions of (MOP) and (MOD), respectively, such that*

(i) *$(f, \mu g)$ is weak strictly pseudoquasi (F, α, ρ, d)-Type I at y,*
(ii) $\lambda\rho^1/\alpha^1(x,y) + \rho^2/\alpha^2(x,y) \geqq 0$.

Then the following cannot hold:

$$f(x) \leq f(y). \tag{13}$$

PROOF. Suppose (13) holds. Then using the inequalities (13) and (10), Hypothesis (i) yields

$$F\left(x,y;\alpha^1(x,y)\xi\right) + \rho^1 d^2(x,y) < 0, \qquad \text{for all } \xi \in \partial^c f(y),$$
$$F\left(x,y;\alpha^2(x,y)\eta'\right) + \rho^2 d^2(x,y) \leqq 0, \qquad \text{for all } \eta' \in \partial^c(\mu g)(y),$$

which imply

$$F(x,y;\lambda\xi) < -\frac{\lambda\rho^1 d^2(x,y)}{\alpha^1(x,y)},$$
$$F(x,y;\eta') \leqq -\frac{\rho^2 d^2(x,y)}{\alpha^2(x,y)}.$$

Therefore, by sublinearity of F, we get

$$\begin{aligned} F\left(x,y;\lambda\xi+\eta'\right) &\leqq F(x,y;\lambda\xi) + F\left(x,y;\eta'\right) \\ &< -\left(\frac{\lambda\rho^1}{\alpha^1(x,y)} + \frac{\rho^2}{\alpha^2(x,y)}\right) d^2(x,y). \end{aligned}$$

Or, using Hypothesis (ii),

$$F\left(x,y;\lambda\xi+\eta'\right) < 0, \qquad \text{for all } \xi \in \partial^c f(y), \quad \eta' \in \partial^c(\mu g)(y).$$

But by Lemma 5.1, for some $\eta' \in \partial^c(\mu g)(y)$ there exists $\eta \in \partial^c g(y)$ such that $\eta' = \mu\eta$. Hence, the above inequality reduces to

$$F(x,y;\lambda\xi+\mu\eta) < 0, \qquad \text{for all } \xi \in \partial^c f(y), \quad \eta \in \partial^c g(y),$$

which contradicts $F(x,y;0) = 0$ as by the dual constraint (9), for some $\bar{\xi} \in \partial^c f(y)$ and $\bar{\eta} \in \partial^c g(y)$, $\lambda\bar{\xi} + \mu\bar{\eta} = 0$. Hence, the result.

The proofs of the following weak and strong duality theorems are similar to Theorems 5.1 and 4.6, respectively, and hence are omitted.

THEOREM 5.2. WEAK DUALITY. *Let x and (y, λ, μ) be feasible solutions of (MOP) and (MOD), respectively, such that*

(i) *$(f, \mu g)$ is pseudoquasi (F, α, ρ, d)-Type I at y,*
(ii) *$\lambda\rho^1/\alpha^1(x,y) + \rho^2/\alpha^2(x,y) \geqq 0$.*

Then the following cannot hold:

$$f(x) < f(y).$$

THEOREM 5.3. STRONG DUALITY. *Let $\bar{x}$ be an efficient solution for (MOP) at which the Cottle constraint qualification is satisfied. Then, there exist $\bar{\lambda} \in R^k$, $\bar{\mu} \in R^m$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is feasible for (MOD) and the two objectives are equal. Furthermore, if weak duality holds between (MOP) and (MOD) for all feasible solutions of the primal and dual problems, then $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is efficient for (MOD).*

THEOREM 5.4. STRICT CONVERSE DUALITY. *Let $\bar{x}$ and $(\bar{y}, \bar{\lambda}, \bar{\mu})$ be feasible for (MOP) and (MOD), respectively, such that*

(i) *$\bar{\lambda} f(\bar{x}) \leqq \bar{\lambda} f(\bar{y})$,*
(ii) *$\rho^1/\alpha^1(\bar{x},\bar{y}) + \rho^2/\alpha^2(\bar{x},\bar{y}) \geqq 0$.*
Also, if either
(a) *$(\bar{\lambda} f, \bar{\mu} g)$ is strictly pseudoquasi (F, α, ρ, d)-Type I at $\bar{y}$, or*
(b) *$(\bar{\lambda} f, \bar{\mu} g)$ is quasi strictly pseudo (F, α, ρ, d)-Type I at $\bar{y}$,*

then, $\bar{x} = \bar{y}$.

PROOF. Since $\mu \geqq 0$, the dual constraint (10) and Hypothesis (a) yield

$$F\left(\bar{x}, \bar{y}; \alpha^2(\bar{x},\bar{y})\eta'\right) \leqq -\rho^2 d^2(\bar{x},\bar{y}), \qquad \text{for all } \eta' \in \partial^c(\bar{\mu} g)(\bar{y}). \tag{14}$$

By the dual constraint (9), there exist some $\bar{\xi} \in \partial^c f(\bar{y})$ and $\bar{\mu} \in \partial^c g(\bar{y})$ such that $\bar{\lambda}\bar{\xi} + \bar{\mu}\bar{\eta} = 0$ which implies

$$F\left(\bar{x}, \bar{y}; \bar{\lambda}\bar{\xi} + \bar{\mu}\bar{\eta}\right) = 0. \tag{15}$$

Also by Lemma 5.1, for $\bar{\xi} \in \partial^c f(\bar{y})$ and $\bar{\eta} \in \partial^c g(\bar{y})$ there exist $\bar{\xi}' \in \partial^c(\bar{\lambda} f)(\bar{y})$ and $\bar{\eta}' \in \partial^c(\bar{\mu} g)(\bar{y})$ such that $\bar{\xi}' = \bar{\lambda}\bar{\xi}$ and $\bar{\eta}' = \bar{\mu}\bar{\eta}$.

Hence (15) gives

$$F\left(\bar{x}, \bar{y}; \bar{\xi}' + \bar{\eta}'\right) = 0,$$

which by sublinearity of F along with (14) and Hypothesis (ii) gives

$$\begin{aligned} F\left(\bar{x}, \bar{y}; \bar{\xi}'\right) &\geqq -F\left(\bar{x}, \bar{y}; \bar{\eta}'\right) \\ &\geqq \frac{\rho^2 d^2(\bar{x},\bar{y})}{\alpha^2(\bar{x},\bar{y})} \\ &\geqq \frac{-\rho^1 d^2(\bar{x},\bar{y})}{\alpha^1(\bar{x},\bar{y})}. \end{aligned}$$

Or,

$$F\left(\bar{x}, \bar{y}; \bar{\xi}'\right) \geqq \frac{-\rho^1 d^2(\bar{x},\bar{y})}{\alpha^1(\bar{x},\bar{y})}.$$

Therefore Hypothesis (a) yields

$$\bar{\lambda} f(\bar{x}) > \bar{\lambda} f(\bar{y}),$$

which contradicts Hypothesis (i). Hence, the result.

If Hypothesis (b) holds, then Hypothesis (i) and the dual constraint (10) give

$$F\left(\bar{x}, \bar{y}; \alpha^1(\bar{x},\bar{y})\xi'\right) + \rho^1 d^2(\bar{x},\bar{y}) \leqq 0, \qquad \text{for all } \xi' \in \partial^c(\bar{\lambda} f)(\bar{y}),$$
$$F\left(\bar{x}, \bar{y}; \alpha^2(\bar{x},\bar{y})\eta'\right) + \rho^2 d^2(\bar{x},\bar{y}) < 0, \qquad \text{for all } \eta' \in \partial^c(\bar{\mu} g)(\bar{y}).$$

Since $\alpha^1(\bar{x},\bar{y}) > 0$, $\alpha^2(\bar{x},\bar{y}) > 0$, sublinearity of F gives

$$F(\bar{x},\bar{y};\xi'+\eta') \leqq F(\bar{x},\bar{y};\xi') + F(\bar{x},\bar{y};\eta') < -\left(\frac{\rho^1}{\alpha^1(\bar{x},\bar{y})} + \frac{\rho^2}{\alpha^2(\bar{x},\bar{y})}\right) d^2(\bar{x},\bar{y}).$$

Using Hypothesis (ii), we get

$$F(\bar{x},\bar{y};\xi'+\eta') < 0,$$

which contradicts $F(\bar{x},\bar{y};0) = 0$. Hence the result holds.

It may be noted that in this section we require f and g to be regular locally Lipschitz functions to use Lemma 5.1. However, we do not need the regularity assumption if the dual constraint (10) is replaced by m constraints $\mu_j g_j(y) \geqq 0$, $j \in M$. In that case the weak and strict converse duality theorems need minor modifications and are stated below. We shall use f^λ and g^μ to denote $(\lambda_1 f_1, \lambda_2 f_2, \dots, \lambda_k f_k)$ and $(\mu_1 g_1, \mu_2 g_2, \dots, \mu_m g_m)$, respectively.

THEOREM 5.5. WEAK DUALITY. *Let x and (y,λ,μ) be feasible solutions of (MOP) and (MOD), respectively, such that*

(i) *(f, g^μ) is weak strictly pseudoquasi (F,α,ρ,d)-Type I at y,*
(ii) $\lambda\rho^1/\alpha^1(x,y) + \sum_{j=1}^m \rho_{k+j}/\alpha^2(x,y) \geqq 0$.

Then the following cannot hold:

$$f(x) \leq f(y).$$

THEOREM 5.6. STRICT CONVERSE DUALITY. *Let $\bar{x}$ and $(\bar{y},\bar{\lambda},\bar{\mu})$ be feasible for (MOP) and (MOD), respectively, such that*

(i) $\bar{\lambda}_i f_i(\bar{x}) \leqq \bar{\lambda}_i f_i(\bar{y})$,
(ii) $\sum_{i=1}^k \rho_i/\alpha^1(\bar{x},\bar{y}) + \sum_{j=1}^m \rho_{k+j}/\alpha^2(\bar{x},\bar{y}) \geqq 0$.
Also, if either
(a) *$(f^{\bar{\lambda}}, g^{\bar{\mu}})$ is strictly pseudoquasi (F,α,ρ,d)-Type I at $\bar{y}$ or,*
(b) *$(f^{\bar{\lambda}}, g^{\bar{\mu}})$ is quasi strictly pseudo (F,α,ρ,d)-Type I at $\bar{y}$,*

then, $\bar{x} = \bar{y}$.

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