Cohesive zone model based analytical solutions for adhesively bonded pipe joints under torsional loading

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1. Introduction

Pipe structures are a very important structural form for energy industry and construction industry. With the advancement of materials science and manufacturing, the mechanical properties of pipe itself has been dramatically improved. Due to limitations of component size imposed by manufacturing process and the requirement for inspection, accessibility, repair, and transportation/assembly necessitates some load carrying joints in most piping systems. However, the limitations of the overall system performance usually come from the capacity of pipe joints. Therefore, the pipe joints play the most important role in the overall integrity of most piping systems.

Joints are divided into two main categories in piping systems: adhesively bonded joints and flanged joints. For the traditional flanged connection, which is based on the shear connection through bolt, fatigue of the connection members is a concern, especially under high stress concentration on the bolts. Another serious problem is the corrosion of connecting bolts. In most adhesively bonded joints, whether metallic or composite, a coupler usually butts-welds the two pipes together. The isometric view, sectional view and side view of a typical pipe joint are illustrated in Fig. 1a and b and Fig. 2, respectively. The loading is transferred by means of the adhesive layer between the two contacting surfaces of the pipe and the coupler. The adhesive bonding is becoming a primary connection method because it can not only effectively lower the stress concentration but is also generally corrosion-free.

Among all the possible loading configurations, torsion loading is one of the fundamental loading type. Some previous works were conducted to analytically investigate the interface behavior of the adhesively bonded pipe joints under torsion loads. Volkersen (1965) first studied the problem of torsional stress in tubular lap joints. In his analysis, the two tubular adherends of the joint were treated by the mechanics of materials approach, in which the presence of the circumferential shear stress was ignored. Adams and Peppiatt (1977) and Graves and Adams (1981) improved Volkersen’s analysis by taking the thickness of the adhesive layer into account. Chon (1982) applied two-dimensional polar theory to the analysis of tubular joints, in which the unknown parameters were related to the composite layers. Based on the variational principle, Chen and Cheng (1992) proposed a stress distribution formulation for the adhesively bonded tubular lap joint under torsion. All

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Based on the mechanics of composite materials, Zhao and Pang (1995) developed an analytical model to investigate the response of laminated composite pipe under torsion. The maximum strain failure criterion was applied in their study to predict the failure of the bonded composite joints. Zou and Taheri (2006) derived a model for the adhesively bonded sandwich pipe joints under torsion based on the general composite shell theory. Cheng and Li (2008) conducted stress analyses of a smart composite pipe joint integrated with piezoelectric composite layers under torsion loading, in order to evaluate the effect of the integrated piezoelectric reinforced polymer composite layer on the joint performance.

These previous efforts helped significantly in modeling and understanding the structural response and stress distribution of the pipe joints under torsion load. However, all the above models focused on the traditional strain–stress analysis. With exact or approximate solutions derived in these models, the classical failure criterion, such as the maximum stress or the maximum strain criteria is then applied to predict the failure load of adhesively bonded pipe joints. In all previous analytical solutions, linear elastic properties are assumed for the entire pipe joints. The linear elastic behavior may be appropriate for the pipes themselves; while for the adhesive layer which is usually the weakest link in the bonded joints and can often suffer from micro-cracking and local damage or softening under torsion, non-linear modeling may be necessary. There are very few analytical models for the pipe joint under torsional load which can consider the non-linear response at the adhesive layer.

The failure analysis of pipe joints are still a matter of controversy with respect to a unified design approach, despite the fact that many exact and finite element solutions have been presented in the literature (Hosseinzadeh et al., 2006). The significantly increased fracture mechanics based models offered a promising alternative to predict the debonding and failure of bonded joints in a more accurate but relatively simpler approach (Hutchinson and Evans, 2000; Rizzi et al., 2000). Fracture studies were usually carried out under several idealized conditions, such as in the case of linear elastic fracture mechanics or the case of small scale yielding. In such cases, the details of the stress–strain around the crack tip are uniquely characterized by a single macroscopic parameter such as the stress intensity factor. These global parameters are related to the corresponding material properties typically the fracture toughness that determines the critical conditions of crack initiation and growth. When the crack tip experiences inelastic damage, the concepts based purely on the theory of elasticity are not valid. Further, for cracks along bimaterial interfaces, the crack tip will no longer be embedded in a square-root singular stress field; leading to a condition that stress intensity factor may either be zero or infinity (Atkinson, 1979).

As an alternative approach to this singularity driven fracture approach, the origins of the concept of cohesive zone model (CZM) goes back to the work of Barenblatt (1959) and Dugdale (1960). CZM has evolved as a preferred method to analyze fracture problems in monolithic and composite material systems not only because it avoids the singularity but also because it can be easily implemented in a numerical method of analysis such as in finite element modeling. Therefore, various CZMs have been proposed to investigate the fracture process in a number of material systems including fiber reinforced polymer composites, metallic materials, ceramic materials, cementitious or concrete materials, and bimaterial systems. All of them start from the assumption that one or more interfaces can be defined, where crack propagation is allowed by the introduction of a possible discontinuity in the displacement field. Various cohesive zone models (cohesive laws) were proposed (Hilleborg et al., 1976; Rose et al., 1983; Needleman, 1987; Tvergaard, 1990; Tvergaard and Hutchinson, 1992; Xu and Needleman, 1993; Camacho and Ortiz, 1996). The main difference between these models lies in the shape of the traction–displacement response, and the parameters used to describe that shape.

It is generally accepted that CZMs can be described by two or three independent parameters (Hutchinson and Evans, 2000). These parameters may be the fracture toughness (the area under the traction–displacement curve, which is typically obtained from experimental test), the cohesive strength σf (or τf) and the shape of the cohesive law. However, many previous studies indicated that the interface cohesive strength and fracture toughness (interface fracture energy) are the most important parameters to describe the fracture behavior as concluded by Williams and Hadavinia (2002); Blackman et al. (2003) and Ouyang and Li (2008). The cohesive zone model (CZM) based method have been extensively implemented in the finite element analysis for investigating the interface fracture behavior of structures or specimens under mode I (tension loading), mode II (in-plane shear loading), and mixed mode I/II. Some analytical solutions have also been developed.
based on beam theory for mode I, mode II or mixed mode I/II (Williams and Hadavinia, 2002; Wu et al., 2002; Blackman et al., 2003; Pan and Leung, 2007; Ouyang and Li, 2008). However, there are very few studies focusing on the analytical solution of mode III interface fracture problems in the literature. In the present study, the cohesive zone model based analytical solutions are developed to solve debonding (fracture) problem for the adhesively bonded pipe joints under torsion load.

In the current work, two cohesive laws are investigated. The first cohesive zone model is based on the equivalent linear elastic cohesive law. The second cohesive zone model is based on the non-linear cohesive law. A typical non-linear model – bilinear cohesive law is derived in order to investigate the effect of non-linear response at the bond interface on the pipe joint under torsion load. Simultaneously, the maximum torsion load capacity is derived as a function of the interface fracture energy in the present study. Compared to the complicated results of most previous analytical solutions, the simple expressions of torsion failure load can be directly used for practical torsion design.

Before the derivations, the following assumptions are made in the current study:

1. Small deformation is considered, and the classical torsion theory is adopted.
2. The radius of the pipe is much larger than the thickness of the pipe; and the thickness of the pipe is much larger than the thickness of the adhesive layer.
3. The torsion load carried by the thin and soft adhesive layer is ignored; and the external torsion load is assumed to be resisted by the main pipe and the coupler only.
4. Local bending effects in the pipe joint under torsion load are neglected, and the specimens are assumed under pure mode III loading condition.
5. The debonding path (cohesive zone) is assumed to develop along the bond interface only.
6. The main pipe and coupler are made of homogeneous and isotropic materials. The pipes remain linear elastic under the external torsional load.

2. Fundamental equations

Consider an adhesively bonded pipe joint as illustrated in Fig. 1. The two main pipes are bonded to the coupler through a thin and soft adhesive layer. The side view of the pipe joint is illustrated in Fig. 2. Due to symmetry, only the right half of the pipe joint is considered. We assume that the distance between the left end of pipe 1 and the right end of the coupler is L. Obviously, the total bond length of the pipe joint is 2L. However, since only the right half of the pipe joint is considered in the current study, the bond length is denoted by L in the following text for the sake of clarification.

The origin of the coordinate x is situated at the left end of the pipe. However, for the derivations in Section 4, the location of the origin will be changed for the sake of convenience. Due to the symmetry of the pipe structures, the effects of warping need not be considered. According to the classical torsion theory, the internal torsion $T_1$ and $T_2$ of the pipe and the coupler can be expressed as follows, respectively:

$$\phi_1' \cdot G_1 J_1 = T_1; \quad \phi_2' \cdot G_2 J_2 = T_2$$

(1)

where $G_1$ and $G_2$, $T_1$ and $T_2$, and $\phi_1$ and $\phi_2$ are the shear modulus, the internal torque, and the rotation angle of the pipe and the coupler, respectively. $J_1$ and $J_2$ are the polar moment of inertia of the thin-walled pipe and coupler, respectively, they can be written by

$$J_1 = 2\pi R_1^3 t_1; \quad J_2 = 2\pi R_2^3 t_2$$

(2)

in which, $t_1$ and $t_2$ are the thickness of the thin-walled pipe and coupler, respectively; $R_1$ and $R_2$ are the average radius of the pipe and the coupler, respectively (see Figs. 1b and 2).

As assumed, the torsion load carried by the soft and thin adhesive layer is ignored. Thus, the equilibrium between external and internal torsion load in the pipe joint requires

$$T_1(x) + T_2(x) \equiv T; \quad \frac{dT_1(x)}{dx} + \frac{dT_2(x)}{dx} = \frac{dT}{dx} = 0$$

(3)

For a given cross-section, if the rotations of the pipe and coupler are identical, there is no relative displacement between them at that cross-section. If at the given cross-section, the rotations of the pipe and the coupler are different from each other, a relative rotation occurs accompanied by a circumferential relative displacement at the bond layer. Let’s introduce the relative interface rotation $\phi$, which equals to the difference of the individual rotation angle of the pipe and the coupler at the cross-section x as illustrated in Fig. 3. Thus, we have

$$\phi = \phi_1 - \phi_2; \quad \phi' = \phi_1' - \phi_2'; \quad \phi'' = \phi_1'' - \phi_2''$$

(4)

The circumferential relative displacement at the bond interface will induce a circumferential interface shear stress as illustrated in Fig. 3. The circumferential interface shear stress thus causes a torque gradient $m(x)$ acting on the pipe and the coupler, respectively. Obviously, the local torsional equilibrium of the infinitely small section dx in the pipe and the coupler requires

$$\phi_1' \cdot G_1 J_1 = \frac{dT_1}{dx} = m(x)$$

(5)

$$\phi_2' \cdot G_2 J_2 = \frac{dT_2}{dx} = -m(x)$$

(6)

It is noted that $dT_1/dx = -dT_2/dx$ as implied by the second term in Eq. (3). Therefore, the torque gradient $m(x)$, which is caused by the circumferential interface shear stress, is identical in quantity, but opposite in direction for the pipe and the coupler, respectively.

Combine Eqs. (5) and (6), it can be derived that

$$\phi_1' \cdot G_1 J_1 + \phi_2' \cdot G_2 J_2 = 0$$

(7)
With the third term in Eqs. (4) and (7), it can be derived that
\[ \phi' \left( \frac{G_{J1}}{G_{J1} + G_{J2}} \right) + \phi'' = \frac{-G_{J1}}{G_{J1} + G_{J2}} \phi'' \]  
(8)

Consider the torsional equilibrium of an infinitely small section \( dx \) in the pipe (main pipe) as illustrated in Fig. 3
\[ 2\pi R \cdot \tau(x) \cdot R \cdot dx = dT_{1} \]
(9)
where \( \tau(x) \) is the interfacial shear stress along the circumferential direction, and \( R \) is the distance from the center of the pipe and mid-height of the adhesive layer. Without loss of generality, assume \( R_{2} > R_{1} \) as seen in Fig. 1b, \( R \) can be calculated by
\[ R = \left( \frac{R_{1} + \frac{L}{2}}{2} \right) + \left( \frac{R_{2} - \frac{L}{2}}{2} \right) \]
(10)
With Eq. (5) and the first term in Eq. (8), it can be derived that
\[ dT_{1} = \phi' G_{J1} dx = \frac{G_{J1} G_{J1}}{G_{J1} + G_{J2}} \phi'' dx \]
(11)
Combine Eqs. (9) and (11), it can be derived
\[ \frac{G_{J1} G_{J1}}{G_{J1} + G_{J2}} \phi'' = 2\pi R \cdot \tau(\delta) \cdot R \]
(12)
Denote this relative displacement (slip) at the bond layer interface along circumferential direction as \( \delta \), as shown in Fig. 3. The interface slip \( \delta \) can thus be expressed as a function of the relative interface rotation \( \phi \) as follow:
\[ \delta = R\phi_{1} - R\phi_{2} = R\phi \]
(13)
By this point, the general constraint equation for the debonding process of the pipe joint under torsion load has been illustrated in Eq. (12). With the given interface cohesive laws \( \tau(\delta) \) (the relationships between \( \tau \) and \( \delta \)), associated with Eq. (13), the governing equation in terms of the relative interface rotation \( \phi \) can be determined. It is also noted the governing Eq. (12) is applicable to any types of interface cohesive laws.

3. Equivalent linear elastic cohesive law

The equivalent linear elastic cohesive zone model which is illustrated in Fig. 4a is discussed firstly in this section; and the bilinear cohesive zone model will be studied in the next section. As discussed before, the two-parameter cohesive zone model (CZM) may well describe the interface debonding process. The two parameters in the linear elastic cohesive law are interface fracture energy \( G_{f} \) and cohesive shear strength \( \tau_{f} \) (or maximum shear stress). Note that this equivalent linear CZM is based on identical interface fracture energy (the area under the slip–stress curve) and identical cohesive shear strength \( \tau_{f} \) with the actual non-linear CZM. According to this equivalent linear cohesive law, the interface shear stress can be correlated to the interface slip (or relative interface rotation \( \phi \)) as follow:
\[ \tau(x) = k_{e} \frac{\delta}{R} \quad (0 \leq \delta \leq \delta_{f}) \]
(14)
where \( \delta_{f} \) is the final circumferential interface slip which is reached when the interface stress \( \tau = \tau_{f} \) (see Fig. 4a), and
\[ k_{e} = \frac{T_{f}^{2}}{2G_{f}} \]
(15)
in which, \( k_{e} \) is the equivalent interface stiffness of the linear elastic cohesive law, \( \tau_{f} \) is the maximum shear stress, and \( G_{f} \) is the interface fracture energy under mode III shear loading.

Eqs. (12) and (14) yield
\[ \frac{G_{J1} G_{J1}}{G_{J1} + G_{J2}} \phi'' = 2\pi R \cdot k_{e} \phi \]
(16)
For the sake of convenience, Eq. (16) can be rewritten by
\[ \phi'' = \alpha^{2} \phi \quad \text{when } 0 \leq \phi \leq \phi_{f} \]
(17)
where \( \phi_{f} \) is the final relative interface rotation when the interface slip \( \delta = \delta_{f} \); and
\[ \alpha = \sqrt{2\pi R} \cdot k_{e} \cdot \frac{G_{J1} + G_{J2}}{G_{J1} G_{J2}} \cdot \phi_{f} = \frac{\delta_{f}}{R} \]
(18)
The general solution of the governing differential Eq. (17) can be expressed by
\[ \phi(x) = B_1 \cdot \exp(-\alpha x) + B_2 \cdot \exp(\alpha x) \]
(19)
With Eq. (19), we can further obtain
\[ \phi'(x) = -\alpha B_1 \cdot \exp(-\alpha x) + \alpha B_2 \cdot \exp(\alpha x) \]
(20)
\[ \phi''(x) = \alpha^{2} B_1 \cdot \exp(-\alpha x) + \alpha^{2} B_2 \cdot \exp(\alpha x) \]
(21)
Note that
\[ G_{J1} \phi_{1}|_{x=0} = T_2 = T; \quad G_{J1} \phi_{1}|_{x=0} = T_1 = 0 \]
(22)
\[ G_{J1} \phi_{2}|_{x=0} = T_2 = 0; \quad G_{J1} \phi_{2}|_{x=0} = T_1 = T \]
(23)
From the two terms in Eq. (22), the boundary condition can be derived as follow:
\[ G_{J2} \phi_{1}|_{x=L} - G_{J2} \phi_{1}|_{x=0} = -G_{J2} \cdot \phi_{2}|_{x=0} = -T \]
(24)
From the two terms in Eq. (23), the boundary condition can be derived as follow:
\[ G_{J1} \phi_{1}|_{x=L} = G_{J1} \phi_{1}|_{x=L} = G_{J1} \cdot \phi_{1}|_{x=L} = T \]
(25)
With the boundary conditions as described by Eqs. (24) and (25), associated with Eq. (20), the two unknown coefficients in the general solution can be derived as follows

Fig. 4. (a) The equivalent linear elastic cohesive zone model and (b) the bilinear cohesive zone model.
Substitute Eq. (26) into Eq. (19), the relative interface rotation \( \phi(x) \) can be expressed by

\[
\phi(x) = T \left[ \frac{1}{\pi \mu_1} + \frac{1}{\pi \mu_2} \exp(zL) \right] \cdot \exp(-2z)
+ T \left[ \frac{1}{\pi \mu_1} + \frac{1}{\pi \mu_2} \exp(-zL) \right] \cdot \exp(2z) \tag{27}
\]

With the determined relative interface rotation \( \phi(x) \), the interface shear stress \( \tau(x) \) can be obtained according to the linear elastic cohesive law as illustrated by Eq. (14).

The relative rotations \( \phi \) at the two ends of the joint can be expressed as follows:

\[
\phi(x)|_{x=0} = T \left[ \frac{1}{\pi \mu_1} + \frac{1}{\pi \mu_2} \exp(zL) \right] \cdot \exp(-zL) \tag{28}
\]

\[
\phi(x)|_{x=L} = T \left[ \frac{1}{\pi \mu_1} + \frac{1}{\pi \mu_2} \exp(-zL) \right] \cdot \exp(zL) \tag{29}
\]

Subtract Eq. (28) by Eq. (29), it can be derived that

\[
\phi(x)|_{x=0} - \phi(x)|_{x=L} = \frac{T}{\pi \mu_1} \left[ \frac{\exp(zL) - \exp(-zL)}{x[\exp(xL) - \exp(-xL)]} \right] \tag{30}
\]

From Eq. (30), it can be observed that if \( G_{J2} = G_{J1} \), the relative interface rotation \( \phi \) (or shear stress \( \tau \), or interface slip \( \delta \)) at the two ends of the joint \( (x=0\) and \( x=L) \) are exactly identical. While for the general conditions that \( G_{J2} \neq G_{J1} \), without lack of generality, we assume that \( G_{J2} > G_{J1} \), note that \( [\exp(zL) + \exp(-zL)] > 2 \), one can see that \( \phi(x=0) < \phi(x=L) \). This means that when \( G_{J2} > G_{J1} \), the right end of the joint \( (x=L) \) will reach its final relative interface rotation \( \phi_L \) first.

The minimum relative interface rotation, which is denoted as \( \phi_m \) in the current study, can be determined from the equation as follows:

\[
\frac{d\phi(x)}{dx} = 0 \tag{31}
\]

With Eq. (31), the distance between the cross-section where the minimum relative interface rotation \( \phi_m \) is located and the left end \( (x=0) \) can be determined as follows:

\[
d = \frac{1}{2} \ln \left[ \frac{\frac{1}{\pi \mu_1} + \frac{1}{\pi \mu_2} \exp(zL)}{\frac{1}{\pi \mu_1} + \frac{1}{\pi \mu_2} \exp(-zL)} \right] \tag{32}
\]

Substitute \( x=d \) into Eq. (27), the minimum relative interface rotation \( \phi_m \) can be determined as follows:

\[
\phi_m = T \left[ \frac{1}{\pi \mu_1} + \frac{1}{\pi \mu_2} \exp(zL) \right] \cdot \exp(-zd) \nonumber
+ T \left[ \frac{1}{\pi \mu_1} + \frac{1}{\pi \mu_2} \exp(-zL) \right] \cdot \exp(zd) \tag{33}
\]

The maximum torsion capacity \( T_{max} \) is reached when the relative interface rotation at the right end \( \phi_L = \phi_f \). Substitute \( \phi(x=L)=\phi_f \) into Eq. (29) for the right end \( (x=L) \), the value of the maximum torsion capacity of the pipe joint can be derived as

\[
T_{max} = \phi_f \left[ \frac{\exp(zL) - \exp(-zL)}{\frac{1}{\pi \mu_1} + \frac{1}{\pi \mu_2} \exp(zL) + \exp(-zL)} \right] \tag{34}
\]

It is important to note that when \( G_{J2} < G_{J1} \), one should use Eq. (28) instead of Eq. (29), to obtain the relationship between the end rotation and the torsion load as follows:

\[
T_{max} = \phi_f \left[ \frac{\exp(zL) - \exp(-zL)}{\frac{1}{\pi \mu_1} + \frac{1}{\pi \mu_2} \exp(zL) + \exp(-zL)} \right] \tag{35}
\]

Obviously, when \( G_{J2} = G_{J1} \), Eqs. (34) and (35) collapse into an identical expression. Eqs. (34) and (35) imply that for the adhesively bonded pipe joints, the torsion load capacity \( T_{max} \) is dependent on the bond length \( L \). However, an effective torsion transfer (bond development) length exists in the bonded pipe joints. In another word, when the bond length \( L \) is longer than a certain value, any further increase in the bond length \( L \) will not improve the torsion load capacity significantly.

We denote the theoretical maximum torsion \( T_{max} \) as the torsion capacity of the pipe joint when the bond length \( L \rightarrow \infty \) (or long enough). Obviously, for the case that \( G_{J2} > G_{J1} \), from Eq. (34), it can be derived that

\[
T_{max} = \phi_f zG_{J1} = \frac{2G_f (2\pi R)}{G_{J2}} (G_{J1} + G_{J2}) \tag{36}
\]

While for the case that \( G_{J2} < G_{J1} \), the theoretical maximum torsion \( T_{max} \) of the pipe joint, can be similarly obtained as

\[
T_{max} = \phi_f zG_{J2} = \frac{2G_f (2\pi R)}{G_{J1}} (G_{J1} + G_{J2}) \tag{37}
\]

From Eqs. (36) and (37), one can also see that when the bond length \( L \) is long enough, the maximum transferable torsion load \( T_{max} \) becomes independent of the maximum shear stress \( \tau \) and the interface stiffness \( k_\delta \). It depends on the interface fracture energy, radius \( R \) (distance between the interface and the center of the joint), and the torsion stiffness \( G_{J1} \) and \( G_{J2} \) of the joint only.

If an effective development length \( l_e \) is defined as the length needed to attain 99% of the theoretical maximum transferable torsion load \( T_{max} \), it can be determined for the case that \( G_{J2} > G_{J1} \) as follow:

\[
\frac{\exp(zL_e) - \exp(-zL_e)}{\frac{1}{\pi \mu_1} + \frac{1}{\pi \mu_2} [\exp(zL_e) + \exp(-zL_e)]} = \frac{0.99 \cdot T_{max}}{\phi_f} = 0.99 \cdot zG_{J1} \tag{38}
\]

For the case that \( G_{J2} < G_{J1} \), one can similarly obtain the effective development length \( l_e \). By solving Eq. (38), the effective development length \( l_e \) can be expressed by

\[
l_e = \frac{1}{2} \ln \left[ \frac{0.99 \cdot \frac{2G_f (2\pi R)}{G_{J2}} (G_{J1} + G_{J2})}{0.99 \cdot zG_{J1}} \right] \tag{39}
\]

in which \( \eta \) represents the ratio of the torsion stiffness between the pipe and coupler. Note that \( \eta = G_{J1}/G_{J2} \) for the case when \( G_{J2} \geq G_{J1} \); while \( \eta = G_{J2}/G_{J1} \) when \( G_{J2} < G_{J1} \).

When the bond length of the pipe is longer than the effective transfer length \( l_e \), one may simply apply Eq. (36) or (37) to calculate the torsion load capacity of the pipe joints. It is also noted that Eq. (39) may be used for the torsion design of the adhesively bonded pipe joints. For the conditions that \( G_{J2} \geq G_{J1} \), or \( G_{J2} < G_{J1} \), the maximum theoretical torsion \( T_{max} \) may be approximated as follows, respectively
Finally, it is noted the developed model in the present study can also be used for the conventional stress analysis. Simply replacing the interface shear stiffness \( k_s \) as described in Eq. (15) by \( G_s/h_s \) \((G_s \text{ and } h_s \text{ are the shear modulus and thickness of the adhesive layer, respectively,})\), and applying Eq. (27), one can obtain the distribution of the relative interface rotation \( \phi(x) \) under the given external torsion load \( T \). With the determined \( \phi(x) \), the interface shear distribution \( \tau(x) \) can be obtained as follows:

\[
\tau(x) = \frac{\phi(x) R}{h_s} G_s
\]

(41)

4. Bilinear cohesive zone model

In this section, the bilinear cohesive zone model is applied to the modeling of interface debonding in the pipe joints. A typical bilinear cohesive zone model consists of a linear elastic branch and a linear softening branch, as illustrated in Fig. 4b. And the typical constitutive relation of the bilinear cohesive zone model (see Fig. 4b) can be written as follows:

\[
\tau(x) = \begin{cases} 
    k_1 \delta & 0 \leq \delta \leq \delta_1 \\
    k_2 (\delta - \delta_1) & \delta_1 < \delta \leq \delta_2 \\
    0 & \delta_2 < \delta 
\end{cases}
\]

(42)

where \( k_1 \) and \( k_2 \) are the interface stiffness as illustrated in Fig. 4b; we also define the interface stiffness ratio \( k = k_1/k_2 \) (see Fig. 4b); and \( \delta_1 \) is the characteristic interface slip, at which the interface shear stress reaches its maximum value \( \tau_1 \); \( \delta_2 \) is the final cohesive slip at which the interface shear stress becomes zero.

Note that there are two special relative interface rotations corresponding to the characteristic interface slip \( \delta_1 \) and final cohesive slip \( \delta_2 \) as follows:

\[
\phi_1 = \delta_1/R; \quad \phi_2 = \delta_2/R
\]

(43)

Due to the presence of a softening zone along the bond length direction, the solution of bilinear cohesive law becomes more complicated than the linear elastic cohesive law. For the sake of convenience, we reassign a coordinate system for the bilinear cohesive zone model different from that for the equivalent linear elastic model as illustrated in Fig. 5. Two coordinates, \( x_1 \) and \( x_2 \) are introduced for the elastic zone and softening zone, respectively. For the equivalent linear elastic model, the origin of the coordinate \( x \) is situated at the left end of the main pipe (or the center of the joint). However, for the bilinear cohesive zone model, the origin of the coordinate \( x_1 \) is always set at the cross-section where the minimum relative interface rotation \( \phi_m \) is located. The origin of the coordinate \( x_2 \) is situated at the cross-section (\( \phi = \phi_1 \)), which separates the elastic zone and the softening zone (see Fig. 5).

With similar method, the governing equations for the elastic zone and softening zone can be expressed as follows, respectively:

\[
\phi'(x_1) = \gamma(t) \phi(x_1) \quad \text{when} \quad 0 \leq \phi \leq \phi_1
\]

\[
\phi'(x_2) = \beta(\phi - \phi(x_2)) \quad \text{when} \quad \phi_1 < \phi < \phi_f
\]

(44)

(45)

where coordinate \( x_1 \) and \( x_2 \) are for elastic zone and softening zone, respectively, in which

\[
\gamma = \sqrt{2\pi R k_1 (G_{J1} + G_{J2})/(G_{J2} G_{J1})}; \quad \beta = \sqrt{2\pi R k_2 (G_{J1} + G_{J2})/(G_{J2} G_{J1})}
\]

(46)

The general solutions for the elastic zone and softening zone can be written as follows, respectively

\[
\phi(x_1) = C_1 \exp(-\gamma x_1) + C_2 \exp(\gamma x_1)
\]

(47)

\[
\phi(x_2) = D_1 \cos(\beta x_2) + D_2 \sin(\beta x_2) + \phi_f
\]

(48)

where \( C_1, C_2, D_1, \) and \( D_2 \) are the unknown coefficients to be determined.

As discussed in the last section, depending on the torsion stiffness ratio \( G_{J2}/G_{J1} \), the softening zone and macro-debonding can occur at one end of the pipe joint firstly, or occur at both ends simultaneously. Therefore, two basic cases will be discussed for the pipe joints under torsion load. Once again, for the sake of convenience, the origin of the coordinate \( x_1 \) (coordinate for elastic zone) is situated at the cross-section where the minimum relative interface rotation \( \phi_m \) is located.

4.1. Case one: \( G_{J1} = G_{J2} \)

Let’s first consider the simple case that \( G_{J2} = G_{J1} \), for the joint under torsion load. Evidently, the distribution of the relative interface rotation \( \phi \) and interface shear stress \( \tau \) is symmetric with respect to the mid-cross-section. There are two stages for this condition. When the relative interface rotation at the two ends (they are identical) are less than \( \phi_1 \), the local deformation is controlled by linear elastic behaviors, which is similar to the derivations in the previous section. The characteristic torsion \( T_0 \), when the relative interface rotation at both ends equal \( \phi_1 \), can be determined as follow:

\[
T_0 = \phi_1 \left[ \frac{\exp(\gamma L) - \exp(-\gamma L)}{\exp(\gamma L) + \exp(-\gamma L)} \right] \quad \text{when} \quad G_{J2} = G_{J1}
\]

(49)

When the left end interface rotation \( \phi_1 \) (at \( x=0 \)) or right end interface rotation \( \phi_2 \) (at \( x=L \)) are larger than \( \phi_1 \) or the external torsion load \( T > T_0 \), which indicates that some portions of the interface have come into the softening zone (see Fig. 5), a governing equation different from elastic behavior should be applied to the softening zone as illustrated by Eq. (48). Obviously, the change rate of the relative interface rotation \( \phi \) at the mid-cross-section must be zero due to the symmetry for the case when \( G_{J2} = G_{J1} \). Note that the relative rotation \( \phi \) at the mid-cross-section (\( x_m = 0 \)) itself is not zero, which is equal to the minimum relative interface rotation \( \phi_m \). However, it can be expected that when \( L \) is long enough, the relative interface rotation at \( x_m = 0 \) must approach zero (\( \phi_m = 0 \)).

The damage zone length \( d_1 \) and elastic zone \( d_0 \) are introduced to describe the half of the softening zone size and elastic zone size, respectively. Obviously, the sum of \( d_1 \) and \( d_0 \) must equal \( L/2 \) for the case when \( G_{J2} = G_{J1} \). Obviously, the symmetric boundary condition for the case when \( G_{J2} = G_{J1} \) is as follow:
\[ \phi'(x_1)|_{x_1=0} = 0 \]  

For the cross-section \((x_1 = 0)\) where the minimum relative interface rotation \(\phi_m\) is located, with Eq. (47), the boundary condition can be written as follow:

\[ \phi(x_1)|_{x_1=0} = C_1 + C_2 = \phi_m \]  

(51)

With Eq. (47), and combining Eqs. (50) and (51), it can be derived that

\[ C_1 = C_2 = \frac{\phi_m}{2} \]  

(52)

According to the definition of elastic zone length \(d_0\), obviously, the relative interface rotation \(\phi(x_1 = d_0) = \phi_1\) for the cross-section \((x_1 = d_0)\). Thus we have

\[ \phi(x_1)|_{x_1=d_0} = \frac{\phi_m}{2} \exp(-\gamma d_0) + \frac{\phi_m}{2} \exp(\gamma d_0) = \phi_1 \]  

(53)

From Eq. (53), the elastic zone length \(d_0\) can be expressed by

\[ d_0 = \frac{1}{\gamma} \arccos \frac{\phi_1}{\phi_m} \]  

(54)

On the other hand, for the coordinate \(x_2\) (softening zone), for the cross-section at \(x_2 = 0\) \((x_1 = d_0)\) and with Eq. (48), it can be derived

\[ \phi(x_2)|_{x_2=0} = D_1 + \phi_f = \phi_1 \]  

(55)

The continuous boundary condition requires

\[ \phi(x_1)|_{x_1=d_0} = \phi(x_2)|_{x_2=0} \]  

(56)

Eq. (56) yields

\[ D_2 = \frac{\phi_0}{\gamma} \cdot \exp(\gamma d_0) - \frac{\phi_0}{2} \cdot \exp(-\gamma d_0) \]  

(57)

Square both sides of Eqs. (53) and (57), and substrate each other. After simplifications, it can be derived that

\[ D_2 = \frac{\gamma}{\beta} \sqrt{\phi_0^2 - \phi_m^2} \]  

(58)

Note that for the right side of the joint \((x_2 > 0)\), the value of \(D_2\) is positive. By this point, \(C_1, C_2, D_1, \) and \(D_2\) are expressed as a function of \(\phi_m\). Substitute Eqs. (55) and (58) into Eq. (47), it can be derived

\[ \phi(x_2) = (\phi_1 - \phi_f) \cos(\beta x_2) + \frac{\gamma}{\beta} \sqrt{\phi_0^2 - \phi_m^2} \sin(\beta x_2) + \phi_f \]  

(59)

Substitute Eq. (52) into Eq. (46), it can be derived

\[ \phi(x_1) = \frac{\phi_m}{2} \cdot \exp(-\gamma x_1) + \frac{\phi_m}{2} \cdot \exp(\gamma x_1) \]  

(60)

Note that \(d_1 = L/2 - d_0\). With Eq. (54), the relative interface rotation \(\phi_\eta\) at the right end of the joint \((x_2 = d_1)\) can be expressed by

\[ \phi(x_2)|_{x_2=d_1} = \phi_f + (\phi_1 - \phi_f) \cos \left[ \frac{\gamma}{\beta} \arccos \frac{\phi_1}{\phi_m} \right] \]  

(61)

\[ + \frac{\gamma}{\beta} \sqrt{\phi_0^2 - \phi_m^2} \sin \left[ \frac{\gamma}{\beta} \arccos \frac{\phi_1}{\phi_m} \right] \]

It is noted that

\[ \phi' = \phi_1 - \phi_\eta = \frac{T_1}{GJ_1} - \frac{T_2}{GJ_2} \]  

(62)

Note that for the loading end \((x_2 = d_1)\), the external torsion \(T = T_1\) and \(T_2 = 0\). With Eqs. (61) and (62), the external torsion \(T\) can be derived as follow:

\[ T = GJ_1 \left[ \left( \phi_1 - \phi_\eta \right) \cdot \phi_1 \sin \left[ \frac{\gamma}{\beta} \arccos \frac{\phi_1}{\phi_m} \right] \right] \]  

(63)

\[ + \gamma \sqrt{\phi_0^2 - \phi_m^2} \cos \left[ \frac{\gamma}{\beta} \arccos \frac{\phi_1}{\phi_m} \right] \]

It is important to note that to utilize Eqs. (61) and (63), the minimum relative interface rotation \(\phi_m\) has to satisfy the condition below

\[ \phi_m \leq \phi_\eta \leq \phi_1 \]  

(64)

where \(\phi_\eta\) is the critical minimum relative interface rotation, which is reached when the end relative interface rotation \(\phi = \phi_1\), and \(\phi_\eta\) can be determined as follow

\[ \phi_\eta = 2\phi_1 \cdot \exp\left(\frac{\phi_1}{\gamma} + \frac{\phi_0}{2} \right) \exp(-\gamma L) + \exp(-\gamma L) + \frac{2}{\gamma} \]  

(65)

When the minimum relative interface rotation \(\phi_m\) is smaller than \(\phi_\eta\), the entire bond length of the pipe joint is controlled by the elastic behavior. One can simply apply the derivations in the previous section to obtain the relationship between the torsion load and relative interface rotation (or interface shear stress). While when \(\phi_0 < \phi_m < \phi_1\), the entire bond length consists of both elastic zone and softening zone.

When the minimum relative interface rotation \(\phi_m \geq \phi_1\), the entire bond length \(L\) becomes the softening zone. It is important to note that the maximum torsion load capacity \(T_{max}\) must be reached before \(\phi_m = \phi_1\). This is because when \(\phi_m > \phi_1\), the entire bond length is within the softening zone, and any increase in \(\phi_m\) will cause further decrease in the interface shear stress, so that the torsion load capacity \(T_{max}\) will monotonically decrease when the entire bond length becomes softening zone.

It is also noted that the torsion load capacity \(T\) monotonically increases with the increase in \(\phi_m\) before the maximum torsion load \(T_{max}\) is reached. As discussed before, the maximum torsion load must be within the corresponding region when \(\phi_0 < \phi_m < \phi_1\). One can simply apply the equation below to find the characteristic relative interface rotation \(\phi_m\) when the maximum external torsion capacity \(T_{max}\) is reached

\[ \frac{\partial T}{\partial \phi_m} = 0 \]  

(66)

With the determined characteristic \(\phi_m\), the maximum torsion load capacity \(T_{max}\) can be obtained correspondingly by applying Eq. (63). Note that when \(\phi_m > \phi_1\), the entire bond length \(L\) becomes the softening zone. If steady crack propagation is allowed, one can obtain the solutions similarly with decreased entire bond length by crack length \(a\). However, for the sake of brevity, the detail will not be discussed in the present study, since the maximum torsion must have been reached before this condition occurs.

4.2 Case two: \(G_1 J_1 \neq G_2 J_2\)

Due to asymmetry, the solution for the case that \(G_2 J_2 > G_1 J_1\) is more complicated than that for the case of \(G_2 J_2 = G_1 J_1\). When \(G_2 J_2 > G_1 J_1\), without lack of generality, we assume \(G_2 J_2 > G_1 J_1\). For the case that \(G_2 J_2 < G_1 J_1\), one can obtain the solutions in the same manner.

For the case that \(G_2 J_2 > G_1 J_1\), there are two possibilities when the final debonding at the right end occurs, depending on how larger is \(G_2 J_2\) than \(G_1 J_1\). The first possibility is that when \(\phi_0 = \phi_1\), \(\phi_\eta\) (the relative rotation at the left end) is still within the elastic zone \((\phi_1 \leq \phi_\eta\)). The first possibility occurs for the pipe joints with configuration that \(G_2 J_2 \gg G_1 J_1\). The second possibility is that when \(\phi_0 = \phi_1\), \(\phi_\eta\) has also come into the softening zone \((\phi_1 \leq \phi_\eta \leq \phi_1\)) which means that the softening zone will be developed at both sides simultaneously. The second possibility happens when \(G_2 J_2\) is larger than \(G_1 J_1\), but not much too larger.

Due to the similarity between these two possibilities when the left end rotation \(\phi_1 < \phi_0\), we will discuss the first possibility firstly. However, when \(\phi_1 > \phi_0\), but \(\phi_0 < \phi_1\), different governing equations should be applied for the second possibility.
4.2.1. When $G_2 J_2 \gg G_1 J_1$

In this possibility ($G_2 J_2 \gg G_1 J_1$), the left side of the pipe joint is always within elastic zone until the macro-debonding occurs at the right end $\phi_1 = \phi_f$. Therefore, we separate the entire bond length $L$ into three parts: $d_{oa}, d_{ob}$ and $d_{1R}$ as illustrated in Fig. 5. In which, $d_{oa}$ represents the elastic zone length on the left side of the cross-section where $\phi_m$ is located; and $d_{1R}$ represents the elastic zone length on the right side of the cross-section where $\phi_m$ is located; $d_{1R}$ represents the softening zone length on the right side of the cross-section where $\phi_m$ is located. Obviously, the entire bond length can be written by

$$L = d_{oa} + d_{ob} + d_{1R} \quad (67)$$

It is important to note that $d_{oa} \leq d_{ob}$, since the elastic zone $d_{oa}$ on the left side is not fully developed ($\phi_1 < \phi_f$) compared to the fully developed elastic zone $d_{ob}$ on the right side when the right end rotation $\phi_m = \phi_f$ (macro-debonding occurs at the right end).

With similar method to Eq. (54), the elastic zone length $d_{oa}$ can be determined by

$$d_{oa} = \frac{1}{\gamma} \arccos h \left( \frac{\phi_1}{\phi_m} \right) \quad (68)$$

Similar to boundary conditions in Eqs. (24) and (25), the boundary conditions for the left end ($x = -d_{oa}$) and the right end ($x = d_{1R}$) can be written by

$$G_2 J_2 \cdot \phi''_{h_{-d_{oa}}} = -T; \quad G_1 J_1 \cdot \phi''_{h_{-d_{oa}}} = T \quad (69)$$

Eq. (69) yields the following two equations, respectively.

$$G_2 J_2 \cdot \gamma \cdot \phi_m \sinh [\gamma d_{oa}] = T \quad (70)$$

$$G_1 J_1 \cdot \left[ \beta (\phi_1 - \phi_m) \sinh [\beta d_{1R}] + \gamma \sqrt{\phi_1^2 - \phi_m^2} \cos [\beta d_{1R}] \right] = T \quad (71)$$

From Eqs. (70) and (71), it can be derived that

$$\frac{\beta (\phi_1 - \phi_m) \sinh [\beta d_{1R}] + \gamma \sqrt{\phi_1^2 - \phi_m^2} \cos [\beta d_{1R}]}{\gamma \cdot \phi_m \sinh [\gamma d_{oa}]} = \frac{G_2 J_2}{G_1 J_1} \quad (72)$$

Substitute Eqs. (68) and (67) into Eq. (72), the softening zone on the right side $d_{1R}$ can be correlated to the minimum relative interface rotation $\phi_m$ as follow:

$$\frac{\beta (\phi_1 - \phi_m) \sinh [\beta d_{1R}] + \gamma \sqrt{\phi_1^2 - \phi_m^2} \cos [\beta d_{1R}]}{\gamma \cdot \phi_m \sinh [\gamma d_{oa}] - \arccos h \left( \frac{\phi_1}{\phi_m} \right)} = \frac{G_2 J_2}{G_1 J_1} \quad (73)$$

For a given minimum relative interface rotation $\phi_m$, there is a corresponding softening zone length $d_{1R}$ according to Eq. (73). With the determined value of $d_{1R}$, one can readily obtain the torsion load $T$ by utilizing the second term in Eq. (69).

However, it is important to clarify the applicable range of $\phi_m$ in Eq. (73). The value of $\phi_m$ in Eq. (73) have to be larger than $\phi_1$, which is the first characteristic minimum relative interface rotation when $\phi_2 = \phi_f$. This condition implies that the right end starts entering into the softening zone. This characteristic minimum relative interface rotation $\phi_{m1}$ can be determined with similar methods as follow:

$$\phi_{m1} = \phi_1 \left[ \frac{\frac{\phi_1}{\gamma} + \frac{J_1}{\gamma J_2} \exp(\gamma L)}{\frac{\phi_1}{J_2} + \frac{J_1}{J_2} \exp(\gamma L) + \exp(-\gamma L)} \right] \exp(-\gamma d_{m1})$$

$$+ \phi_1 \left[ \frac{\frac{1}{\gamma J_2} + \frac{\phi_1}{\gamma J_1} \exp(-\gamma L)}{\frac{1}{\gamma J_2} + \frac{\phi_1}{\gamma J_1} \exp(\gamma L) + \exp(-\gamma L)} \right] \exp(\gamma d_{m1}) \quad (74)$$

in which

$$d_{m1} = \frac{1}{2\gamma} \ln \left[ \frac{\frac{1}{\gamma J_2} + \frac{\phi_1}{\gamma J_1} \exp(\gamma L)}{\frac{1}{\gamma J_2} + \frac{\phi_1}{\gamma J_1} \exp(-\gamma L)} \right] \quad (75)$$

4.2.2. When $G_2 J_2$ the same order as $G_1 J_1$

When $G_2 J_2$ is not too much larger than $G_1 J_1$, or the interface stiffness ratio $K = \frac{k_1}{k_2}$ is relatively small, the second possibility can happen. There are two stages for the second possibility. When the left end rotation $\phi_1 \leq \phi_f \ (stage \ 1)$, the governing equations are exactly identical to that of the first possibility. When $\phi_1 > \phi_f \ (stage \ 2)$, different governing equation should be used. Note that the critical condition for these two stages is $\phi_f = \phi_1$. Obviously, when $\phi_1 > \phi_f$, the elastic zone on the left side has been fully developed, which means $d_{oa} = d_{ob}$. When $\phi_1 = \phi_f$, combine this condition that $d_{oa} = d_{ob}$ with Eq. (72), the right side elastic zone length $d_{oa}$ can be correlated to the minimum rotation $\phi_m$ as follow:

$$\frac{\beta (\phi_1 - \phi_m) \sinh [\beta (L - 2d_{oa})] + \gamma \sqrt{\phi_1^2 - \phi_m^2} \cos [\beta (L - 2d_{oa})]}{\gamma \cdot \phi_m \sinh [\gamma d_{oa}]} = \frac{G_2 J_2}{G_1 J_1} \quad (76)$$

Let’s denote the minimum rotation $\phi_m$ in Eq. (76) as $\phi_m^{(1)}$ for the critical condition that $\phi_1 = \phi_f$. Substitute Eq. (68) into Eq. (76), and rewrite $\phi_m^{(1)}$ in Eq. (76) by $\phi_m^{(2)}$, the governing equation for solving the characteristic rotation $\phi_m^{(2)}$ can be determined as below

$$\frac{\beta (\phi_1 - \phi_m^{(2)}) \sinh [\beta (L - 2d_{oa})] + \gamma \sqrt{\phi_1^2 - \phi_m^{(2)}^2} \cos [\beta (L - 2d_{oa})]}{\gamma \cdot \phi_m^{(2)} \sinh [\gamma d_{oa}]} = \frac{G_2 J_2}{G_1 J_1}$$

$$\quad \text{if} \quad \phi_m^{(2)} < \phi_1 \quad (77)$$

Note that only the first positive root of Eq. (77) within the range $(\phi_m < \phi_m^{(2)} < \phi_1)$ gives the true value of $\phi_m^{(2)}$. For the second possibility, when $\phi_m > \phi_m^{(2)}$, different governing equations are required. To consider the stage that $\phi_m > \phi_m^{(2)}$ for the second possibility up to the right end rotation $\phi_1$, let’s introduce another parameter $d_{1R}$ to represent the softening zone length on the left side of the joint as following:

$$L = 2d_{oa} + d_{1R} + d_{1R} \quad (78)$$

Note again that $d_{1R} = d_{oa}$ in Eq. (78) because both elastic zones on the right and left side are fully developed when $\phi_m \geq \phi_m^{(2)}$. The softening zone size $d_{1R}$ on the left side and the softening zone size $d_{1R}$ on the right side can be determined by the boundary condition as follows, respectively

$$G_2 J_2 \cdot \phi''_{l_{-d_{oa}}} = -T; \quad G_1 J_1 \cdot \phi''_{l_{-d_{oa}}} = T \quad (79)$$

With similar methods to the first possibility, the governing equation can be derived as

$$\frac{\beta (\phi_1 - \phi_m^{(2)}) \sinh [\beta (L - 2d_{oa} - d_{1R})] + \gamma \sqrt{\phi_1^2 - \phi_m^{(2)}^2} \cos [\beta (L - 2d_{oa} - d_{1R})]}{\gamma \cdot \phi_m^{(2)} \sinh [\gamma (d_{oa} + d_{1R})]} = \frac{G_2 J_2}{G_1 J_1} \quad (80)$$

in which, $d_{oa}$ can be determined by Eq. (68). Fundamentally, Eq. (80) gives the relationship between $\phi_m$ and $d_{1R}$. Any given $\phi_m$ corresponds to a softening zone length $d_{1R}$ as described by Eq. (80). With this corresponding softening zone length $d_{1R}$ and the second term in Eq. (79), the external torsion load $T$ can be determined.

Note that the applicable condition of $\phi_m$ in Eq. (80) is that $\phi_2 \geq \phi_m$. When $\phi_1 \leq \phi_m \leq \phi_2$, one can simply apply the derivations for the first possibility to obtain the relationship between $\phi_m$ and external torsion load $T$.

4.3. Simplified maximum torsion load for bilinear CZM

By this point, the exact solutions of the bilinear cohesive zone model (CZM) have been derived for arbitrary bond length $L$. For
the sake of design consideration, a simplified result of the maximum torsion load capacity $T_{\text{max}}$ of the pipe joints with long bond length $L$ will be derived within the frame of bilinear cohesive zone model.

Obviously, the maximum torsion load capacity $T_{\text{max}}$ approaches the theoretical maximum torsion load $T_{\text{max}}$ when the bond length $L \rightarrow \infty$ (or long enough). For the case that $G_2 J_2 \geq G_1 J_1$, the maximum torsion load is reached when the relative interface rotation at the right end $\phi_{\text{R}} = \phi_{\text{f}}$. We assume that the softening zone length on the right side is $d_{\text{IR}}$ when the maximum torsion load capacity is reached.

Substitute this boundary condition that $\phi(x_2) = d_{\text{IR}}$ into Eq. (59), we have

$$\beta(\phi_{G} - \phi_{L}) \cos(\beta d_{\text{IR}}) + \gamma(\phi_{G}^2 - \phi_{L}^2) \sin(\beta d_{\text{IR}}) = 0$$

(81)

According to the boundary condition that

$$G J_1 \cdot \phi(x_2) = T_{\text{max}}$$

(82)

and combining Eqs. (59) and (82), the following equation can be derived:

$$\frac{T_{\text{max}}}{G J_1} = \beta(\phi_{G} - \phi_{L}) \sin(\beta d_{\text{IR}}) + \gamma(\phi_{G}^2 - \phi_{L}^2) \cos(\beta d_{\text{IR}})$$

(83)

By squaring both sides of Eqs. (81) and (83), and adding them together, it is interesting to find

$$\left(\frac{T_{\text{max}}}{G J_1}\right)^2 = \beta^2(\phi_{G} - \phi_{L})^2 + \gamma^2(\phi_{G}^2 - \phi_{L}^2)$$

(84)

When the bond length $L$ is long enough, evidently, the minimum relative interface rotation $\phi_{\text{R}}$ must approach zero. Therefore, Eq. (84) can be further reduced to

$$T_{\text{max}} = G J_1 \sqrt{\beta^2(\phi_{G} - \phi_{L})^2 + \gamma^2(\phi_{G}^2 - \phi_{L}^2)}$$

when $G J_2 \geq G J_1$.

(85)

According to the definition of $\beta$ and $\gamma$ as described by Eq. (46) and the definition of $k_1$ and $k_2$ as illustrate by Eq. (42), and note that $2G = \tau_0$ after simplifications, Eq. (85) can be rewritten as follow:

$$T_{\text{max}} = \sqrt{2G J_1 (2\pi R)} \frac{G J_1}{G J_2} (G J_1 + G J_2); \quad \text{when } G J_2 \geq G J_1$$

(86)

For the case that $G J_2 \leq G J_1$, one can similarly derive that

$$T_{\text{max}} = \sqrt{2G J_1 (2\pi R)} \frac{G J_2}{G J_1} (G J_1 + G J_2); \quad \text{when } G J_2 \leq G J_1$$

(87)

Compare Eqs. (86) and (87) to Eqs. (36) and (37), it is interesting to find that when the bond length $L$ is long enough, the expressions of the maximum torsion load capacity $T_{\text{max}}$ are identical for the equivalent linear elastic and bilinear cohesive zone models. It is also noted that these simplified equations can be applicable to arbitrary types of bi-linear laws. Therefore, the maximum torsion load capacity $T_{\text{max}}$ of the adhesively bonded composite joints is dependent on the torsion stiffness of the pipe and coupler, the radius $R$, and the interface fracture energy $G_0$ only, but independent of the shape of the cohesive laws if $L$ is long enough.

It can also be observed from Eqs. (86) and (87) that for a given torsion stiffness of the main pipe $G J_1$, radius $R$, and interface fracture energy $G_0$, the torsion load capacity $T_{\text{max}}$ reaches its maximum value when $G J_2 = G J_1$. Due to the simplicity of Eqs. (86) and (87) as compared to most previous solutions, they may be considered for practical torsion design of adhesively bonded pipe joints. As to be discussed in the next section, the bond length $L$ to achieve 99% of the theoretical maximum torsion load capacity for the bilinear model is always shorter than that for the equivalent linear elastic model. Therefore, one can simply apply Eq. (39) to obtain a safe design bond length $L$.

5. Validation and parametric studies

In this section, the comparison with finite element analysis result will be conducted to validate the model developed in the current study. Based on the verified models, comprehensive parametric studies are then implemented.

5.1. Finite element analysis validation

Hosseinzadeh et al. (2006) conducted a finite element analysis (FEA) of the adhesively bonded pipe joints by assuming both pipe walls and adhesive layer are linear elastic. The analytical results derived in the present study are compared to their numerical results. The material and geometric properties of their finite element model are listed in Table 1 and Table 2, respectively. These FEA inputs are exactly identical to those for the analytical results. The detail of the FEA model can refer to their original paper (Hosseinzadeh et al., 2006). Replacing the interface shear stiffness $k_s$ as described in Eq. (15) by $G J_2 h_a$ ($G_a$ and $h_a$ are the shear modulus and thickness of the adhesive layer, respectively), and applying Eq. (27), the interface shear distribution $\tau(x)$ can be obtained. The numerical and analytical results with different thicknesses of the adhesive layer are plotted in Fig. 6. The good agreement between

<table>
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<tr>
<th>Table 1</th>
<th>Material specification of the pipe joint components for FEA input.</th>
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<tr>
<td>Material</td>
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<td>Young’s modulus (GPa)</td>
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<td>Shear strength (MPa)</td>
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<td>Poisson’s ratio</td>
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<th>Table 2</th>
<th>Geometrical and loading parameters for FEA input.</th>
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</thead>
<tbody>
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<td>Torque (N m)</td>
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</tr>
<tr>
<td>$t_2$ (mm)</td>
<td>2</td>
</tr>
<tr>
<td>$R$ (mm)</td>
<td>16.8</td>
</tr>
<tr>
<td>$L$ (mm)</td>
<td>20</td>
</tr>
</tbody>
</table>

Fig. 6. Comparison of the interface shear stress between analytical and FEA results with different thicknesses of the adhesive layer.
FEA results and the analytical results validate the model developed in the current study.

5.2. Parametric studies

In the parametric studies, three different bond lengths \(L = 80, 160\) and \(320\) \(\text{mm}\) are considered. The maximum shear stress \(\tau_f = 20\) \(\text{MPa}\) is selected for the linear elastic cohesive model. The interfacial fracture energy \(G_f = 1.2\) \(\text{N/mm}\), and its effect on the interface debonding behavior of the pipe will be discussed in the parametric study. The typical inner diameter and thickness are assumed to be \(290\) and \(10\) \(\text{mm}\) for the main pipe, respectively. And the inner diameter and thickness of the coupler are assumed to be \(311\) and \(15\) \(\text{mm}\), respectively. With these geometric configurations, one can readily see that \(R_1 = 150\) \(\text{mm}\), \(R_2 = 163\) \(\text{mm}\), and the thickness of the adhesive layer is \(0.5\) \(\text{mm}\). In the discussions below, the geometric configurations are fixed for all the conditions, which means that the \(J_1\) and \(J_2\) will not change for all the conditions. We can adjust the value of the shear modulus \(G_1\) and \(G_2\), so that the value of \(G_1 J_1\) and \(G_2 J_2\) can be correspondingly changed for the parametric studies.

6. Effect of bond length

The shear moduli \(G_1\) and \(G_2\) are assumed to be \(28\) \(\text{GPa}\) for the pipe and the coupler, respectively. A typical interfacial fracture energy \(G_f\) is assumed to be \(1.2\) \(\text{N/mm}\) for Mode III fracture. The maximum shear stress \(\tau_f\) is assumed to be \(20\) \(\text{MPa}\). Obviously, for this configuration, \(G_2 J_2 > G_1 J_1\). Therefore, the relative interface rotation at the right end \((x = L)\) is larger than that at the left end.

Fig. 7 shows the distribution of the interface shear stress along the bond length \(L = 80\) \(\text{mm}\) for the equivalent linear elastic and bilinear cohesive zone models, respectively. One can see that for the relatively short bond length \((L = 80\) \(\text{mm})\), the minimum relative interface rotation \(\phi_m\) cannot be ignored. At the cross-section where \(\phi_m\) is located, the interface stress \(\tau(x)\) is approximately \(8\) and \(7\) \(\text{MPa}\) for linear and bilinear models. This implies that the minimum relative interface rotation \(\phi_m\) is relatively large. Another feature of the short bond length joint \((L = 80\) \(\text{mm})\) is that the distribution of the interface shear stress is relatively uniform along the entire bond length.

Fig. 8 shows the distribution of the interface shear stress along the bond length \(L = 160\) \(\text{mm}\) for the equivalent linear elastic and bilinear cohesive zone models, respectively. One can see that for the medium bond length \((L = 160\) \(\text{mm})\), the minimum relative interface rotation \(\phi_m\) becomes less important. For linear model, the minimum interface stress \(\tau(x)\) is approximately \(2.6\) \(\text{MPa}\), which is \(13\)\% of the maximum shear stress \(\tau_f\) at the right end. For bilinear model, the minimum interface stress \(\tau(x)\) almost equals zero. And the distribution of the interface shear stress becomes more non-uniform along the entire bond length.

Fig. 9 shows the distribution of the interface shear stress along the bond length \(L = 320\) \(\text{mm}\) for the equivalent linear elastic and bilinear cohesive zone models, respectively. One can see that for the long bond length \((L = 320\) \(\text{mm})\), the minimum relative interface rotation \(\phi_m\) becomes ignorable. And the distribution of the interface shear stress becomes very non-uniform along the entire bond length.

In order to investigate the configuration that \(G_1 J_1 = G_2 J_2\), the value of the shear modulus \(G_2\) of the coupler is adjusted to be \(14.55\) \(\text{GPa}\), so that \(G_1 J_1 = G_2 J_2\). The distributions of the interface shear stresses (bilinear model and \(L = 160\) \(\text{mm}\)) under different external torsional loads are plotted in Fig. 10. It is also seen that the distributions of the interface shear stresses are symmetric with respect to the mid-cross-section. It is worth noting that the torsion load \(T = 55.6, 160.7, 166.7\) \(\text{kN m}\) corresponds to the right end rotation \(\phi_f = \phi_2, 0.75\phi_2 \) and \(\phi_f\), respectively. The value of \(k_1\) in the bilinear cohesive zone is \(1500\) \(\text{N/mm}^3\) when the interface stiffness ratio \(K = 8\). For the adhesive layer with typical thickness \(h_a = 0.5\) \(\text{mm}\) and typical shear modulus \(G_a = 800\) \(\text{MPa}\), the interface stiffness \(k_1\) largely reflects the linear elastic behavior of the pipe joints under torsional loads. If the traditional maximum stress failure criterion is applied, the torsion load capacity \(T\) is only about \(55.6\) \(\text{kN m}\) (which corresponds to \(\phi_f = \phi_2\)) when the maximum shear stress is reached at the right end \((\tau_f = 20\) \(\text{MPa}\)) as illustrated in Fig. 10. This is much lower than the torsion load capacity.
predicted by the cohesive zone model (166.7 kNm). A similar underestimation of the critical stress criterion based model has been observed by previous study (Rizzi et al., 2000). Rizzi et al. (2000) experimentally tested three-point bending specimens made of syntactic foam. They also conducted the numerical simulations based on the modified Drucker–Prager model and cohesive zone model, respectively. The prediction of the peak load based on Drucker–Prager model was only about 44% of the experimental value; while there was a very good agreement between the test data and the prediction based on cohesive zone model.

6.1. Effect of interface stiffness ratio $K$

The effect of the interface stiffness ratio $K = k_1/k_2$ for the bilinear cohesive zone model is plotted in Fig. 11. From Fig. 11, the softening zone is developed at both ends when $K = 8$ and 16. The softening zone lengths at both ends are slightly increased when $K$ changes from 8 to 16. While for the bilinear cohesive law with $K = 1$, the softening zone is only developed at the right end of the pipe joint when the macro-debonding starts propagating at the right end. It can also be observed that the bilinear model with larger $K$ has smaller minimum relative interface rotation $\phi_{e1}$ (or smaller minimum interface shear stress) than the model with smaller $K$ value.

The maximum torsion load capacity for the bilinear models with different $K$ and $L$ (shear modulus $G_1 = G_2 = 28$ GPa) are plotted in Fig. 12. One can see that when the bond length $L$ is larger than a certain value, any further increases in bond length $L$ cannot bring any significant increase in $T_{\text{max}}$. In another word, the effective development (transfer) length is an important design consideration for the pipe joint under torsion load. As shown in Fig. 12, for the given bond length $L$, the model with larger $K$ has higher torsion capacity. However, when the bond length $L$ is long enough, the results of $T_{\text{max}}$ obtained from different models converge to an identical value as described by Eqs. (86) or (87). Fig. 12 also implies a safe, simple and effective method for the bond length design of adhesively bonded pipe joints under torsional loads. Note that larger $K$ value requires shorter effective transfer length $L_e$ (shorter bond length is required to develop the torsion load) as illustrated by Fig. 12. Therefore, one may simply apply Eq. (39) to design the bond length of the adhesively bonded joints under torsion load. This design length is always safer than the required bond length based on any bilinear model.

6.2. Effect of torsion stiffness ratio

As seen in Fig. 12, for the same bond length $L$, the only difference in torsion load capacity among linear and bilinear models is that higher $K$ value induces higher $T_{\text{max}}$, and with the increase of the bond length, such difference gradually disappears. When the bond length $L$ is long enough, there is no difference in torsion load capacity among different models.

Therefore, only the torsion load capacity results based on the linear model is presented in the following text of this study for the sake of brevity. Based on the equivalent linear elastic model, Fig. 13 gives the distributions of interface shear stresses with different values of $G_2/J_2$ when the rotation of either end reaches $\phi_{\text{e1}}$. In Fig. 13, we fixed the value of $G_1 J_1$, and changed the value of $G_2 J_2$ by adjusting the shear modulus $G_2$ of the coupler. One can see that for the identical $G_1 J_1$, with the change of $G_2 J_2$,
the end rotation can reach $\phi_1$ at the left end, right end, or both ends simultaneously. It is noted that when $G_2 J_2 = G_1 J_1$, the area under the stress curve reaches its maximum. This implies that for given geometric and material properties ($G_1, J_1$) of the main pipe (pipe 1), the maximum torsion load capacity is achieved when $G_2 J_2 = G_1 J_1$.

Although Fig. 13 is based on the configuration that $L = 160$ mm, it is still valid for any bond length $L$. Fig. 14a and b plot the maximum torsion load capacity $T_{\text{max}}$ as a function of different torsion stiffness ratios $G_2 J_2/G_1 J_1$ with different bond lengths of $L = 80, 160$ and $320$ mm, respectively. Note that in Fig. 14a and Fig. 14b, we keep the value of the torsion stiffness $G_1 J_1$ unchanged, while change the value of $G_2 J_2$ to achieve different torsion stiffness ratios. From Fig. 14, it is seen that the maximum torsion load capacity $T_{\text{max}}$ is reached when $G_2 J_2/G_1 J_1 = 1$ regardless of the bond length $L$.

It is important to note that $R_2$ (radius of the coupler) must be larger than $R_1$ (radius of the main pipe) in the pipe joints, so that $J_2$ is much larger than $J_1$ due to the cubic relation between the radius and polar moment of inertia as seen in Eq. (2). Therefore, it is worth noting that for the torsion design of joint, the thickness of the coupler pipe (pipe 2) must be thinner than that of the main pipe (pipe 1) in order to achieve the maximum torsion load capacity if the identical materials are used for both pipes. On the other hand, if the same thickness is adopted for both pipes, in order to maximize the torsion load capacity, the shear modulus $G_2$ of the coupler pipe (pipe 2) must be adjusted so that the condition $G_2 J_2 = G_1 J_1$ can be satisfied ($G_2$ must be smaller than $G_1$).

![Fig. 14. (a) Relationship between torsion load capacity $T_{\text{max}}$ and torsion stiffness ratio when $G_2 J_2 \geq G_1 J_1$ with different $L$. (b) Relationship between torsion load capacity $T_{\text{max}}$ and torsion stiffness ratio when $G_2 J_2 \leq G_1 J_1$ with different $L$.](image)

6.3. Effect of interface fracture energy

Finally, based on the equivalent linear model, the parametric study on the effects of the interface fracture energy $G_f$ is shown in Fig. 15. Larger interface fracture energy leads to larger torsion load capacity $T_{\text{max}}$. From Fig. 15, it can be seen that $T_{\text{max}}$ is not proportional to the root square of $G_f$ although this is true for the theoretical maximum torsion load capacity $T_{\text{max}}$ as seen in Eqs. (36) and (37). It is also observed in Fig. 15 that different bond lengths $L$ also affect the relationship between $T_{\text{max}}$ and $G_f$. However, when the bond length is long enough, the effect becomes ignorable.

7. Conclusions

In the current study, the first cohesive zone model (upon the open literature) based theoretical models are successfully developed for the pipe joints under torsional loads. With the equivalent linear elastic and bilinear cohesive laws, the analytical solutions of the torsion load $T$, relative interface rotation angle $\phi$, and distribution of interface shear stress $\tau$ are derived. The models can be extended to other types of cohesive law, such as multi-linear cohesive laws with similar methodology by applying continuity conditions. Although the derivations in the current models are based on isotropic materials, it is believed that by first understanding how different parameters affect the global load capacity of isotropic pipe joints, a better perspective of the response of composite joints could be gained. By modifying the torsion stiffness of the pipes, the present models may be further extended to orthotropic materials, such as fiber-reinforced composite pipes joints. Based on the derivations in the current study, some important conclusions are summarized as follows:

1. Given that the bond length of a pipe joint is large enough, the expressions for the maximum transferable torsion load capacity $T_{\text{max}}$ is dependent only on the values of the interfacial fracture energy, the pipe’s torsion stiffness, and the radius $R$ (the distance between the bond interface and center of the pipe joint). The maximum torsion load capacity $T_{\text{max}}$ is independent of the shape of the stress–slip laws when the bond length $L$ of the pipe joint is long enough (identical expressions for bilinear cohesive zone model and equivalent linear elastic model).

2. The brief interface fracture energy based formulation of the torsion load capacity derived in the present work can be directly used for the practical torsion design of adhesively bonded pipe joints.
(3) An effective torsion transfer (development) length $l_e$ exists for the adhesively bonded pipe joint. When the bond length $L$ is longer than a certain value ($l_e$), any increase in $L$ cannot bring any significant increase in the torsion load capacity $T_{\text{max}}$. The brief expression of the effective transfer length $l_e$ derived in the present study may be used for practical torsion design of adhesively bonded pipe joints.

(4) Depending on the value of $G_1 J_1/G_2 J_2$, the debonding can occur firstly at the right end, left end or simultaneously at both ends. When $G_1 J_1 \neq G_2 J_2$, the relative interface rotation and interface shear stress is not symmetric with respect to the mid-cross-section ($x = L/2$).

(5) For a given torsion stiffness $G_1 J_1$ of the main pipe, the maximum torsion load capacity $T_{\text{max}}$ reaches its maximum value when $G_1 J_1 = G_2 J_2$, regardless of the bond length $L$. This conclusion implies that for identical thickness of the main pipe and coupler, the shear modulus of the coupler must be designed to be smaller than that of the main pipe in order to achieve the maximum torsion load capacity. On the other hand, for the pipe joints made of identical materials, the thickness of the coupler must be designed to be thinner than that of the main pipe for achieving the maximum torsion load capacity of the joints.

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References


