Some inequalities for Laplace transforms

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Abstract

Recently we established Matysiak and Szablowski’s conjecture [V. Matysiak, P.J. Szablowski, Some inequalities for characteristic functions, Theory Probab. Appl. 45 (2001) 711–713] about a lower bound of real-valued characteristic functions. In this paper, we investigate the counterparts for Laplace transforms of non-negative random variables. Surprisingly, the resulting inequalities hold true on the right half-line. Besides, we show some more inequalities by applying the convex/concave properties of the remainder in Taylor’s expansion for the exponential function.

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1. Introduction

Let \( X \) be a random variable with characteristic function \( f \) and let \( \alpha_j \) denote its \( j \)th moment. Based on an investigation of numerous examples, Matysiak and Szablowski [7] posed the following conjecture.

**Matysiak and Szablowski’s conjecture.** Let \( X \) be a symmetric random variable with \( \alpha_6 < \infty \) and characteristic function \( f \). Then there exists a constant \( \delta = \delta(\alpha_2, \alpha_4, \alpha_6) > 0 \) such that the following inequality holds (if the support of \( X \) contains at least four points):

\[
f(t) \geq p_1 \cos(y_1 t) + p_2 \cos(y_2 t) \quad \text{for } |t| \leq \delta,
\]

where

\[
p_1 = \frac{\sqrt{r^2 - 4s} + (r - 2\alpha_2)}{2\sqrt{r^2 - 4s}}, \quad p_1 + p_2 = 1, \quad y_1 = \left(\frac{1}{2}(r - \sqrt{r^2 - 4s})\right)^{1/2},
\]

\[
y_2 = \left(\frac{1}{2}(r + \sqrt{r^2 - 4s})\right)^{1/2}, \quad r = \frac{\alpha_6 - \alpha_2 \alpha_4}{\alpha_4 - \alpha_2^2}, \quad s = \frac{\alpha_2 \alpha_6 - \alpha_4^2}{\alpha_4 - \alpha_2^2}.
\]

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Recently, we proved that under the moment condition $\alpha_8 < \infty$, the above inequality (1) holds true. Moreover, by solving a differential inequality, we are able to find the range of argument for which inequality (1) holds under a stronger moment condition $E(|X|^9) < \infty$. Actually, in two previous papers we have obtained some general results for lower and upper bounds of the real part of characteristic functions (Hu and Lin [5, 6]).

In this paper, we shall investigate Laplace transforms of non-negative random variables and obtain the counterparts of the previous results about characteristic functions. Surprisingly, the resulting inequalities for Laplace transforms hold true on the right half-line. Besides, we show some more inequalities by applying the convex/concave properties of the remainder in Taylor’s expansion for the exponential function. The main results are stated in Section 2, while all the proofs and auxiliary lemmas are given in Section 3.

2. The main results

Consider the Laplace transform of a non-negative random variable $X$, namely, $L(s) = E(\exp(-sX))$, $s \geqslant 0$. Since the function $G_s(x) = \exp(-sx)$, $x \geqslant 0$, is convex on $[0, \infty)$, we have, by Jensen’s inequality, that

$$L(s) = E(G_s(X)) \leqslant G_s(\alpha_1) = \exp(-s\alpha_1) \quad \text{for all} \quad s \geqslant 0$$

(see also Rossberg, Jesiak and Siegel [8, p. 74]). This is the simplest form of the lower bounds for $L(s)$. Two general results are given in Theorems 1 and 2 below. For convenience, denote the support of $L(s)$ (see also Rossberg, Jesiak and Siegel [8, p. 74]).

**Theorem 1.** Let $X \geqslant 0$ be a random variable with $\alpha_{2m-1} < \infty$ and $|\text{supp}(X)| \geqslant m + 1$ for some integer $m \geqslant 1$. Let $p_1, p_2, \ldots, p_m$ and $y_1, y_2, \ldots, y_m$ be a set of real numbers satisfying

1. $p_1 + p_2 + \cdots + p_m = 1$,
2. $p_1 y_{1k}^k + p_2 y_{2k}^k + \cdots + p_m y_{mk}^k = \alpha_k$, \quad $k = 1, 2, \ldots, 2m - 1$,
3. $0 < p_i$, \quad $i = 1, 2, \ldots, m$, \quad $0 < y_1 < y_2 < \cdots < y_m$.

Then

$$L(s) \geqslant \sum_{i=1}^m p_i e^{-s y_i} \quad \text{for} \quad s \geqslant 0.$$  

(8)

**Theorem 2.** Let $X \geqslant 0$ be a random variable with $\alpha_{2m} < \infty$ and $|\text{supp}(X)| \geqslant m + 1$ for some integer $m \geqslant 1$. Let $p_1, p_2, \ldots, p_{m+1}$ and $y_1, y_2, \ldots, y_m$ be a set of real numbers satisfying

1. $p_1 + p_2 + \cdots + p_{m+1} = 1$,
2. $p_1 y_{1k}^k + p_2 y_{2k}^k + \cdots + p_{m+1} y_{mk}^k = \alpha_k$, \quad $k = 1, 2, \ldots, 2m$,
3. $0 < p_i$, \quad $i = 1, 2, \ldots, m$, \quad $0 < y_1 < y_2 < \cdots < y_m$.

Then

$$L(s) \leqslant p_{m+1} + \sum_{i=1}^m p_i e^{-s y_i} \quad \text{for} \quad s \geqslant 0.$$  

(12)

**Remark 1.** The RHS of (8) is exactly the Laplace transform of a random variable $X_0$ with $P(X_0 = y_i) = p_i$, $i = 1, 2, \ldots, m$, while the RHS of (12) is that of a random variable $X_*$ with $P(X_* = y_i) = p_i$, $i = 1, 2, \ldots, m$, and $P(X_* = 0) = p_{m+1}$ (the bound in (12) is sharp). If $X$ is positive and $|\text{supp}(X)| = m \geqslant 1$, the solution (consisting of $\{p_i\}$ and $\{y_i\}$) of (5)–(7) is unique. In this case, $X$ is distributed as $X_0$ and the equality in (8) holds for all $s \geqslant 0$.

We next claim without proof the existence and uniqueness of the solution of (5)–(7) and of (9)–(11). The proofs are in fact similar to those of Propositions 1 and 2 in Hu and Lin [6] (precisely, replace $y_i^2$, $\alpha_{2k}$, $x^2$ and $G(x)$ by $y_i$, $\alpha_k$, $x$ and $F(x)$, respectively).
Proposition 1. Let \( X \geq 0 \) be the same as in Theorem 1. Then there exists exactly a set of real numbers \( p_1, p_2, \ldots, p_m \) and \( y_1, y_2, \ldots, y_m \) satisfying conditions (5)–(7).

Proposition 2. Let \( X \geq 0 \) be the same as in Theorem 2. Then there exists exactly a set of real numbers \( p_1, p_2, \ldots, p_{m+1} \) and \( y_1, y_2, \ldots, y_m \) satisfying conditions (9)–(11).

When \( m = 1 \), Theorem 1 reduces to the well-known result (4) above, while Theorem 2 reduces to the next corollary. When \( m = 2 \), Theorems 1 and 2 reduce to Corollaries 2 and 3, respectively. We note that Eckberg [1, pp. 138–139] obtained the bounds in (13) and (14) below by a different approach and then applied them to various problems in queueing and traffic theory (see also Guljaš, Pearce and Pečarić [2]).

Corollary 1. Let \( X \geq 0 \) be a random variable with \( \alpha_2 < \infty \) and \( |\text{supp}(X)| \geq 2 \). Then

\[
L(s) \leq 1 - \frac{\alpha_1^2}{\alpha_2} + \frac{\alpha_1^2}{\alpha_2} e^{-(\alpha_2/\alpha_1)s} \quad \text{for } s \geq 0.
\]  

(13)

Corollary 2. Let \( X \geq 0 \) be a random variable with \( \alpha_3 < \infty \) and \( |\text{supp}(X)| \geq 3 \). Then

\[
L(s) \geq p_1 e^{-y_1 s} + p_2 e^{-y_2 s} \quad \text{for } s \geq 0,
\]

where

\[
p_1 = \frac{\sqrt{u^2 - 4v} + (u - 2\alpha_1)}{2\sqrt{u^2 - 4v}}, \quad p_1 + p_2 = 1, \quad y_1 = \frac{1}{2}(u - \sqrt{u^2 - 4v}),
\]

\[
y_2 = \frac{1}{2}(u + \sqrt{u^2 - 4v}), \quad u = \frac{\alpha_3 - \alpha_1\alpha_2}{\alpha_2 - \alpha_1^2}, \quad v = \frac{\alpha_1\alpha_3 - \alpha_2^2}{\alpha_2 - \alpha_1^2}.
\]

Corollary 3. Let \( X \geq 0 \) be a random variable with \( \alpha_4 < \infty \) and \( |\text{supp}(X)| \geq 3 \). Then

\[
L(s) \leq p_3 + p_1 e^{-y_1 s} + p_2 e^{-y_2 s} \quad \text{for } s \geq 0,
\]

where

\[
p_1 = \frac{(1 - p_3)(u + \sqrt{u^2 - 4v} - 2\alpha_1)}{2\sqrt{u^2 - 4v}}, \quad p_2 = \frac{2\alpha_1 - (1 - p_3)(u - \sqrt{u^2 - 4v})}{2\sqrt{u^2 - 4v}},
\]

\[
p_3 = \frac{(\alpha_4 - \alpha_2^2)(\alpha_2 - \alpha_1^2) - (\alpha_3 - \alpha_1\alpha_2)^2}{\alpha_2\alpha_4 - \alpha_3^2}, \quad y_1 = \frac{1}{2}(u - \sqrt{u^2 - 4v}),
\]

\[
y_2 = \frac{1}{2}(u + \sqrt{u^2 - 4v}), \quad u = \frac{\alpha_1\alpha_4 - \alpha_2\alpha_3}{\alpha_1\alpha_3 - \alpha_2^2}, \quad v = \frac{\alpha_2\alpha_4 - \alpha_3^2}{\alpha_1\alpha_3 - \alpha_2^2}.
\]

Remark 2. To compare the bounds for Laplace transforms, we consider the standard exponential distribution which has Laplace transform \( L(s) = 1/(1 + s) \), \( s \geq 0 \), and moments \( \alpha_n = n! \), \( n \geq 1 \). Let \( \ell_1(s) \), \( u_1(s) \), \( \ell_2(s) \) and \( u_2(s) \) denote the bounds in (4), (13)–(15), respectively. Then with the help of Maple we have

\[
\ell_1(s) \leq \ell_2(s) \leq L(s) \leq u_2(s) \leq u_1(s) \quad \text{for } s \in [0, 4],
\]

and

\[
\frac{u_2(s) - \ell_2(s)}{u_1(s) - \ell_1(s)} = \frac{1}{3} s^3 - \frac{1}{3} s^3 + \frac{41}{180} s^4 + O(s^5) \quad \text{as } s \to 0,
\]

\[
\frac{u_2(1) - \ell_2(1)}{L(1)} = 0.0576 < 0.3996 = \frac{u_1(1) - \ell_1(1)}{L(1)},
\]

\[
\frac{u_2(2) - \ell_2(2)}{L(2)} = 0.3538 < 1.1215 = \frac{u_1(2) - \ell_1(2)}{L(2)}.
\]

This shows that the improvement in the bounds is significant if more information about the distribution is available.
To give more inequalities for Laplace transforms, we shall apply the convex/concave properties of the remainder in Taylor’s expansion for the exponential function. See also Zubkov [9, p. 679] for Theorem 4 and Hu and Lin [4, Remark 3] for Theorem 5, while Theorem 6 is the counterpart of Theorem 7 in Hu and Lin [5].

**Theorem 3.** Let $X \geq 0$ be a random variable with $\alpha_2 \in (0, \infty)$. If $y > \alpha_2/\alpha_1$, then there exists a constant $\delta = \delta(y, \alpha_1, \alpha_2) > 0$ such that $L(s) \geq 1 - \alpha_1^2/\alpha_2^2 + (\alpha_1^2/\alpha_2^2)e^{-ys}$ for $s \in [0, \delta)$.

**Theorem 4.** Let $X \geq 0$ be a random variable with $\alpha_n < \infty$ for some positive integer $n$.

(a) If $n$ is odd,

$$L(s) \geq \sum_{k=0}^{n} \frac{(-1)^k s^k}{k!} \alpha_k \quad \text{for } s \geq 0.$$ 

(b) If $n$ is even, the inequality in (a) is reversed.

**Theorem 5.** Let $X \geq 0$ be a random variable with $\alpha_n < \infty$ for some integer $n \geq 2$.

(a) If $n$ is odd,

$$L(s) \geq e^{-\alpha_1 s} + \sum_{k=2}^{n} \frac{(-s)^k}{k!} (\alpha_k - \alpha_1^k) \quad \text{for } s \geq 0.$$ 

(b) If $n$ is even, the inequality in (a) is reversed.

**Theorem 6.** Assume that $0 \leq X \leq c$ almost surely for some positive constant $c$. Then there exists a constant $\delta > 0$ such that $L(s) \geq 1 - \alpha_1 s + \alpha_2 \tau(cs)s^2$ for $s \in [0, \delta/c]$, where $\tau(s) = (e^{-x} - 1 + s)/s^2 \geq 0$ for $s > 0$ and $\tau(0) = 1/2$.

For $\gamma > 0$, define the function $\eta_{\gamma}$ by

$$\eta_{\gamma}(s) = \frac{1 - e^{-s}}{s^\gamma}, \quad s > 0, \quad (16)$$

and $\eta_1(0) = 1$, $\eta_1(0) = +\infty$ or 0 according to whether $\gamma > 1$ or $\gamma \in (0, 1)$. Then it is seen that (i) $\eta_{\gamma}$ decreases on $[0, \infty)$ if $\gamma > 1$, and (ii) $\eta_{\gamma}$ increases on $[0, a_{\gamma}]$ and decreases on $[a_{\gamma}, \infty)$ if $\gamma \in (0, 1)$, where $a_{\gamma} > 0$ is the unique solution of the equation $e^{a} = 1 + a/\gamma$, $a > 0$. The next result is the counterpart of Theorem 8 in Hu and Lin [5].

**Theorem 7.** Let $\eta_{\gamma}$ be the function defined in (16). Assume that $0 \leq X \leq c$ almost surely for some positive constant $c$.

(a) If $\gamma \geq 1$ and $A > 0$, then $L(s) \leq 1 - \alpha_{\gamma} \eta_{\gamma}(A)s^\gamma$ for $s \in [0, A/c]$, where $\alpha_{\gamma} = E(X^\gamma)$.

(b) If $\gamma \in (0, 1)$ and $A \in (0, a_{\gamma})$, then $L(s) \geq 1 - \alpha_{\gamma} \eta_{\gamma}(A)s^\gamma$ for $s \in [0, A/c]$, where $a_{\gamma}$ is the unique solution of the equation $e^{a} = 1 + a/\gamma$, $a > 0$.

For $n \geq 1$ and $\gamma \in (0, 1]$, define the function $\eta_{n, \gamma}$ by

$$\eta_{n, \gamma}(s) = \frac{(-1)^{n+1}}{s^\gamma} \left( e^{-s} - \sum_{k=0}^{n} \frac{(-1)^k s^k}{k!} \right) \equiv \frac{(-1)^{n+1}}{s^\gamma} R_n(s), \quad s > 0, \quad \text{and } \eta_{n, \gamma}(0) = 0. \quad (17)$$

Then $\eta_{n, \gamma}$ increases on $[0, \infty)$ and we have the following:

**Theorem 8.** Let $\gamma \in (0, 1]$, $A > 0$ and integer $n \geq 1$. Let $\eta_{n, \gamma}$ be the function defined in (17). Assume further that $0 \leq X \leq c$ almost surely for some positive constant $c$. 

(a) If \( n \) is even, then
\[
L(s) \geq \sum_{k=0}^{n} \frac{(-1)^k}{k!} \alpha_k s^k - \eta_n,\gamma(A) \alpha, s^n, \quad 0 \leq s \leq \frac{A}{c}.
\]
(b) If \( n \) is odd, then
\[
L(s) \leq \sum_{k=0}^{n} \frac{(-1)^k}{k!} \alpha_k s^k + \eta_n,\gamma(A) \alpha, s^n, \quad 0 \leq s \leq \frac{A}{c}.
\]

**Theorem 9.** Let \( X \geq 0 \) be a random variable and let \( \beta_{\gamma c} = E(X^{\gamma} I[X \leq c]) \), where \( \gamma \) and \( c \) are positive constants and \( I \) denotes the indicator function.

(a) If \( \gamma \geq 1 \), then \( L(s) \geq 1 - \beta_{\gamma c}/c^\gamma + (\beta_{\gamma c}/c^\gamma)e^{-sc} \) for \( s \geq 0 \).
(b) If \( \gamma \in (0, 1) \), then \( L(s) \geq 1 - \beta_{\gamma c}/c^\gamma + (\beta_{\gamma c}/c^\gamma)e^{-sc} - P(X > c) \) for \( s \in [0, a_\gamma/c] \), where \( a_\gamma > 0 \) is the unique solution of the equation \( e^a = 1 + a/\gamma, a > 0 \).

### 3. Lemmas and proofs of main results

To prove the main results, we need some useful lemmas. One key point in the proofs below is to count the changes of sign of a function. Let \( \phi \) be a real-valued and Lebesgue measurable function on \( (a, b) \). We say that \( \phi \) has \( n \) changes of sign in \( (a, b) \) if there exists a disjoint partition \( I_1 < I_2 < \cdots < I_{n+1} \) of \( (a, b) \) such that (i) \( \phi \) has opposite signs on subsequential intervals \( I_j \) and \( I_{j+1} \) for \( j = 1, 2, \ldots, n \), and (ii) \( \int_{I_j} \phi(t) \, dt \neq 0 \) for all \( j \), where \( I_j \cap I_{j+1} = \emptyset \) means that \( I_j \) lies on the left hand side of \( I_{j+1} \). The first two lemmas are fundamental and their proofs are available in Hu and Lin [6].

**Lemma 1.** Let \( X \geq 0 \) be a random variable with distribution \( F \). Assume further that \( k \) is a continuous function on \( [0, \infty) \) such that the Lebesgue–Stieltjes integral \( g(x) = \int_{1,x,\infty}^x k(t) \, dF(t) \) is finite for \( x \geq 0 \). Then the following properties hold:

(a) the function \( g \) is left-continuous on \( (0, \infty) \), and
(b) if, in addition, \( g(0) = 0 = g(\infty) \equiv \lim_{x \to \infty} g(x) \) and the function \( k \) has \( n \geq 1 \) changes of sign in \( (0, \infty) \), \( g \) has at most \( n - 1 \) changes of sign in \( (0, \infty) \).

**Lemma 2.** Let \( g \) be a left-continuous function on \( [0, \infty) \) such that (i) \( g(0) = g(\infty) = 0 \) and (ii) the integral \( g_1(x) = \int_x^\infty g(t) \, dt \) is finite for \( x \geq 0 \). Then the following properties hold:

(a) the function \( g_1 \) is continuous on \( [0, \infty) \), and
(b) if, in addition, the function \( g \) has \( n \geq 1 \) changes of sign in \( (0, \infty) \), \( g_1 \) has at most \( n - 1 \) changes of sign in \( (0, \infty) \).

We next recall an important property of the moment matrix. For given integers \( 0 \leq k_1 \leq k_2 \leq \cdots \leq k_m \), assume \( E(X^{2k_m}) < \infty \) and let \( A = [a_{i,j}] \) be the moment matrix of random variables \( X^{k_1}, X^{k_2}, \ldots, X^{k_m} \), where \( a_{i,j} = E(X^{k_i+k_j}) \). Then the \( m \times m \) symmetric matrix \( A \) is non-negative definite and hence its determinant \( \det(A) \geq 0 \). This is an immediate consequence of the fact that
\[
E \left( \left( \sum_{i=1}^{m} X^{k_i} t_i \right)^2 \right) = \sum_{i=1}^{m} \sum_{j=1}^{m} E(X^{k_i+k_j}) t_i t_j = t^\top A t \geq 0 \quad \text{for all } t_1, t_2, \ldots, t_m \in \mathcal{R} \equiv (-\infty, \infty),
\]
where \( t^\top = (t_1, t_2, \ldots, t_m) \) is the transpose of \( t \in \mathcal{R}^m \).
Lemma 3. Let X ≥ 0 be a random variable and m a positive integer.

(a) Assume α2m < ∞ and |supp(X)| ≥ m. Let A2m = [αi,j] be the moment matrix of random variables 1, X, X2, . . . , Xm. Then det(A2m) ≥ 0 with equality holding if and only if |supp(X)| = m.

(b) Assume α2m+1 < ∞ and |supp(X)| ≥ m + 1. Let B2m+1 = [βi,j] be the moment matrix of random variables X1/2, X3/2, . . . , Xm+1/2. Then det(B2m+1) ≥ 0 with equality holding if and only if (i) |supp(X)| = m + 1 and (ii) 0 ∈ supp(X).

(c) Assume α2m < ∞ and |supp(X)| ≥ m. Let C2m = [γi,j] be the moment matrix of random variables X, X2, . . . , Xm. Then det(C2m) ≥ 0 with equality holding if and only if (i) |supp(X)| = m and (ii) 0 ∈ supp(X).

Proof. Clearly, det(A2m) ≥ 0 and the equality det(A2m) = 0 means that \( \sum_{i=0}^{m} X^i t_i = 0 \) almost surely for some constants \( t_0, t_1, \ldots, t_m \), not all zero. Since |supp(X)| ≥ m, the conclusion of part (a) follows immediately. For part (b), note that det(B2m+1) ≥ 0 and the equality holds if and only if \( \sum_{i=0}^{m} X^{i+1/2} t_i = X^{1/2} \sum_{i=0}^{m} X^i t_i = 0 \) almost surely for some constants \( t_0, t_1, \ldots, t_m \), not all zero. The conclusion follows because |supp(X)| ≥ m + 1. The proof of part (c) is similar and omitted. □

Lemma 4. Let X ≥ 0 be the same as in Theorem 1.

(a) Given moments \( \alpha_k, k = 1, 2, \ldots, 2m - 1 \), the system of equations

\[
\alpha_k = \sum_{j=1}^{m} (-1)^{j-1} s_j \alpha_{k-j}, \quad k = m, m+1, \ldots, 2m-1,
\]

has exactly a solution consisting of \( s_j = \det(A_{2m,m+1-j})/\det(A_{2m,m+1}) > 0, \) \( j = 1, 2, \ldots, m \), where the \( m \times m \) matrix \( A_{2m,i} \) is formed by deleting the \( (m+1) \)th (last) column and \( i \)th row of the moment matrix \( A_{2m} \) defined in Lemma 3(a). Consequently, \( \det(A_{2m,i}) > 0 \) for each \( i \).

(b) For given positive constants \( s_j \) in (a), the equation \( H_m(x) = x^m - s_1 x^{m-1} + \alpha_2 x^{m-2} - \cdots + (-1)^m s_m = 0, x \in \mathcal{R} \), has \( m \) distinct positive solutions, say \( 0 < y_1 < y_2 < \cdots < y_m \), namely, \( H_m(x) = \prod_{i=1}^{m} (x - y_i), x \in \mathcal{R} \).

(c) For given positive constants \( s_j \) in (a), \( L^{(m)}(s) + s_1 L^{(m-1)}(s) + \cdots + s_m L(s) \geq 0, s \geq 0 \).

Proof. Note that the function \( H_m(x) \) in part (b) is the Chebyshev polynomial

\[
\frac{1}{\det(A_{2m,m+1})} \begin{vmatrix} 1 & \alpha_1 & \cdots & \alpha_{m-1} & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_m & x \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{m-1} & \alpha_m & \cdots & \alpha_{2m-2} & x^{m-1} \\ \alpha_m & \alpha_{m+1} & \cdots & \alpha_{2m-1} & x^m \end{vmatrix}
\]

(A) Applying Cramér rule and the property \( \det(A) = \det(A^T) \) for any square matrix \( A \), we have that \( s_j = \det(A_{2m,m+1-j})/\det(A_{2m,m+1}) \), \( j = 1, 2, \ldots, m \). To prove \( s_m > 0 \), recall that \( A_{2m,m+1} = A_{2m-2} \) is the moment matrix of \( X, X^2, \ldots, X^{m-1} \). By Lemma 3(a), we have \( \det(A_{2m,m+1}) > 0 \) because \( |supp(X)| \geq m + 1 \). Similarly, by Lemma 3(b), we have \( \det(A_{2m,1}) > 0 \), because \( A_{2m,1} = B_{2m-1} \) is the moment matrix of \( X^{1/2}, X^{3/2}, \ldots, X^{m-1/2} \). Therefore, \( s_m > 0 \). It remains to prove \( s_j > 0 \) for \( 1 \leq j \leq m - 1 \). For convenience, let us define the functions

\[
g_0(x) = \int_{[x, \infty)} H_m(t) dF(t) \quad \text{and} \quad g_n(x) = \int_{x}^{\infty} g_{n-1}(t) dt, \quad x \geq 0, \quad n = 1, 2, \ldots, m,
\]

where \( H_m \) is defined in part (b), \( F \) is the distribution of \( X \) and \( g_0 \) is left-continuous on \([0, \infty)\). Then we have that

\[
g_1(\infty) = 0, \quad g_i(0) = \frac{1}{i!} \left( \alpha_{m+i} - s_1 \alpha_{m+i-1} + \cdots + (-1)^m s_m \alpha_i \right) = 0, \quad 0 \leq i \leq m - 1.
\]

Suppose on the contrary that \( s_1 \leq 0 \). Then the function \( H_m \) has at most \( m - 1 \) changes of sign in \((0, \infty)\), because \( H_m^{(m-1)}(x) = m! x - (m - 1)! s_1 \geq 0 \) on \([0, \infty)\). This implies by Lemma 1 that \( g_0 \) has at most \( m - 2 \) changes of sign in
(0, ∞) because \( g_0(0) = g_0(∞) = 0 \). Similarly, for \( i = 1, 2, \ldots, m-1 \), the function \( g_i \) has at most \( m - 2 - i \) changes of sign in \((0, ∞)\) by Lemma 2 and the fact that \( g_i(0) = g_i(∞) = 0 \). In particular, the function \( g_{m-2} \) has no changes of sign in \((0, ∞)\) and hence either \( g_{m-2}(x) \geq 0 \) on \([0, ∞)\) or \( g_{m-2}(x) \leq 0 \) on \([0, ∞)\). This together with the fact \( g_{m-1}(0) = g_{m-1}(∞) = 0 \) implies that \( g_{m-1}(x) = 0 \) on \([0, ∞)\). The latter in turn implies that \( g_i(x) = 0 \) on \([0, ∞)\) for all \( i = 0, 1, \ldots, m - 2 \). Especially, \( g_0(x) = ∫_{[x, ∞)} H_m(y) dF(y) = 0 \) for all \( x \geq 0 \), which is impossible because \(|\text{supp}(X)| \geq m + 1 \). Therefore \( s_1 > 0 \). Applying the facts \( s_1 > 0, s_m > 0, H_m(∞) = ∞ \) and proceeding along the same lines, we can prove that \( s_2 > 0 \). By induction, we finally have \( s_j > 0 \) for \( 3 \leq j \leq m - 1 \). The proof of part (a) is completed.

(B) To prove part (b), suppose on the contrary that the equation \( H_m(x) = 0, x \in \mathcal{R} \), has at most \( m - 1 \) distinct positive solutions. Then the function \( H_m \) has at most \( m - 1 \) changes of sign in \((0, ∞)\), due to the fact that \( s_m > 0 \) and \( H_m(x) → ∞ \) as \( x → ∞ \). Proceeding along the same lines as in the proof of part (a), we have that \( g_{m-1}(x) = \cdots = g_0(x) = 0, x \geq 0 \), which is impossible because \(|\text{supp}(X)| \geq m + 1 \). The proof of part (b) is completed.

(C) To prove part (c), we calculate that

\[
g_m(0^+) = \lim_{x \to 0^+} g_m(x) = \frac{1}{m!}(α_2α_m - s_1α_{2m-1} + \cdots + (-1)^m s_mα_m) = \frac{1}{m!} \det(A_{2m}) > 0 \]

(the function \( g_m \) is monotone on \([0, ∞)\) as shown below). The last inequality is due to Lemma 3(a) and the assumption \(|\text{supp}(X)| \geq m + 1 \) (note that \( g_m(0^+) = ∞ \) if \( α_2α_m = ∞ \)). By part (b), the equation \( H_m(x) = 0 \) has exactly \( m \) distinct positive solutions. This in turn implies that the function \( g_0 \) has at most \( m - 1 \) changes of sign in \((0, ∞)\) due to Lemma 1 and the fact \( g_0(0) = g_0(∞) = 0 \). By induction, \( g_{m-1} \) has no changes of sign in \((0, ∞)\). This together with the fact \( g_{m-1}(0) = g_{m-1}(∞) = 0 \) implies either (i) \( g_{m-1}(x) \geq 0 \) on \([0, ∞)\) or (ii) \( g_{m-1}(x) \leq 0 \) on \([0, ∞)\). Since \( g_m(0^+) > 0 \), we conclude that \( g_{m-1}(x) ≥ 0 \) on \([0, ∞)\). By integration by parts, we finally have

\[
L^{(m)}(s) + s_1L^{(m-1)}(s) + \cdots + s_mL(s)
= (-1)^m \int_{[0, ∞)} e^{-sx} H_m(x) dF(x) = (-1)^m (-1) \int_{[0, ∞)} e^{-sx} dF(x) = (-1)^m (-1) \lim_{x→∞} s e^{-sx} g_0(x) dx
= (-1)^m (-1)^2 \int_0^∞ s^2 e^{-sx} g_1(x) dx = \cdots = (-1)^{2m} \int_0^∞ s^m e^{-sx} g_{m-1}(x) dx ≥ 0, \quad s ≥ 0.
\]

The proof is completed. \( \square \)

**Lemma 5.** Let \( X ≥ 0 \) be the same as in Theorem 2.

(a) Given moments \( α_k, k = 1, 2, \ldots, 2m \), the system of equations

\[
α_k = \sum_{j=1}^m (-1)^{j-1} s_j α_{k-j}, \quad k = m + 1, m + 2, \ldots, 2m,
\]

has exactly a solution consisting of \( s_j = \det(B_{2m+1,m+1-j}) / \det(B_{2m+1,m+1}) > 0 \), \( j = 1, 2, \ldots, m \), where the \( m \times m \) matrix \( B_{2m+1,i} \) is formed by deleting the \((m+1)\)th (last) column and \( i \)th row of the moment matrix \( B_{2m+1} \) defined in Lemma 3(b). Consequently, \( \det(B_{2m+1,i}) > 0 \) for each \( i \).

(b) For given positive constants \( s_j \) in (a), the equation \( H_m(x) = x^m - s_1x^{m-1} + s_2x^{m-2} - \cdots + (-1)^m s_m = 0, x \in \mathcal{R} \), has \( m \) distinct positive solutions, say \( 0 < y_1 < y_2 < \cdots < y_m \), namely, \( H_m(x) = \prod_{i=1}^m (x - y_i), x \in \mathcal{R} \).

(c) For given positive constants \( s_j \) in (a), \( L^{(m+1)}(s) + s_1L^{(m)}(s) + \cdots + s_mL'(s) ≤ 0, s ≥ 0 \).

**Proof.** (A) Recall that \( B_{2m+1,m+1} = B_{2m-1} \) is the moment matrix of \( X^{1/2}, X^{3/2}, \ldots, X^{m-1/2} \). By Lemma 3(b), we have \( \det(B_{2m+1,m+1}) > 0 \) because \(|\text{supp}(X)| \geq m + 1 \). Clearly, \( s_j = \det(B_{2m+1,m+1-j}) / \det(B_{2m+1,m+1}) \) for \( j = 1, 2, \ldots, m \). As in the proof of Lemma 4(a), we first claim that \( s_m > 0 \). To see this, note by Lemma 3(c) that
det(B_{2m+1,1}) > 0$, because $B_{2m+1,1} = C_{2m}$ is the moment matrix of $X$, $X^2, \ldots, X^m$ and $|\text{supp}(X)| \geq m + 1$. Therefore, $s_m > 0$. Next, we prove $s_j > 0$ for $1 \leq j \leq m - 1$. Let us consider the functions

$$g_n^*(x) = \int_{[x, \infty)} t H_m(t) dF(t) \quad \text{and} \quad g_n^*(x) = \int_0^\infty g_{n-1}^*(t) dt, \quad x \geq 0, \quad n = 1, 2, \ldots, m,$$

where $H_m$ is defined in part (b) and $F$ is the distribution of $X$. It is seen that $g_n^*(\infty) = 0$ for $n \geq 0$ and

$$g_n^*(0) = \frac{1}{1!} (\alpha_m + s_m \alpha_{m+1} + \cdots + (-1)^m s_m \alpha_{m+1}) = 0, \quad 0 \leq i \leq m - 1.$$

The remaining part of the proof is similar to that of Lemma 4(a) and is omitted.

(B) The proof of part (b) is similar to that of Lemma 4(b) and is also omitted.

(C) To prove part (c), we recall that

$$g_m^*(0^+) = \frac{1}{m!} (\alpha_{2m+1} - s_1 \alpha_{2m} + \cdots + (-1)^m s_m \alpha_{m+1}) = \frac{1}{m!} \det(B_{2m+1}) \geq 0$$

(the function $g_m^*$ is monotone on $[0, \infty)$ as shown below). There are two possible cases: (i) $g_m^*(0^+) = 0$ and (ii) $g_m^*(0^+) > 0$. We next prove that in either case, $g_m^*(x) \geq 0$ on $[0, \infty)$. By part (b), the equation $H_m(x) = 0$ has exactly $m$ distinct positive solutions. This in turn implies that the function $g_m^*$ has at most $m - 1$ changes of sign in $(0, \infty)$ due to the fact that $g_m^*(0) = g_m^*(\infty) = 0$. Proceeding along the same lines as in the proof of Lemma 4(c), we have either $g_m^*(x) \geq 0$ on $[0, \infty)$ or $g_m^*(x) \leq 0$ on $[0, \infty)$. For case (i) above, we conclude that $g_m^*(x) = 0$ on $[0, \infty)$, because $g_m^*(0^+) = g_m^*(\infty) = 0$ and $g_m^*$ is monotone on $[0, \infty)$. This implies that $g_m^*(x) = 0$ on $[0, \infty)$. On the other hand, for case (ii), we have that $g_m^*(x) \geq 0$ on $[0, \infty)$. Therefore, for $s \geq 0$,

$$L^{(m+1)}(s) + s_1 L^{(m)}(s) + \cdots + s_m L'(s) = (-1)^{m+1} \int_{[0, \infty)} e^{-sx} x H_m(x) dF(x)$$

$$= (-1)^{m+1} \int_0^\infty s^m e^{-sx} g_{m-1}^*(x) dx \leq 0.$$

The proof is completed. \[\square\]

To prove Theorems 1 and 2, we need some more notations and auxiliary lemmas. For given positive real numbers $y_1 < y_2 < \cdots < y_m$ and positive integer $j \leq m$, define the symmetric sums

$$s_1 j = y_1 + \cdots + y_j,$$

$$s_2 j = y_1 y_2 + \cdots + y_{j-1} y_j,$$

$$\vdots$$

$$s_{jj} = y_1 \cdots y_j,$$

which together form the function

$$H_{jm}(x) \equiv \prod_{i=1}^j (x - y_i) = x^j - s_1 j x^{j-1} + s_2 j x^{j-2} - \cdots + (-1)^j s_{jj}, \quad x \geq 0. \quad (18)$$

If $\alpha_m < \infty$, denote further $c_0 = 1$ and $c_j = \alpha_j - s_1 j \alpha_{j-1} + \cdots + (-1)^j s_{jj}, 1 \leq j \leq m$. Moreover, let $L_0 = L$ (the Laplace transform of $X \geq 0$) and define the functions $L_j$, $1 \leq j \leq m$, by

$$L_j(s) = L^{(j)}(s) + s_1 j L^{(j-1)}(s) + \cdots + s_{jj} L(s), \quad s \geq 0.$$

The next lemma is fundamental and requires no proof.
Lemma 6. Under the above setting, if \( \alpha_m < \infty \), the following identities hold:
\[
L_j(s) = L'_{j-1}(s) + y_j L_{j-1}(s), \quad s \geq 0, \\
L_j(0) = (-1)^j c_j, \quad j = 1, 2, \ldots, m.
\]

Lemma 7. Let \( a_1, a_2, \ldots, a_m \) be \( m \) distinct positive real numbers and let \( c, b_1, b_2, \ldots, b_m \) be real numbers.

(a) Assume that the function \( h \) satisfies the differential inequality:
\[
h'(s) + a_1 h(s) \geq b_1 + b_2 e^{-a_2 s} + \cdots + b_m e^{-a_m s} \quad \text{for} \quad s \geq 0, \\
h(0) = c.
\]
Then it has the lower bound:
\[
h(s) \geq \frac{b_1}{a_1} + \left( c - \frac{b_1}{a_1} + \sum_{j=2}^{m} \frac{b_j}{a_j - a_1} \right) e^{-a_1 s} + \sum_{j=2}^{m} \frac{-b_j}{a_j - a_1} e^{-a_j s} \quad \text{for} \quad s \geq 0.
\]

(b) If the inequality in (19) is reversed, so is that in (20).

Proof. By the assumption (19), we have
\[
\frac{d}{ds} \left( h(s) e^{a_1 s} \right) = e^{a_1 s} \left( h'(s) + a_1 h(s) \right) \geq b_1 e^{a_1 s} + \sum_{j=2}^{m} b_j e^{-(a_j - a_1) s}, \quad s \geq 0.
\]
Taking integration from 0 to \( s \) yields that
\[
h(s) e^{a_1 s} - h(0) \geq \frac{b_1}{a_1} e^{a_1 s} - 1 + \sum_{j=2}^{m} \frac{b_j}{a_j - a_1} \left( 1 - e^{-(a_j - a_1) s} \right), \quad s \geq 0,
\]
which in turn implies the required inequality (20). This proves part (a).

The proof of part (b) is similar and omitted. \( \square \)

Proof of Theorem 1. Since the real numbers \( p_1, p_2, \ldots, p_m \) and \( y_1, y_2, \ldots, y_m \) together satisfy conditions (5)–(7), the set of symmetric sums \( s_1, s_2, \ldots, s_m \) of \( y_1, \ldots, y_m \) (namely, \( s_j = s_{jm} \) for each \( j \)) is exactly the solution of the system of equations in Lemma 4(a) (by mimicking the proof of Lemma 3 in Hu and Lin [5]). This implies that the set \( \{y_i\}_{i=1}^{m} \) is the same as in Lemma 4(b). By Lemmas 4(c) and 6, we have
\[
L'_{m-1}(s) + y_m L_{m-1}(s) = L_m(s) = L^{(m)}(s) + s_{1m} L^{(m-1)}(s) + \cdots + s_{mm} L(s) \geq 0, \quad s \geq 0.
\]
Lemma 7 then implies that
\[
L_{m-1}(s) \geq L_{m-1}(0) e^{-y_m s} \geq 0, \quad s \geq 0.
\]
If \( m = 1 \), the above inequality is exactly the required result (8). Suppose now \( m \geq 2 \). Then by Lemmas 6 and 7 again, we obtain that
\[
L_{m-2}(s) \geq \left( L_{m-2}(0) + \frac{L_{m-1}(0)}{y_m - y_{m-1}} \right) e^{-y_m s} + \frac{-L_{m-1}(0)}{y_m - y_{m-1}} e^{-y_m s}, \quad s \geq 0.
\]
Applying the same procedure \( m - 2 \) more times, we finally get that
\[
L_0(s) \geq \left( 1 - \sum_{j=2}^{m} d_j \right) e^{-y_1 s} + \sum_{j=2}^{m} d_j e^{-y_j s}, \quad s \geq 0,
\]
where
\[
d_j = \sum_{i=1}^{m-j+1} \left( \prod_{k=1,k \neq j}^{m} \frac{1}{s_j - y_k} \right) (-1)^{j+i-2} L_{j+i-2}(0), \quad j = 2, 3, \ldots, m.
\]
It remains to prove \( d_j = p_j \) for \( j \geq 2 \). To do this, we write
\[
(-1)^j L_j(0) = c_j = \alpha_j - s_1 \alpha_{j-1} + \cdots + (-1)^j s_{jj}
\]
\[
= \sum_{i=1}^{m} p_i y_i^j - s_1 j \sum_{i=1}^{m} p_i y_i^{j-1} + \cdots + (-1)^j s_{jj} \sum_{i=1}^{m} p_i
\]
\[
= p_1 H_{jm}(y_1) + \cdots + p_m H_{jm}(y_m)
\]
\[
= p_{j+1} H_{jm}(y_{j+1}) + \cdots + p_m H_{jm}(y_m), \quad 1 \leq j \leq m - 1,
\]
in which \( H_{jm}(y_i) = 0 \) for \( i \leq j \) by the definition in (18). Equivalently,
\[
p_{j+1} \prod_{i=1}^{j} (y_{i+1} - y_i) + \cdots + p_m \prod_{i=1}^{j} (y_m - y_i) = (-1)^j L_j(0), \quad 1 \leq j \leq m - 1.
\]
Solving the above system of \( m - 1 \) equations leads to
\[
p_j = \sum_{i=1}^{m-j+1} \left( \prod_{k=1, k \neq j}^{j+i-1} \frac{1}{y_{j} - y_{k}} \right) (-1)^{j+i-2} L_{j+i-2}(0) = d_j, \quad 2 \leq j \leq m.
\]
This completes the proof. \( \square \)

**Proof of Theorem 2.** Since the real numbers \( p_1, p_2, \ldots, p_m+1 \) and \( y_1, y_2, \ldots, y_m \) together satisfy conditions (9)–(11), we can show, proceeding along the same lines as in the proof of Lemma 3 in Hu and Lin [5], that the set of symmetric sums \( s_1, s_2, \ldots, s_m \) of \( y_1, \ldots, y_m \) (namely, \( s_j = s_{jm} \)) is exactly the solution of the system of equations in Lemma 5(a). This implies that the set \( \{ y_i \}_{i=1}^{m} \) is the same as in Lemma 5(b). By Lemma 5(c), we have
\[
L^{(m+1)}(s) + s_1 L^{(m)}(s) + \cdots + s_m L'(s) \leq 0, \quad s \geq 0.
\]
Taking integration from 0 to \( s \) yields that
\[
L^{(m)}(s) + s_1 m L^{(m-1)}(s) + \cdots + s_m m L(s) \leq L^{(m)}(0) + s_1 m L^{(m-1)}(0) + \cdots + s_m m L(0)
\]
\[
= (-1)^m (\alpha_m - s_1 \alpha_{m-1} + \cdots + (-1)^m s_{mm})
\]
\[
= s_m m p_{m+1} = \left( \prod_{i=1}^{m} y_i \right) p_{m+1}, \quad s \geq 0,
\]
in which the penultimate equality is due to the fact
\[
\alpha_m - s_1 \alpha_{m-1} + \cdots + (-1)^m s_{mm} (p_1 + p_2 + \cdots + p_m) = 0.
\]
Therefore, by Lemma 6,
\[
L'_{m-1}(s) + y_m L_{m-1}(s) = L_m(s) \leq \left( \prod_{i=1}^{m} y_i \right) p_{m+1}, \quad s \geq 0.
\]
Using Lemma 7 then yields
\[
L_{m-1}(s) \leq \frac{1}{y_m} \left( \prod_{i=1}^{m} y_i \right) p_{m+1} + \left( L_{m-1}(0) - \frac{1}{y_m} \left( \prod_{i=1}^{m} y_i \right) p_{m+1} \right) e^{-y_m s}, \quad s \geq 0.
\]
If \( m = 1 \), the above inequality is exactly the required result (12). Suppose now \( m \geq 2 \). Then applying the same procedure \( m - 1 \) more times, we get
\[
L_0(s) \leq p_{m+1} + \left( 1 - p_{m+1} - \sum_{j=2}^{m} \frac{d^*_j}{y_j} \right) e^{-y_1 s} + \sum_{j=2}^{m} \frac{d^*_j}{y_j} e^{-y_j s}, \quad s \geq 0,
\]
where
\[ p_{m+1} = L_m(0)/\prod_{i=1}^m y_j \quad \text{and} \quad d_j^n = \sum_{i=1}^{m-j+1} \left( \prod_{k=1, k \neq j}^{j+i-1} \frac{1}{y_j - y_k} \right) (-1)^{j+i-1} L'_{j+i-2}(0). \]

It remains to prove \( d_j^n/y_j = p_j \) for \( j = 2, 3, \ldots, m \). By definition of \( L_j \), we have
\[ L'_j(0) = L^{(j+1)}(0) + s_{1j} L^{(j)}(0) + \cdots + s_{jj} L'(0) \]
\[ = (-1)^{j+1} (\alpha_{j+1} - s_{1j} \alpha_j + \cdots + (-1)^j s_{jj} \alpha_1). \]
This in turn implies that
\[ (-1)^{j+1} L'_j(0) = \alpha_{j+1} - s_{1j} \alpha_j + \cdots + (-1)^j s_{jj} \alpha_1 \]
\[ = \sum_{i=1}^m p_i y_j^{i-1} - s_{1j} \sum_{i=1}^m p_i y_j^i + \cdots + (-1)^j s_{jj} \sum_{i=1}^m p_i y_i \]
\[ = p_1 y_1 H_{jm}(y_1) + \cdots + p_m y_m H_{jm}(y_m) \]
\[ = p_j + y_{j+1} H_{jm}(y_{j+1}) + \cdots + p_m y_m H_{jm}(y_m), \quad 1 \leq j \leq m - 1. \]

Solving the above system of \( m - 1 \) equations yields the required result:
\[ p_j = \frac{1}{y_j} \sum_{i=1}^{m-j+i-1} \left( \prod_{k=1, k \neq j}^{j+i-1} \frac{1}{y_j - y_k} \right) (-1)^{j+i-1} L'_{j+i-2}(0) = \frac{d_j^n}{y_j}, \quad 2 \leq j \leq m. \]

To prove Theorem 3, we recall that the remainder \( R_n \) in (21) below is convex or concave on \([0, \infty)\) according to whether \( n \) is odd or even. Lemma 8 can be proved by induction on \( n \); see also Hardy, Littlewood and Pólya [3, p. 104].

**Lemma 8.** For each \( n \geq 1 \), the function
\[ G_n(s) = (-1)^{n+1} \left( e^{-s} - \sum_{k=0}^n \frac{(-1)^k s^k}{k!} \right) = (-1)^{n+1} R_n(s) \geq 0, \quad s \geq 0, \tag{21} \]
is convex on \([0, \infty)\), where \( R_n \) is the remainder in Taylor’s expansion for the exponential function \( g(s) = e^{-s}, s \geq 0 \).

**Proof of Theorem 3.** It follows from (21) that
\[ 0 \leq e^{-sx} - (1 - sx) \leq \frac{1}{2} (sx)^2 \quad \text{for all} \ s \ \text{and} \ x \in [0, \infty). \]
Hence we have
\[ 0 \leq L(s) - (1 - \alpha_1 s) \leq \frac{1}{2} \alpha_2 s^2 \quad \text{for all} \ s \in [0, \infty), \]
and for \( y \geq 0 \)
\[ \frac{\alpha_1}{\alpha_2} e^{-sy} = \frac{\alpha_1}{\alpha_2} - \frac{\alpha_1}{\alpha_2} y s + O(s^2) \quad \text{as} \ s \to 0. \]
These together imply that as \( s \to 0 \)
\[ L(s) - \frac{\alpha_1}{\alpha_2} e^{-sy} = 1 - \frac{\alpha_1}{\alpha_2} - \left( \alpha_1 - \frac{\alpha_1}{\alpha_2} y \right) s + O(s^2), \]
which proves the required results. \( \square \)

**Proof of Theorems 4 and 5.** Theorem 4 follows immediately from the fact that \( G_n \geq 0 \) in (21), while Theorem 5 follows from Jensen’s inequality and the convex/concave properties of the function \( G_n \) on \([0, \infty)\). \( \square \)
**Proof of Theorem 6.** Note that the function \( \tau \) is decreasing in some interval \([0, \delta]\) with \( \delta > 0 \). Therefore \( L(s) = 1 - \alpha_1 s + s^2 E(Y_2 \tau(s X)) \geq 1 - \alpha_1 s + \alpha_2 \tau(cs)s^2 \) for \( s \in [0, \delta/c] \). The proof is completed.

**Proof of Theorems 7 and 8.** Note first that the function \( \eta_{n, \gamma} \) increases on \([0, \infty)\) due to Lemma 8 above and Lemma 6(a) in Hu and Lin [5]. Then mimicking the proof of Theorem 6, we can prove Theorems 7 and 8 by the monotone properties of functions \( \eta_\gamma \) and \( \eta_{n, \gamma} \), respectively. The proof is completed.

**Proof of Theorem 9.** By definition of \( \eta_\gamma \) in (16), we have
\[
e^{-sx} = 1 - (sx)^\gamma \eta_\gamma(sx) \leq 1 - (sx)^\gamma \eta_\gamma(sx)I[X \leq c] \quad \text{for all} \; s, x \geq 0.
\] (A) If \( \gamma \geq 1 \), the function \( \eta_\gamma \) decreases on \([0, \infty)\). Therefore, for \( s \geq 0 \),
\[
E(Y^\gamma \eta_\gamma(sX)I[X \leq c]) \geq \beta_\gamma c \eta_\gamma(sc).
\] Combining (22) and (23) yields that for \( s \geq 0 \),
\[
L(s) \leq 1 - s^\gamma \beta_\gamma c \eta_\gamma(sc) = 1 - \frac{\beta_\gamma c}{c^\gamma} + \frac{\beta_\gamma c}{c^\gamma} e^{-sc}.
\]
(B) If \( \gamma \in (0, 1) \), the function \( \eta_\gamma \) increases on \([0, a_\gamma] \). Therefore, for \( s \in [0, a_\gamma/c] \),
\[
E(Y^\gamma \eta_\gamma(sX)I[X \leq c]) \leq \beta_\gamma c \eta_\gamma(sc)
\] and
\[
L(s) \geq E(e^{-sX}I[X \leq c]) = E(Y^\gamma \eta_\gamma(sX)I[X \leq c]) = E(I[X \leq c]) - s^\gamma E(Y^\gamma \eta_\gamma(sX)I[X \leq c]).
\] Combining (24) and (25) yields the required result. The proof is completed.

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**References**