JOURNAL OF DIFFERENTIAL EQUATIONS 36, 117–138 (1980)

Ważewski's Principle for Retarded Functional Differential Equations*

KRZYSZTOF P. RYBAKOWSKI

Lefschetz Center for Dynamical Systems, Division of Applied Mathematics, Brown University, Providence, Rhode Island 02912

Received June 7, 1979

Ważewski Principle is an important tool in the study of the asymptotic behavior of solutions of ordinary differential equations. A direct extension of this principle to retarded functional differential equations (RFDEs) can be obtained by noticing that solutions of RFDEs generate processes on $C = C([-r, 0], \mathbb{R}^n)$ and by using the general version of Ważewski Principle for flows on topological spaces. The resulting method is of little use in applications, due to the infinite-dimensionality of the space C. This paper presents a "Razumikhin-type" extension of Ważewski's Principle, which is widely applicable to concrete examples. The main results are Corollaries 3.1 and 3.2. Also, an extension of the method to RFDEs with a merely continuous right-hand side is given, and a few examples illustrate the use of the method. Throughout the paper, a standard notation is used.

1. INTRODUCTION

Ważewski's Principle [9] is an important tool in the study of the asymptotic behavior of solutions of ordinary differential equations (ODEs). A direct extension of this principle to retarded functional differential equations (RFDEs) can be obtained by noticing that solutions of RFDEs generate processes on $C = C([-r, 0], \mathbb{R}^n)$ (cf. [3], p. 76), and by using the general version of Ważewski's Principle for flows on topological spaces (see e g. [2], p. 24). The resulting method is of little use in applications, due to the infinite-dimensionality of the space C.

This paper presents a "Razumikhin-type" extension of Ważewski's Principle, which is widely applicable to concrete examples.

The main results are Corollaries 3.1 and 3.2. Also, an extension of the method to RFDEs with a merely continuous right-hand side is given, and a few examples illustrate the use of the method. In particular, a result in [8] is generalized.

* This research was supported in part by the United States Army Research Office under DAAG27-76-G-0294 and in part by the National Science Foundation under ENG 77-12946.

0022-0396/80/040117-22$02.00/0

Throughout this paper, standard notation is used. Let us only remark that A^0, $\bar{A}$ and ∂A denote the interior, the closure and the boundary of A, respectively. (A being a subset of some topological space.) If x is a continuous mapping from an interval $[-r + t, t]$ into $\mathbb{R}^n$, then x_t denotes the element of $C = C([-r, 0], \mathbb{R}^n)$ defined as $x_t(\theta) = x(t + \theta)$, $\theta \in [-r, 0]$.

2. Ważewski's Principle for Systems of Curves

In this section, an abstract version of Ważewski's Principle is stated. It appeared earlier in a different form in [7]. However, the version presented here has an intuitively more appealing geometrical character, and it leads more directly to our main result in Section 3. We need two definitions:

Definition 2.1. Let X be a topological space, and let $A \subset X$. We consider A endowed with the relative topology of X.

(a) A is called a retract of X, if there exists a continuous mapping $f: X \to A$ such that $f(x) = x$ for all $x \in A$. f is called a retraction of X onto A.

(b) A is called a strong deformation retract of X if there is a continuous mapping $F: [0, 1] \times X \to X$ such that:

(i) $F(0, x) = x$ for all $x \in X$,

(ii) $F(1, x) \in A$ for all $x \in X$,

(iii) $F(s, x) = x$ for all $s \in [0, 1]$ and all $x \in A$.

F is called a strong deformation retraction of X onto A.

Remark. This definition of a strong deformation retract follows Conley [2]. Borsuk [1] uses a weaker definition.

Definition 2.2. Let Λ be a convergence space, let $\Omega \subset \mathbb{R} \times \Lambda$ be open in $\mathbb{R} \times \Lambda$, and let x be a mapping, associating with every $(\sigma, \lambda) \in \Omega$ a function $x(\sigma, \lambda): D_{\sigma,\lambda} \to \mathbb{R}^n$ where $D_{\sigma,\lambda}$ is an interval in $\mathbb{R}$ (closed, open, or half-open).

Assume 1, through 3.

[SC 1] $\sigma \in D_{\sigma,\lambda}$;

[SC 2] if $(\sigma_n, \lambda_n) \in \Omega$, $(\sigma, \lambda) \in \Omega$, t_n, $t \in \mathbb{R}$, $t \in D^0_{\sigma,\lambda}$, and if $\sigma_n \to \sigma$, $\lambda_n \to \lambda$, $t_n \to t$ as $n \to \infty$, then there is an n_0 such that for all $n \geqslant n_0$: $t_n \in D_{\sigma_n,\lambda_n}$;

[SC 3] If $(\sigma_n, \lambda_n) \in \Omega$, $(\sigma, \lambda) \in \Omega$, $\sigma_n \to \sigma$, $\lambda_n \to \lambda$ and $t_n \in D_{\sigma_n, \lambda_n}$, $t \in D_{\sigma, \lambda}$, $t_n \to t$, then $x(\sigma_n, \lambda_n)(t_n) \to x(\sigma, \lambda)(t)$.

If all the above conditions are satisfied, then (Λ, Ω, x) is called a system of curves in $\mathbb{R}^n$.

THEOREM 2.1. *Let (Λ, Ω, x) be a system of curves in $\mathbb{R}^n$. Let ω, W, Z be sets. Assume that conditions* 1 *through* 4, *below hold*:

(1) (α) *ω is open in $\mathbb{R} \times \mathbb{R}^n$, $W \subset \partial\omega$,*

(β) *$Z \subset \omega \cup W$, $B = \bar{Z} \cap (Z \cup W)$, $Z \cap W$ is a retract of W, $Z \cap W$ is not a retract of Z.*

(2) *There is a continuous map p: $B \to \Lambda$ such that for any $z = (\sigma, y) \in B$: $(\sigma, p(z)) \in \Omega$, and if also $z \in W$, then $x(\sigma, p(z))(\sigma) = y$.*

(3) *Let A be the set of all $z = (\sigma, y) \in Z \cap \omega$ such that there is a t, $t > \sigma$, $t \in D_{\sigma, p(z)}$ and such that $(t, x(\sigma, p(z))(t)) \notin \omega$.*

Assume that for every $z = (\sigma, y) \in A$ there is a $t(z)$, $t(z) > \sigma$, such that:

(α) *$t(z) \in D_{\sigma, p(z)}$ and for all t, $\sigma \leqslant t < t(z)$: $(t, x(\sigma, p(z))(t)) \in \omega$,*

(β) *$(t(z), x(\sigma, p(z))(t(z))) \in W$,*

(γ) *For any $\delta > 0$, there is a $t = t(\delta, z)$, $t(z) < t \leqslant t(z) + \delta$,* such that *$t \in D_{\sigma, p(z)}$ and $(t, x(\sigma, p(z))(t)) \notin \bar{\omega}$.*

(4) *For any $z = (\sigma, y) \in W \cap B$, and all $\delta > 0$, there is a $t = t(\delta, z)$ such that $\sigma < t \leqslant \sigma + \delta$, $t \in D_{\sigma, p(z)}$ and $(t, x(\sigma, p(z))(t)) \notin \bar{\omega}$.*

If all the above assumptions are satisfied, then there is a $z_0 = (\sigma_0, y_0) \in Z \cap \omega$ such that for every $t > \sigma_0$, $t \in D_{\sigma_0, p(z_0)}$: $(t, x(\sigma_0, p(z_0))(t)) \in \omega$.

Proof. To prove the theorem, define the map g on $A \cup W$ as follows ($z = (\sigma, y)$):

$$g(z) = \begin{cases} (t(z), x(\sigma, p(z))(t(z))), & \text{if } z \in A \\ z, & \text{if } z \in W. \end{cases}$$

Then g is well-defined, because $t(z) \in D_{\sigma, p(z)}$ for $z \in A$.

Using Definition 2.2 and assumptions (3) and (4), it is easily proved that $g[A \cup W] \subset W$ and g is continuous. Hence g is a retraction of $A \cup W$ onto W. If the theorem were not true, then $Z \cap \omega = A$ from the definition of A, hence W would be a retract of $Z \cup W$. But since $Z \cap W$ is a retract of W by assumption (1), $Z \cap W$ would also be a retract of Z, which contradicts assumption (1).

This completes the proof of Theorem 2.1.

Theorem 2.2 is very similar to Theorem 2.1 and the proof repeats that of Theorem 2.1 with obvious modifications. Theorem 2.2 is of grater use than-

Theorem 2.1 in applications to autonomous RFDEs and this is why it is formulated separately.

THEOREM 2.2. *Let* (Λ, Ω, x) *be a system of curves in* $\mathbb{R}^n$. *Let* ω, W, Z *be sets. Assume that conditions* (1) *through* (4) *below hold*;

(1) (α) ω *is open in* $\mathbb{R}^n$, $W \subset \partial\omega$,

(β) $Z \subset \omega \cup W$, $B := \bar{Z} \cap (Z \cup W)$, $Z \cap W$ *is a retract of* W, *but* $Z \cap W$ *is not a retract of* Z.

(2) *There is a continuous map* $p\colon B \to \Lambda$ *such that for any* $z \in B$:$(0, p(z)) \in \Omega$, *and if also* $z \in W$, *then* $x(0, p(z))(0) = z$.

(3) *Let* A *be the set of all* $z \in Z \cap \omega$ *such that there is a* t, $t > 0$, $t \in D_{0,p(z)}$ *and such that* $x(0, p(z))(t) \notin \omega$. *Assume that for every* $z \in A$ *there is a* $t(z)$, $t(z) > 0$, *such that*:

(α) $t(z) \in D_{0,p(z)}$, *and for all* t, $0 \leqslant t < t(z)$: $x(0, p(z))(t) \in \omega$,

(β) $x(0, p(z))(t(z)) \in W$,

(γ) *For all* $\delta > 0$ *there is a* t, $t(z) < t \leqslant t(z) + \delta$, *such that* $t \in D_{0,p(z)}$ *and* $x(0, p(z))(t) \notin \bar{\omega}$.

(4) *For all* $z \in W \cap B$ *and all* $\delta > 0$ *there is a* $t = t(\delta, z)$ *such that* $0 < t \leqslant \delta$, $t \in D_{0,p(z)}$ *and* $x(0, p(z))(t) \notin \bar{\omega}$.

If all the above assumptions are satisfied, there exists a $z_0 \in Z \cap \omega$ *such that for all* $t > 0$, $t \in D_{0,p(z_0)}$: $x(0, p(z_0))(t) \in \omega$.

Remarks. (1) In this work, we shall consider the following application of Theorems 2.1 and 2.2: Let $\Lambda = C = ([-r, 0], \mathbb{R}^n)$ and let Ω be open in $\mathbb{R} \times C$. Let $F\colon \Omega \to \mathbb{R}^n$ be a continuous mapping.

Consider the equation:

$$\dot{x}(t) = F(t, x_t). \tag{2.1}$$

Assume that through each $(\sigma, \phi) \in \Omega$ there is a unique solution $x(\sigma, \phi)$ of Eq.(2.1) defined on a maximal interval $[\sigma, a)$, $\sigma < a \leqslant \infty$. Let $D_{\sigma,\phi} = [\sigma, a)$. Then (Λ, Ω, x) is a system of curves in $\mathbb{R}^n$, this being a consequence of the continuous dependence of solutions of Eq. (2.1) on initial data.

Henceforth, except for Section 4, we shall tacitly assume that all systems of curves which will be considered are generated in the way just described.

(2) If assumption (2) is replaced by the stronger assumption (2′):

(2′) There is a continuous mapping $p\colon \omega \cup W \to \Lambda$ such that for any $z \in (\sigma, y) \in \omega \cup W$: $(\sigma, p(z)) \in \Omega$ and $x(\sigma, p(z))(\sigma) = y$,

and if, at the same time, (1β) is replaced by the weaker assumption (1β'):

(1β') W is not a strong deformation retract of $\omega \cup W$, then the conclusion of Theorem 2.1 remains true. (In the definition of A, "Z" is replaced by "$\omega \cup W$", and in assumption (4), $W \cap B$ is replaced by "W".) If not, define $G\colon [0, 1] \times (\omega \cup W) \to \omega \cup W$ to be

$$G(s, \sigma, y) = \begin{cases} (\sigma + s \cdot (t(z) - \sigma), & x(\sigma, p(z))(\sigma + s \cdot (t(z) - \sigma))) \\ & \text{if} \quad z = (\sigma, y) \in \omega = A \\ (\sigma, y) & \text{if} \quad z = (\sigma, y) \in W \end{cases}$$

It is easily proved that G is a strong deformation retraction, which contradicts assumption ($1\beta'$) and proves our statement. An analogous remark applies to Theorem 2.2.

(3) For autonomous RFDEs, there is a simple relation between Theorem 2.2 and the direct extension of Ważewski's Principle mentioned in the introduction. Let us formulate the latter as Theorem DE:

THEOREM DE (cf. Conley [2], p. 24). *Let Ω be open in C, and let $F\colon \Omega \to \mathbb{R}^n$ be continuous. Consider Eq.* (2.2):

$$\dot{x} = F(x_t). \tag{2.2}$$

Assume uniqueness of solutions of Eq. (2.2). *Let ω^* be an open subset of Ω. Let W^* be a subset of $\partial\omega^* \cap \Omega$.*

Define A^* to be the set of all $\phi \in \omega^*$ such that $x_t(0, \phi) \notin \omega^*$ for some $t > 0$, where $x_t(0, \phi) \in C$ is the solution of Eq. (2.2) *through* $(0, \phi)$. *Assume that* (3) *and* (4) *below hold*:

(3) *For every $\phi \in A^*$ there is a $t(\phi) > 0$ such that*:

(α) *the solution through $(0, \phi)$ is defined at $t(\phi)$ and for all $0 \leqslant t < t(\phi)$*:

$$x_t(0, \phi) \in \omega^*.$$

(β) $x_{t(\phi)}(0, \phi) \in W^*$.

(4) *For all $\phi \in W^*$ and all $\delta > 0$ there is a $t = t(\delta, \phi)$, $0 < t \leqslant \delta$, such that the solution through $(0, \phi)$ is defined at t and $x_t(0, \phi) \notin \bar{\omega}^*$.*

Then W^* is a strong deformation retract of $A^* \cup W^*$.

Now suppose the system (A, Ω, x) is generated by Eq. (2.2) and that assumptions (1) and (2) of Theorem 2.2 hold. Let ω^* be the set of all $\phi \in C$ such that $\phi(\theta) \in \omega$ for $\theta \in [-r, 0]$. It is easily seen that ω^* is open in C. Suppose that ω^* satisfies the following condition (CND):

(CND) For every $\phi \in C$ such that $\phi(\theta) \in \bar{\omega}$ for $\theta \in [-r, 0]$, there is a sequence $\{\phi_\nu\}$, $\phi_\nu \in \omega^*$, such that $\phi_\nu \to \phi$ as $\nu \to \infty$ (in the topology of C).

Now assume that $\omega^* \subset \Omega$ and let W^* be a subset of $\partial\omega^* \cap \Omega$. Let A^* be defined as in Theorem DE. If $\phi(0) \in W$ for every $\phi \in W^*$, and if $p(z) \in W^*$

for every $z \in W \cap B$, then a simple argument shows that assumptions (3) and (4) of Theorem DE imply assumptions (3) and (4) of Theorem 2.2.

Condition (CND) is satisfied if ω has the following property (PR):

(PR) For every $x \in \bar{\omega}$ and every $\epsilon > 0$ there is an open set U_x containing x, with diameter $< \epsilon$ and such that $U_x \cap \omega$ is connected.

This is proved by using simple topological arguments.

Property (PR) cannot, in general, be dispensed with: e.g., if $\omega = [(0, 1) \times (0, 1)] \backslash [\{1/2\} \times [0, 1/2]\}$.

3. Polyfacial Sets and Ważewski's Principle for RFDEs

In this section, our main results are stated and proved. They rest on the following concept.

Definition 3.1. Let l^i, m^j, $i = 1,..., p$, $j = 1,..., q$ be real-valued C^1-functions defined on $\mathbb{R} \times \mathbb{R}^n$. The set ω: $\omega = \{(t, y) \in \mathbb{R} \times \mathbb{R}^n \mid l^i(t, y) < 0$, $m^j(t, y) < 0$, for all $i, j\}$ is called a *(time-dependent) polyfacial set*. If l^i, m^j are C^1-functions defined on $\mathbb{R}^n$, then

$$\omega = \{y \in \mathbb{R}^n \mid l^i(y) < 0,\ m^j(y) < 0, \text{ for all } i, j\}$$

is called a *time-independent polyfacial set*.

A *(time-dependent) polyfacial* set will be called *regular with respect to Eq.* (2.1), if (α), (β), (γ) below hold:

(α) If $(t, \phi) \in \mathbb{R} \times C$ and if $(t + \theta, \phi(\theta)) \in \omega$ for all $\theta \in [-r, 0)$, then $(t, \phi) \in \Omega$.

(β) For all $i = 1,..., p$, all $(t, y) \in \partial\omega$ for which $l^i(t, y) = 0$, and all $\phi \in C$ for which $\phi(0) = y$ and $(t + \theta, \phi(\theta)) \in \omega$ for $\theta \in [-r, 0)$, it follows that $(t, \phi) \in \Omega$ and:

$$\sum_{r=1}^{n} \frac{\partial l^i}{\partial y_r}(t, y) \cdot F_r(t, \phi) + \frac{\partial l^i}{\partial t}(t, y) > 0.$$

(γ) For all $j = 1,..., q$, all $(t, y) \in \partial\omega$ for which $m^j(t, y) = 0$, and all $\phi \in C$ for which $\phi(0) = y$ and $(t + \theta, \phi(\theta)) \in \omega$ for $\theta \in [-r, 0)$, it follows that $(t, \phi) \in \Omega$ and:

$$\sum_{r=1}^{n} \frac{\partial m^j}{\partial y_r}(t, y) F_r(t, \phi) + \frac{\partial m^j}{\partial t}(t, y) < 0.$$

If l^i, m^j do not depend on t, ω is of the form $\omega = \mathbb{R} \times \tilde{\omega}$, where $\tilde{\omega}$ is a time-independent polyfacial set.

If ω is regular with respect to Eq. (2.1), $\tilde{\omega}$ is also called *regular with respect to Eq.* (2.1).

Remarks. (1) Uniqueness of solutions of Eq. (2.1) is not used in Definition 3.1 and the concept defined there makes sense for arbitrary RFDEs.

(2) Onuchic's definition of a regular polyfacial set with respect to Eq. (2.1) (see [8]) requires that the inequalities in (β) and (γ) be satisfied for *any* $\phi \in C$ such that $\phi(0) = y$ (and (t, y) as above). This very stringent condition makes his results applicable only to very special types of RFDEs.

(3) The concepts defined in Definition 3.1 are related to, and were, in fact, inspired by the concept of a Razumikhin-type Liapunov function:

A Razumikhin-type Liapunov function with respect to Eq. (2.1) is a continuous function $V: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that $\limsup_{h\to 0^+}(1/h)[V(t + h, x(t, \phi)(t + h)) - V(t, \phi(0))] \leqslant 0$, for every $(t, \phi) \in \Omega$ such that $V(t + \theta, \phi(\theta)) \leqslant V(t, \phi(0))$ for $\theta \in [-r, 0]$.

Another related concept is that of a guiding function. See [3], p. 139, for a definition.

(4) In [6], the author considers the scalar delay equation $\dot{x}(t) = f(t, x(t), x(t - h),..., x(t - mh))$, where $h > 0$ and m is an integer >0. She imposes conditions on f implicitly ensuring that for some $R > 0$, the interval $[-R, R]$ is a regular polyfacial set with respect to this equation (in our sense). $[-R, R]$ has pure "exit" behavior on the boundary, i e. no functions of type m^j are involved.

THEOREM 3.1. *Let ω be a (time-dependent) polyfacial set, regular with respect to Eq.* (2.1) *and let W be defined as follows*:

$$W := \{(t, y) \in \partial\omega \mid m^j(t, y) < 0, \text{ for all } j = 1,..., q\}.$$

Moreover, let Z be a subset of $W \cup \omega$, and let $p: B = \bar{Z} \cap (Z \cup W) \to C$ *be continuous such that if $z = (t, y) \in B$, then $(t, p(z)) \in \Omega$, and*:

(1) *If $z = (t, y) \in A$ (A as in Theorem* 2.1), *then $(t + \theta, p(z)(\theta)) \in \omega$ for $\theta \in [-r, 0]$.*

(2) *If $z = (t, y) \in W \cap B$, then $p(z)(0) = y$ and $(t + \theta, p(z)(\theta)) \in \omega$ for $\theta \in [-r, 0)$.*

Then assumptions (3) *and* (4) *of Theorem* 2.1 *are satisfied.*

Proof. Let $z = (\sigma, y) \in A$, and let $t(z)$ be the smallest of all $t \geqslant \sigma$ such that $t \in D_{\sigma, p(z)}$ and $(t, x(\sigma, p(z))(t)) \notin \omega$. Since $(\sigma, x(\sigma, p(z))(\sigma)) = (\sigma, p(z)(0)) \in \omega$, it follows that $\sigma < t(z) < \infty$. Obviously, $(t(z), x(\sigma, p(z))(t(z))) \in \partial\omega$ and for

$\sigma \leqslant t < t(z)$: $(t, x(\sigma, p(z))(t)) \in \omega$, hence (3α) holds. Let $\phi = x_{t(z)}(\sigma, p(z)) \in C$. Then $(t(z), \phi) \in \Omega$, $(t(z), \phi(0)) = (t(z), x(\sigma, p(z))(t(z))) \in \partial\omega$, and

$$(t(z) + \theta, \phi(\theta)) \in \omega, \qquad \text{for} \quad \theta \in [-r, 0).$$

We shall show that $(t(z), \phi(0)) \in W$. Suppose on the contrary that $(t(z), \phi(0)) \notin W$. Since $(t(z), \phi(0)) \in \partial\omega$ it follows that for some j_0, $m^{j_0}(t(z), \phi(0)) = 0$. Hence the inequality in (γ) of Definition 3.1 is satisfied. Since $x(\sigma, p(z))(t)$ is differentiable in t for $t > \sigma$, this inequality becomes

$$\frac{d}{dt} m^{j_0}(t, x(\sigma, p(z))(t)) \bigg|_{t=t(z)} < 0,$$

hence, for some $\delta > 0$ and all $0 < h < \delta$:

$$\begin{aligned} &m^{j_0}(t(z) - h, x(\sigma, p(z))(t(z) - h)) \\ &\qquad > m^{j_0}(t(z), x(\sigma, p(z))(t(z))) = m^{j_0}(t(z), \phi(0)) = 0. \end{aligned}$$

Hence $(t(z) - h, x(\sigma, p(z))(t(z) - h)) \notin \bar{\omega}$, which is a contradiction to (3α) (which we have already proved).

Hence, indeed, $(t(z), \phi(0)) \in W$ and, therefore, (3β) in Theorem 2.1 is satisfied. It follows that $l^{i_0}(t(z), \phi(0)) = 0$ for some i_0. Applying (β) of Definition 3.1 we see that

$$\frac{d}{dt} l^{i_0}(t, x(\sigma, p(z))(t)) \bigg|_{t=t(z)} > 0.$$

Hence, for some $\delta > 0$ and all $0 < h < \delta$

$$\begin{aligned} &l^{i_0}(t(z) + h, x(\sigma, p(z))(t(z) + h)) \\ &\qquad > l^{i_0}(t(z), x(\sigma, p(z))(t(z))) = l^{i_0}(t(z), \phi(0)) = 0. \end{aligned}$$

Hence, $(t(z) + h, x(\sigma, p(z))(t(z) + h))) \notin \bar{\omega}$, which is even stronger than (3γ) of Theorem 2.1. Hence (3γ) is, indeed, satisfied.

Now assume $z = (\sigma, y) \in W \cap B$. Hence $l^{i_0}(\sigma, y) = 0$ for some i_0. Let $\phi = p(z)$, Then $(t + \theta, \phi(\theta)) \in \omega$, for all $-r \leqslant \theta < 0$. Hence

$$\frac{d}{dt} l^{i_0}(t, x(\sigma, p(z))(t))\bigg|_{t=\sigma} > 0$$

where "d/dt" denotes the right-hand derivative. But this implies the existence of $\delta > 0$ such that for all $0 < h < \delta$

$$\begin{aligned} &l^{i_0}(\sigma + h, x(\sigma, p(z))(\sigma + h)) > l^{i_0}(\sigma, x(\sigma, p(z))(\sigma)) \\ &\qquad = l^{i_0}(\sigma, \phi(0)) = l^{i_0}(\sigma, y) = 0. \end{aligned}$$

Hence $(\sigma + h, x(\sigma, p(z))(\sigma + h)) \notin \bar{\omega}$, for $0 < h < \delta$, which, a fortiori, implies (4) of Theorem 2.1.

The theorem is proved.

COROLLARY 3.1. *If all assumptions of Theorem* 3.1 *are satisfied, and if, in addition,* $Z \cap W$ *is a retract of* W, *but* $Z \cap W$ *is not a retract of* W, *then there exists a* $z_0 = (\sigma_0, y_0) \in Z \cap \omega$ *such that* $(t, x(\sigma_0, p(z_0)))(t)) \in \omega$ *for every* $t \geqslant \sigma_0$, $t \in D_{\sigma_0, p(z_0)}$.

Proof. Combine Theorems 2.1 and 3.1.

The following theorem is a reformulation of Theorem 3.1 for time-independent polyfacial sets. The proof follows the same lines as that of Theorem 3.1 and is, therefore, omitted.

THEOREM 3.2. *Let* ω *be a time-independent polyfacial set in* $\mathbb{R}^n$, *regular with respect to Eq.* (2.1), *and let* W *be defined as follows*:

$$W = \{ y \in \partial\omega \mid m^j(y) < 0, \textit{ for all } j = 1,\dots, q\}.$$

Moreover, let Z *be a subset of* $\omega \cup W$, *and let* $p\colon B = \bar{Z} \cap (Z \cup W) \to C$ *be continuous and such that if* $z \in B$, *then* $(0, p(z)) \in \Omega$, *and*:

(1) *If* $z \in A$ (A *as in Theorem* 2.2), *then* $p(z)(\theta) \in \omega$ *for* $\theta \in [-r, 0]$.

(2) *If* $z \in W \cap B$, *then* $p(z)(0) = z$ *and* $p(z)(\theta) \in \omega$ *for* $\theta \in [-r, 0)$.

Then assumptions (3) *and* (4) *of Theorem* 2.2 *are satisfied.*

COROLLARY 3.2. *If all assumptions of Theorem* 3.2 *are satisfied, and if, in addition,* $Z \cap W$ *is a retract of* W, *but* $Z \cap W$ *is not a retract of* Z, *then there exists a* $z_0 \in Z \cap \omega_0$ *such that* $x(0, p(z_0))(t) \in \omega$ *for every* $t \geqslant 0$, $t \in D_{0,p(z_0)}$.

Remarks. By using Remark 2 after Theorem 2.2 a modification of Corollaries 3.1 and 3.2 can be obtained. Obvious details are omitted.

EXAMPLE 3.1. Consider the scalar equation (3.1):

$$\dot{x} = -ax(t) - bx(t - r), \qquad \text{where} \quad a, b \in \mathbb{R}, \quad a \neq 0, \quad r \geqslant 0. \tag{3.1}$$

Let $\omega = (-\alpha, \alpha)$, $\alpha > 0$. Then any easy computation shows that ω is a regular time-independent polyfacial set with respect to Eq. (3.1) if and only if $| b | \leqslant | a |$.

If $| b | < | a |$, then it is intuitively clear that ω is regular not only with respect to Eq. (3.1) but also with respect to all sufficiently small perturbations of Eq. (3.1). However, if $| a | = | b |$, then even the slightest perturbation $a, b \to a', b'$ can yield $| a' | < | b' |$ and then ω is no longer a regular polyfacial set with respect to the perturbed equation. In fact, there is *no* regular polyfacial set with respect to the perturbed equation which contains 0. This motivates the following concept:

DEFINITION 3.2. Let $\Omega \subset \mathbb{R} \times C$ be open, and $F: \Omega \to \mathbb{R}^n$ be continuous. Suppose ω is a (time-dependent) regular polyfacial set with respect to Eq. (3.2):

$$\dot{x} = F(t, x_t). \tag{3.2}$$

Then ω is called *stable at* F if there is an $\epsilon > 0$ such that ω is regular with respect to Eq. (3.3):

$$\dot{x} = F'(t, x_t), \tag{3.3}$$

whenever $F': \Omega \to \mathbb{R}^n$ is continuous and $\sum_{r=1}^{n} | F_r(t, \phi) - F'_r(t, \phi)| < \epsilon$, for all $(t, \phi) \in \Omega$ for which $(t, \phi(0)) \in \partial\omega$ and $(t + \theta, \phi(\theta)) \in \omega$, $\theta \in [-r, 0)$.

The next proposition establishes a simple characterization of stability of ω at F.

PROPOSITION 3.1. *Let ω be a regular polyfacial set with respect to Eq.* (3.2). *Then ω is stable at F if and only if there is $\epsilon > 0$ such that for all i, j, and all $(t, \phi) \in \mathbb{R} \times C$ for which $(t, \phi(0)) \in \partial\omega$, and $(t + \theta, \phi(\theta)) \in \omega$, $\theta \in [-r, 0)$, the following conditions hold*:

$$(1) \quad \epsilon \sum_{r=1}^{n} \left| \frac{\partial l^i}{\partial y_r}(t, \phi(0)) \right| < \left| \sum_{r=1}^{n} \frac{\partial l^i}{\partial y_r}(t, \phi(0)) \cdot F_r(t, \phi) + \frac{\partial l^i}{\partial t}(t, \phi(0)) \right|, \quad \textit{whenever} \quad l^i(t, \phi(0)) = 0.$$

$$(2) \quad \epsilon \sum_{r=1}^{m} \left| \frac{\partial m^j}{\partial y_r}(t, \phi(0)) \right| < \left| \sum_{r=1}^{n} \frac{\partial m^j}{\partial y_r}(t, \phi(0)) \cdot F_r(t, \phi) + \frac{\partial m^j}{\partial t}(t, \phi(0)) \right|, \quad \textit{whenever} \quad m^j(t, \phi(0)) = 0.$$

The proof of Proposition 3.1 is a simple computation, which is, therefore, omitted.

If $F(\phi) = -a\phi(0) - b\phi(-r)$, i.e. in case of Eq. (3.1), Proposition 3.1 easily implies that $\mathbb{R} \times (-\alpha, \alpha)$ is, indeed, stable at F if $|b| < |a|$, and not stable at F if $|b| = |a|$.

In Examples 3.2 through 3.5 below, uniqueness of solutions is assumed.

EXAMPLE 3.2. Consider Eq. (3.4):

$$\begin{aligned} \dot{x}_1 &= -ax_1(t) - bx_1(t - r) + f_1(t, x_{1t}, x_{2t}), \\ x_2 &= -cx_2(t) - dx_2(t - r) + f_2(t, x_{1t}, x_{2t}), \end{aligned} \tag{3.4}$$

where $a, c \neq 0$, $C = C([-r, 0], \mathbb{R})$, and $f = (f_1, f_2): \mathbb{R} \times C \times C \to \mathbb{R}^2$ is continuous and such that for some $M, N > 0$:

$$|f_1(t, \phi, \psi)| \leqslant (|a| - |b|) M \qquad \text{for all } (t, \phi, \psi) \in \mathbb{R} \times C \times C$$

for which

$$|\phi(0)| = M, \quad |\psi(0)| \leqslant N, \quad |\phi(\theta)| < M, \quad |\psi(\theta)| < N, \quad \theta \in [-r, 0),$$

and

$$|f_2(t, \phi, \psi)| \leqslant (|c| - |d|) N \qquad \text{for all } (t, \phi, \psi) \in \mathbb{R} \times C \times C$$

for which

$$|\psi(0)| = N, \quad |\phi(0)| \leqslant M, \quad |\psi(\theta)| < N, \quad |\phi(\theta)| < M, \quad \theta \in [-r, 0).$$

Then $\omega = (-M, M) \times (-N, N)$ is a regular polyfacial set with respect to Eq. (3.4). Hence, there is a solution $x(t) = (x_1(t), x_2(t))$ of Eq. (3.4) such that $x(t)$ is defined for $t \geqslant 0$ and $x(t) \in \omega$ for $t \geqslant 0$.

Example 3.2 is easily generalized to arbitrary dimensions.

EXAMPLE 3.3. If $y = \binom{y_1}{y_2} \in \mathbb{R}^2$ and $k \in \mathbb{R}$, define the "kth power of y" to be

$$y^k = \begin{pmatrix} y_1{}^k \\ y_2{}^k \end{pmatrix}.$$

Consider Eq. (3.5):

$$\begin{aligned} \dot{x}(t) = Ax(t) + Bx(t-r) + Px^2(t) + Qx^2(t-r) \\ + Cx^3(t) + Dx^3(t-r) + E(t) = F(t, x_t) \end{aligned} \tag{3.5}$$

where $x = \binom{x_1}{x_2} \in \mathbb{R}$, A, B, P, Q, C, D are 2×2 matrices,

$$C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}, \qquad D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \qquad |c_1| > |d_1|, \qquad |c_2| > |d_2|,$$

$E: \mathbb{R} \to \mathbb{R}^2$ is continuous.

It is a matter of trivial computation to see that for $M > 0$ large enough, the square $\omega = (-M, M) \times (-M, M)$ is a regular polyfacial set with respect to Eq. (3.5), and hence there is a solution $x(t)$ of Eq. (3.5) defined for $t \geqslant 0$ such that $x(t) \in \omega$ for $t \geqslant 0$.

EXAMPLE 3.4. The following example is a simple illustration of the usefulness of time-dependent polyfacial sets.

Let $\alpha: \mathbb{R} \to \mathbb{R}$ and $\beta: \mathbb{R} \to \mathbb{R}$ be C^1-functions such that $\alpha(t) \to 0$, $\beta(t) \to 0$ as $t \to + y$. Let

$$\begin{aligned} l^1(t, y) &= y - \alpha(t) \\ l^2(t, y) &= \beta(t) - y \\ m(t, y) &= -t \end{aligned}$$

Assume that $\beta(t) < \alpha(t)$ for $t \geqslant 0$. Let

$$\omega = \{(t, y) \in \mathbb{R} \times \mathbb{R} \mid l^1(t, y) < 0, l^2(t, y) < 0, m(t, y) < 0\}.$$

Consider Eq. (3.6)

$$\dot{x} = F(t, x_t), \tag{3.6}$$

where $F: \mathbb{R} \times C([-r, 0], \mathbb{R}) \to \mathbb{R}$ is continuous and such that:

$$\alpha'(t) < F(t, \phi) \qquad \text{for all } (t, \phi) \in \mathbb{R} \times C$$

for which $t > r$, $\phi(0) = \alpha(t)$, $\beta(t + \theta) < \phi(\theta) < \alpha(t + \theta)$, $\theta \in [-r, 0)$, and

$$F(t, \phi) < \beta'(t) \qquad \text{for all } (t, \phi) \in \mathbb{R} \times C$$

for which

$$t > r, \quad \phi(0) = \beta(t), \quad \alpha(t + \theta) > \phi(\theta) > \beta(t + \theta), \quad \theta \in [-r, 0).$$

Then ω is a regular polyfacial set with respect to Eq. (3.6), and there is a solution $x(t)$ of Eq. (3.6) such that $x(t)$ is defined for all $t \geqslant 0$, and $\beta(t) < x(t) < \alpha(t)$ for $t \geqslant 0$, hence $x(t) \to 0$ as $t \to \infty$.

The following example generalizes Theorem 3 in [8].

EXAMPLE 3.5. Let $C = C([-r, 0], \mathbb{R}^n)$ and let $F: \mathbb{R} \times C \times C \to R^n$ be a continuous mapping. Consider the second-order RFDE

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= F(t, x_t, y_t) \end{aligned} \tag{3.7}$$

Assume that

$$\phi(0) \cdot F(t, \phi, \psi) + \psi(0) \cdot \psi(0) > 0 \tag{3.8}$$

is satisfied for all $t \geqslant T + r$, and all $\phi, \psi \in C$ for which:

$$\phi(0) \cdot \psi(0) > 0$$

and

$$\phi(\theta) \cdot \psi(\theta) < \phi(0) \cdot \psi(0) \qquad \text{for} \quad \theta \in [-r, 0).$$

(The dot denotes the scalar product in $\mathbb{R}^n$.) Assume also that uniqueness holds for solutions of Eq. (3.7).

Let

$$z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Then for every $t_0 > T + r$ there is a family of initial values $(\tilde{\phi}, \tilde{\psi}) \in C \times C$, depending on at least n parameters, such that $x(t_0, \tilde{\phi}, \tilde{\psi})(t) \cdot x(t_0, \tilde{\phi}, \tilde{\psi})(t)$ is nonincreasing for $t \geqslant t_0$, as long as it is defined.

Proof. Let $b > 0$, and $T + r < t_1 < t_0$.
Let

$$\left.\begin{aligned} l_b(t, x, y) &= x \cdot y - b \\ m_b(t, x, y) &= t_1 - t \end{aligned}\right\} \qquad \text{for } t \in \mathbb{R}, (x, y) \in \mathbb{R}^n \times \mathbb{R}^n,$$

$$\omega_b := \{(t, x, y) \mid m_b(t, x, y) < 0, l_b(t, x, y) < 0\},$$

$$\begin{aligned} W_b &:= \{(t, x, y) \in \partial\omega_b \mid m_b(t, x, y) < 0\} \\ &= \{(t, x, y) \mid t > t_1, x \cdot y = b\}. \end{aligned}$$

Then ω_b is a polyfacial set and assumption (3.8) just implies that ω_b is regular with respect to Eq. (3.7). Let Z_b be the line segment connecting two points (t_0, ξ_1, η_1) and (t_0, ξ_2, η_2) located in two distinct components of W_b, with $(t_0, 0, 0) \notin Z_b$.

Now a simple geometrical consideration shows that a continuous mapping $p: Z \to C$ can be constructed which satisfies all assumptions of Theorem 3.1. Hence, there exists a $(x_b, y_b) \in Z_b \cap \omega$ such that the solution $z(t_0, p(t_0, x_b, y_b))(t)$ remains in ω_b for $t \geqslant t_0$, as long as it is defined. Now take a decreasing sequence $b_m \to 0$ and use a compactness argument to obtain a (x_0, y_0) such that the solution $z(t_0, p(t_0, x_0, y_0))(t)$ satisfies $x(t_0, p(t_0, x_0, y_0))(t) \cdot y(t_0, p(t_0, x_0, y_0)) \times (t) \leqslant 0$ for all $t \geqslant 0$, as long as it is defined.

Since for $b_1 \neq \bar{b}_1$, Z_{b_1} can be chosen to be disjoint from $Z_{\bar{b}_1}$, it follows that $\{(x_0, y_0)\}$ depends on at least n parameters and the example is proved.

Remark. Onuchic's Theorem 3 in [8] yields a somewhat stronger conclusion. However, his assumptions are much stronger than those of Example 3.5. Onuchic assumes that (3.8) be satisfied for all $\phi, \psi \in C$, $\psi(0) \neq 0$, $t \geqslant T$;

E.g. Let $C = C([-r, 0], \mathbb{R})$. Let $F(t, \phi, \psi) = \phi(0). (\phi(0) \cdot \psi(0) - \phi(-r) \cdot \psi(-r))$, then the assumption (3.8) is satisfied for all $\phi, \psi \in C$ such that $\phi(0) \cdot \psi(0) > 0$ and

$$\phi(-r)\, \psi(-r) < \phi(0) \cdot \psi(0).$$

Even the stronger inequality

$$\phi(0) \cdot F(t, \phi, \psi) > 0$$

holds for all such ϕ, ψ.

However, Onuchic's assumption is not satisfied, since $\phi(0) \cdot F(t, \phi, \psi) + \psi(0) \cdot \psi(0)$ can be made negative by proper choice of $\phi, \psi \in C$, $\psi(0) \neq 0$.

4. Extension of Some Results to the Case in Which Uniqueness of Solutions Does not Hold

In the following two sections, we shall extend Theorems 3.1 and 3.2 to the case in which F is continuous, but the corresponding Eq. (4.1)

$$\dot{x} = F(t, x_t) \tag{4.1}$$

does not necessarily satisfy the uniqueness property of its solutions.

To this end, we need the following approximation theorem:

Theorem 4.1. *Let Ω be open in $\mathbb{R} \times C$, $F: \Omega \to \mathbb{R}^n$ be continuous. Suppose ω is a regular polyfacial set with respect to Eq.* (4.1).

Then there is an open subset $\tilde{\Omega}$ of Ω, and a sequence of continuous mappings $\tilde{F}_\nu : \tilde{\Omega} \to \mathbb{R}^n$ such that uniqueness holds for solutions of equation $\dot{x} = F_\nu(t, x_t)$ for all ν, $\tilde{F}_\nu \to F$, as $\nu \to \infty$, uniformly on $\tilde{\Omega}$, and ω is a regular polyfacial set with respect to Eq. (4.2) *and Eq.* (4.3$_\nu$) *for all ν:*

$$\dot{x} = \tilde{F}(t, x_t), \qquad \text{where} \quad \tilde{F} = \tilde{F} \mid \tilde{\Omega}, \tag{4.2}$$

$$\dot{x} = \tilde{F}_\nu(t, x_t). \tag{4.3$_\nu$}$$

The proof of Theorem 4.1 is given at the end of this section.

Let us next state a lemma:

Lemma 4.1. *Let Ω be open in $\mathbb{R} \times C$, $F_\nu : \Omega \to \mathbb{R}^n$ and $F: \Omega \to \mathbb{R}^n$ be continuous and that $F_\nu \to F$ as $\nu \to \infty$, uniformly on Ω.*

Consider Eq. (4.4) *and Eq.* (4.5$_\nu$):

$$\dot{x} = F(t, x_t), \tag{4.4}$$

$$\dot{x} = F_\nu(t, x_t). \tag{4.5$_\nu$}$$

Let $x^\nu(t)$ be a noncontinuable solution of Eq. (4.5$_\nu$), *through (σ^ν, ϕ^ν), where $(\sigma^\nu, \phi^\nu) \in \Omega$. Suppose $(\sigma^\nu, \phi^\nu) \to (\sigma^0, \phi^0) \in \Omega$ as $\nu \to \infty$. Then there exists a noncontinuable solution $x^0(t)$ of Eq.* (4.4) *through (σ^0, ϕ^0) and a subsequence (x^{ν_k}) of (x^ν) such that whenever b is such that x^0 is defined on $[\sigma^0 - r, b]$, then for all $\epsilon > 0$ there is a $k_1(\epsilon)$ such that for $k \geqslant k_1(\epsilon)$, x^{ν_k} is defined on $[\sigma^0 - r + \epsilon, b]$ and $x^{\nu_k} \to x_0$ uniformly on $[\sigma^0 - r + \epsilon, b]$.*

The proof of the lemma uses Zorn's lemma and standard arguments from the theory of RFDEs, and is omitted. (Cf. Hale [3]).

THEOREM 4.2. *Let Ω be open in $\mathbb{R} \times C$, $F: \Omega \to \mathbb{R}^n$ be continuous and ω be a regular polyfacial set with respect to Eq.* (4.1).

Let W be defined as in Theorem 3.1, *suppose Z is a compact subset of $\omega \cup W$, and there is a mapping p, $p: B = \bar{Z} \cap (Z \cup W) \to C$, satisfying all assumptions of Theorem* 3.1, *with "A" replaced by "$Z \cap \omega$" in the statement of that theorem. Note that $B = Z$ in this case. Finally, assume that one of the following cases holds:*

Either:

1° *$Z \cap W$ is a retract of W and $Z \cap W$ is not a retract of Z,*

Or:

2° *p satisfies $p(z)(0) = y$ for $z \in \omega \cup W$, $Z = \omega \cup W$, and W is not a strong deformation retract of ω.*

Then there is a $z_0 = (\sigma_0, y_0) \in Z \cap \omega$ and a noncontinuable solution $x^0(t)$ of Eq. (4.1) *through $(\sigma_0, p(z_0))$ such that $(t, x^0(t)) \in \omega$ for all $t \geqslant \sigma_0$, as long as t is in the domain of definition of x^0.*

Proof. We can find a sequence $\tilde{F}_\nu$ satisfying the conclusions of Theorem 4.1. Since Eq. (4.3_ν) satisfies the uniqueness property of solutions and all assumptions of the corollary to Theorem 3.1 (see also the remark following that corollary) are satisfied, we obtain for every ν a $z^\nu = (\sigma^\nu, y^\nu) \in Z \cap \omega$ such that the solution $(t, x^\nu(\sigma^\nu, p(z^\nu))(t))$ of Eq. (4.3_ν) remains in ω for all $t \geqslant \sigma^\nu$, as long as it is defined.

Since Z is compact, we can assume that $(\sigma^\nu, y^\nu) \to (\sigma^0, y^0) \in Z$ as $\nu \to \infty$. Let $\phi^\nu = p(z^\nu)$ and $\phi^0 = p(z^0)$, $z^0 = (\sigma^0, y^0)$. Apply Lemma 4.1 to obtain a solution of the restricted equation Eq. (4.2) with properties listed in the lemma.

Let us first show that $z_0 \in Z \cap \omega$. If not, $z_0 \in Z \cap W$. Keeping in mind that ω is a regular polyfacial set with respect to Eq. (4.2) and proceeding as in the proof of Theorem 3.1, we see that there is a t, $t > \sigma^0$, such that $x^0(t)$ is defined and $(t, x^0(t)) \notin \bar{\omega}$. But then Lemma 4.1 implies that for k sufficiently large, $x^{\nu_k}(t)$ is defined and $(t, x^{\nu_k}(t)) \notin \bar{\omega}$, which contradicts the choice of x^{ν_k}. Hence, indeed, $z_0 \in Z \cap \omega$.

Suppose now that for some $t \geqslant \sigma^0$, $x^0(t)$ is defined and $(t, x^0(t)) \notin \omega$. Then $t > \sigma^0$ and by the same argument as above, there is a $\tilde{t} > \sigma^0$ such that $x^0(\tilde{t})$ is defined and $(\tilde{t}, x^0(\tilde{t})) \notin \bar{\omega}$. Now the application of Lemma 4.1 again leads to a contradiction.

So far we know that x^0 is a noncontinuable solution of Eq. (4.2). Hence it is also a solution of Eq. (4.1). We must show that x^0 is also noncontinuable as a solution of Eq. (4.1). But this is obvious by (α) of Definition 3.1, since ω is a regular polyfacial set with respect to both Eq. (4.1) and Eq. (4.2). The theorem is proved completely.

Remark. Theorem 3.2 can analogously be extended to the "nonuniqueness" case. The details are omitted.

We shall now prove Theorem 4.1: Let us introduce some notation:
Let:

$$\Gamma = \{(t, \phi) \in \mathbb{R} \times C \mid (t, \phi(0)) \in \partial\omega, (t + \theta, \phi(\theta)) \in \omega, \theta \in [-r, 0)\}$$

$$\pi = \{(t, \phi) \in \mathbb{R} \times C \mid (t + \theta, \phi(\theta)) \in \omega, \theta \in [-r, 0]\}$$

$$L^i = \{(t, x) \in \partial\omega \mid l^i(t, x) = 0\}$$

$$M^j = \{(t, x) \in \partial\omega \mid m^j(t, x) = 0\}$$

By (α) of Definition 3.1, it follows that $\Gamma \cup \pi \subset \Omega$.

For every $(t, \phi) \in \Omega$ there is an open set $U(t, \phi) \subset \Omega$ containing (t, ϕ) and such that:

(1) For all $i = 1,..., p$, if $(t, \phi(0)) \notin L^i$, then for all $(t', \phi') \in U(t, \phi)$: $(t', \phi'(0)) \notin L^i$.

For all $j = 1,..., q$, if $(t, \phi(0)) \notin M^j$, then for all $(t', \phi') \in U(t, \phi)$: $(t', \phi'(0)) \notin M^j$.

(2) If $(t, \phi) \in \Gamma$ and $(t, \phi(0)) \in L^i$, then for all $(t', \phi'), (t'', \phi'') \in U(t, \phi)$: $\sum_{r=1}^{n} (\partial l^i/\partial y_r)(t', \phi'(0))$.

$$F_r(t'', \phi'') + \frac{\partial l^i}{\partial t}(t', \phi'(0)) > 0.$$

If $(t, \phi) \in \Gamma$ and $(t, \phi(0)) \in M^j$, then for all $(t', \phi'), (t'', \phi'') \in U(t, \phi)$: $\sum_{r=1}^{n}$ $(\partial m^j/\partial y_r)(t', \phi'(0))$.

$$F_r(t'', \phi'') + \frac{\partial m^j}{\partial t}(t', \phi'(0)) < 0.$$

The existence of $U(t, \phi)$ satisfying (1) and (2) above follows immediately from the definition of Γ, L^i, M^j and Definition 3.1.

Now let

$$\tilde{\omega}_\Gamma = \bigcup_{(t,\phi)\in\Gamma} U(t, \phi),$$

$$\tilde{\omega}_\pi = \bigcup_{(t,\phi)\in\pi} U(t, \phi),$$

and let $\epsilon > 0$ be arbitrary.

Then there exist two families of open sets, $\mathscr{V}_\Gamma$, $\mathscr{V}_\pi$ such that:

(a) $\mathscr{V}_\Gamma$ is a locally finite refinement of $\{U(t, \phi) \mid (t, \phi) \in \Gamma\}$;
$\mathscr{V}_\pi$ is a locally finite refinement of $\{U(t, \phi) \mid (t, \phi) \in \pi\}$;

(b) if $V \in \mathscr{V}_\Gamma \cup \mathscr{V}_\pi$, then $V \neq \phi$ and for

$$(t', \phi'), (t'', \phi'') \in V: \sum_{r=1}^{n} |F_r(t', \phi') - F_r(t'', \phi'')| < \frac{\epsilon}{2}.$$

This follows from the continuity of F and from general topology.

Let us recall the following concept:

DEFINITION 4.1. Let X, Y be normed linear spaces, A a subset of X, and $f: A \to Y$ a mapping.

f is called Lipschitzian, if there is an $L > 0$ such that

$$\|f(x) - f(y)\| \leqslant L \|x - y\| \qquad \text{for} \quad x, y \in A.$$

f is called locally Lipschitzian, if for every $x \in A$ there is an open neighborhood U of x, such that $f \mid A \cap U$ is Lipschitzian.

Now the following well-known result holds (cf. the proof of Lemma 1, p. 4 in [5])

LEMMA 4.1. *If $\mathscr{V}$ is a locally finite family of nonempty open sets in a normed space X, then there exists a family $(\phi_V)_{V \in \mathscr{V}}$ of locally Lipschitzian functions, constituting a partition of unity subordinated to $\mathscr{V}$.*

Now let $G: \mathscr{V}_\Gamma \cup \mathscr{V}_\pi \to \mathbb{R} \times C$ be a selector, i.e. $G(V) \in V$ for all $V \in \mathscr{V}_\Gamma \cup \mathscr{V}_\pi$. Selectors exist by the axiom of choice. Define F_ϵ on $\tilde{\Omega} = \tilde{\omega}_\Gamma \cup \tilde{\omega}_\pi$ to be:

$$F_\epsilon(t', \phi') = \Big(\sum_{V \in \mathscr{V}_\Gamma \cup \mathscr{V}_\pi} \phi_V(t', \phi') \Big)^{-1} \cdot \sum_{V \in \mathscr{V}_\Gamma \cup \mathscr{V}_\pi} \phi_V(t', \phi') \cdot F(G(V))$$

where $(\phi_V)_{V \in \mathscr{V}_\Gamma}$ and $(\phi_V)_{V \in \mathscr{V}_\pi}$ are locally Lipschitzian partitons of unity subordinated to $\mathscr{V}_\Gamma$ and $\mathscr{V}_\pi$, respectively.

It follows easily that F_ϵ is locally Lipschitzian. Hence, by results in Hale [3], the RFDE generated by F_ϵ satisfies the uniqueness property of its solutions. Hence Theorem 4.1 will be proved if we can show that

1. $\sum_{r=1}^{n} |F_{\epsilon,r}(t', \phi') - F_r(t', \phi')| < \epsilon \qquad$ for $\quad (t', \phi') \in \tilde{\Omega}$.

2. ω is a regular polyfacial set with respect to the equations (Eq. (4.6), Eq. (4.7):

$$\dot{x} = \tilde{F}(t, x_t), \qquad \text{where} \quad \tilde{F} = F \mid \tilde{\Omega}, \tag{4.6}$$

$$\dot{x} = F_\epsilon(t, x_t). \tag{4.7}$$

$$\sum_{r=1}^{n} |F_{\epsilon,r}(t',\phi') - F_r(t',\phi')|$$
$$= \Big(\sum_{V\in\mathscr{V}_\Gamma\cup\mathscr{V}_\pi} \phi_V(t',\phi')\Big)^{-1}$$
$$\sum_{V\in\mathscr{V}_\Gamma\cup\mathscr{V}_\pi} \phi_V(t',\phi')\Big(\sum_{r=1}^{n} |F_r(G(V)) - F_r(t',\phi')|\Big) < \epsilon$$

by (b) above.

Since $\Gamma \cup \pi \subset \tilde{\Omega}$, it follows that ($\alpha$) of Definition 3.1 is fulfilled. It remains to be verified that (β) and (γ) of Definition 3.1 are satisfied for Eq. (4.7).

Let $(t',\phi') \in \Gamma$ and $l^i(t',\phi'(0)) = 0$. Then $(t',\phi') \in \tilde{\Omega}$ and, by assumption 1 above, $(t',\phi') \notin V$ for all $V \in \mathscr{V}_\pi$. Hence,

$$\sum_{r=1}^{n} \frac{\partial l^i}{\partial y_r}(t',\phi'(0)) \cdot F_{\epsilon,r}(t',\phi') + \frac{\partial l^i}{\partial t}(t',\phi'(0))$$
$$= \sum_{V\in\mathscr{V}_\Gamma} \phi_V(t',\phi')\Big\{\sum_{r=1}^{n} \frac{\partial l^i}{\partial y_r}(t',\phi'(0))\,F_r(G(V)) + \frac{\partial l^i}{\partial t}(t',\phi'(0))\Big\}$$

(since $\sum_{V\in\mathscr{V}_\Gamma}\phi_V(t',\phi') = 1$).

Since $G(V) \in V \subset U(t,\phi)$ for some $(t,\phi) \in \Gamma$, $(t,\phi(0)) \in L^i$, it follows from assumption 2 that the expression in braces is >0, hence (β) of Definition 3.1 is satisfied. (γ) of Definition 3.1 is verified in a similar way.

Hence, Theorem 4.1 is proved.

In the situation of Theorem 4.1, $\tilde{\Omega} \subset \Omega$. When can $\tilde{\Omega}$ be chosen to be equal to Ω? This can be done if an additional condition (AC) is imposed on the geometry of the polyfacial set ω:

(AC): For every $(t,x) \in \partial\omega$ there is a $\phi \in C$ such that $\phi(0) = x$ and $(t+\theta, \phi(\theta)) \in \omega$ for $\theta \in [-r, 0)$.

THEOREM 4.3. *If all assumptions of Theorem* 4.1 *are satisfied and if* (*AC*) *is true, then all conclusions of Theorem* 4.1 *hold with* $\tilde{\Omega} = \Omega$.

Proof. We shall modify the proof of Theorem 4.1 by introducing (AC). Fix $\epsilon > 0$. Choose for every $(t,\phi) \in \Omega$ an open set $U(t,\phi) \subset \Omega$ containing (t,ϕ) and such that (1) and (2) in the proof of Theorem 4.1 as well as (3), (4), and (5) below hold for $U(t,\phi)$:

(3) For every $(t',\phi'), (t'',\phi'') \in U(t,\phi)$,

$$\sum_{r=1}^{n} |F_r(t',\phi') - F_r(t'',\phi'')| < \frac{\epsilon}{2}.$$

(4) If $(t,\phi) \notin \bar{\Gamma}$, then $U(t,\phi) \cap \bar{\Gamma} = \phi$, where Γ is defined in the proof of Theorem 4.1.

(5) If $(t, \phi) \in \bar{\Gamma} \backslash \Gamma$, then there is a $\tilde{y} \in \mathbb{R}^n$ such that for all i, j:

(a) if $(t, \phi(0)) \in L^i$ and

$$\sum_{r=1}^{n} \frac{\partial l^i}{\partial y_r}(t, \phi(0)) \cdot F_r(t, \phi) + \frac{\partial l^i}{\partial y_r}(t, \phi) < 0,$$

then for all $(t', \phi') \in U(t, \phi)$:

$$\sum_{r=1}^{n} \frac{\partial l^i}{\partial y_r}(t', \phi'(0)) \cdot F_r(t', \phi') + \frac{\partial l^i}{\partial t}(t', \phi'(0)) < 0$$

$$\sum_{r=1}^{n} \frac{\partial l^i}{\partial y_r}(t', \phi'(0)) \cdot \tilde{y}_r + \frac{\partial l^i}{\partial t}(t', \phi'(0)) < 0;$$

(b) if $(t, \phi(0)) \in L^i$ and

$$\sum_{r=1}^{n} \frac{\partial l^i}{\partial y_r}(t, \phi(0)) \cdot F_r(t, \phi) + \frac{\partial l^i}{\partial t}(t, \phi(0)) \geqslant 0,$$

then for all $(t', \phi') \in U(t, \phi)$:

$$\sum_{r=1}^{n} \frac{\partial l^i}{\partial y_r}(t', \phi'(0)) \cdot \tilde{y}_r + \frac{\partial l^i}{\partial t}(t', \phi'(0)) > 0;$$

(c) all statements obtained from (a) and (b) by replacing "L^i" by "M^j", "l^i" by "m^j", "$<$" by "$>$", "$>$" by "$<$", and "$\geqslant$" by "$\leqslant$" hold true;

(d) for all $(t', \phi') \in U(t, \phi)$:

$$\sum_{r=1}^{n} | F_r(t', \phi') - \tilde{y}_r | < \frac{\epsilon}{2}.$$

(3) can be satisfied since F is continuous, (4) can be satisfied since $\bar{\Gamma}$ is closed. To see that (5) can also be satisfied, fix $(t, \phi) \in \bar{\Gamma} \backslash \Gamma$. Then $(t, \phi(0)) \in \partial\omega$, and, by condition (AC), there exists a $\tilde{\phi} \in C$, such that $\tilde{\phi}(0) = \phi(0)$, and $(t + \theta, \tilde{\phi}(\theta)) \in \omega$ for $\theta \in [-r, 0)$.

From (α) of Definition 3.1 it follows that $(t, \tilde{\phi}) \in \Omega$. Define $\tilde{y}$ to be

$$\tilde{y} = \rho F(t, \phi) + (1 - \rho) F(t, \tilde{\phi}).$$

If ρ is sufficiently close to 1, then it is easily seen that (5) can be satisfied, due to the fact that ω is a regular polyfacial set with respect to Eq. (4.1).

Now set

$$\tilde{\omega}_\Gamma = \bigcup_{(t,\phi)\in\Gamma} U(t, \phi)$$

$$\tilde{\omega}_{\bar{\Gamma}\backslash\Gamma} = \bigcup_{(t,\phi)\in\bar{\Gamma}\backslash\Gamma} U(t, \phi)$$

$$\tilde{\omega}_{\Omega\backslash\bar{\Gamma}} = \bigcup_{(t,\phi)\in\Omega\backslash\bar{\Gamma}} U(t, \phi).$$

As before, there are locally finite refinements $\mathscr{V}_{\Gamma}$, $\mathscr{V}_{\bar{\Gamma}\setminus\Gamma}$, and $\mathscr{V}_{\Omega\setminus\bar{\Gamma}}$ of $\{U(t, \phi) \mid (t, \phi) \in \Gamma\}$, $\{U(t, \phi) \mid (t, \phi) \in \bar{\Gamma}\setminus\Gamma\}$, and $\{U(t, \phi) \mid (t, \phi) \in \Omega\setminus\bar{\Gamma}\}$, respectively. Also, let $(\phi_V)_{V\in\mathscr{V}_{\Gamma}}$, $(\phi_V)_{V\in\mathscr{V}_{\bar{\Gamma}\setminus\Gamma}}$, $(\phi_V)_{V\in\mathscr{V}_{\Omega\setminus\bar{\Gamma}}}$ be corresponding locally Lipschitzian partitions of unity.

We assume that $V \neq \phi$ for $V \in \mathscr{V}_{\Gamma} \cup \mathscr{V}_{\bar{\Gamma}\setminus\Gamma} \cup \mathscr{V}_{\Omega\setminus\bar{\Gamma}}$. If $V \in \mathscr{V}_{\Omega\setminus\bar{\Gamma}}$, then there is a $(t, \phi) \in \Omega\setminus\bar{\Gamma}$ such that $V \subset U(t, \phi)$. Hence, by (4), $V \cap \bar{\Gamma} = \phi$. Let $G(V) := F(t, \phi)$.

If $V \in \mathscr{V}_{\bar{\Gamma}\setminus\Gamma}$, then there is a $(t, \phi) \in \bar{\Gamma}\setminus\Gamma$ such that $V \subset U(t, \phi)$ and a corresponding $\tilde{y}$ satisfying (5). Let $G(V) = \tilde{y}$. If $V \in \mathscr{V}_{\Gamma}$, then there is a $(t, \phi) \in \Gamma$, such that $V \subset U(t, \phi)$. Let $G(V) = F(t, \phi)$. Then $G: \mathscr{V}_{\Gamma} \cup \mathscr{V}_{\bar{\Gamma}\setminus\Gamma} \cup \mathscr{V}_{\Omega\setminus\bar{\Gamma}} \to \mathbb{R}^n$.

Now let us note that $\Omega = \tilde{\omega}_{\Gamma} \cup \tilde{\omega}_{\bar{\Gamma}\setminus\Gamma} \cup \tilde{\omega}_{\Omega\setminus\bar{\Gamma}}$. For $(t', \phi') \in \Omega$, define $F_\epsilon(t', \phi')$ as follows:

$$F_\epsilon(t', \phi') = \Big(\sum_{V\in\mathscr{V}_{\Gamma}\cup\mathscr{V}_{\bar{\Gamma}\setminus\Gamma}\cup\mathscr{V}_{\Omega\setminus\bar{\Gamma}}} \phi_V(t', \phi')\Big)^{-1} \cdot \sum_{V\in\mathscr{V}_{\Gamma}\cup\mathscr{V}_{\bar{\Gamma}\setminus\Gamma}\cup\mathscr{V}_{\Omega\setminus\bar{\Gamma}}} \phi_V(t', \phi') \cdot G(V).$$

It follows that F_ϵ is defined on Ω and is locally Lipschitzian.

If $(t', \phi) \in V$, then, $\sum_{r=1}^{n} \mid G_r(V) - F_r(t', \phi')| < \epsilon/2$ by (3) and (5d).

Hence, as in the proof of Theorem 4.1,

$$\sum_{r=1}^{n} |F_{\epsilon,r}(t', \phi') - F_r(t', \phi')| < \epsilon.$$

Now it remains to be shown that ω is a regular polyfacial set with respect to the equation $\dot{x} = F_\epsilon(t, x_t)$. Let (t', ϕ') is such that $(t', \phi'(0)) \in \partial\omega$ and $(t' + \theta, \phi'(\theta)) \in \omega$ for $\theta \in [-r, 0)$. Suppose e.g. that $(t', \phi'(0)) \in L^i$ for some $i = 1,..., p$. Since for $V \in \mathscr{V}_{\Omega\setminus\bar{\Gamma}}$, $(t', \phi') \notin V$ (by (4)), it follows that

$$\sum_{r=1}^{n} \frac{\partial l^i}{\partial y_r}(t', \phi'(0)) \cdot F_{\epsilon,r}(t', \phi') + \frac{\partial l^i}{\partial t}(t', \phi'(0))$$

$$= \Big(\sum_{V\in\mathscr{V}_{\Gamma}\cup\mathscr{V}_{\bar{\Gamma}/\Gamma}} \phi_V(t', \phi')\Big)^{-1}$$

$$\cdot \sum_{V\in\mathscr{V}_{\Gamma}\cup\mathscr{V}_{\bar{\Gamma}\setminus\Gamma}} \phi_V(t', \phi') \Big\{\sum_{r=1}^{n} \frac{\partial l^i}{\partial y_r}(t', \phi'(0)) \cdot G_r(V) + \frac{\partial l^i}{\partial t}(t', \phi'(0))\Big\}.$$

If $V \in \mathscr{V}_{\Gamma}$, then $V \subset U(t, \phi)$ for some $(t, \phi) \in \Gamma$, $G(V) = F(t, \phi)$ and $(t, \phi(0)) \in L^i$ by (1). Now (2) implies that the expression in braces >0.

If $V \in \mathscr{V}_{\bar{\Gamma}\setminus\Gamma}$, then $V \subset U(t, \phi)$ for some $(t, \phi) \in \bar{\Gamma}\setminus\Gamma$, $G(V) = \tilde{y}$, $(t, \phi(0)) \in L^i$.

Obviously (5a) cannot hold, because this would contradict the fact that $(t', \phi') \in U(t, \phi)$ and ω is a regular polyfacial set with respect to $\dot{x} = F(t, x_t)$. Hence, the assumption and, therefore, the conclusion of (5b) holds, i.e. the expression in braces is >0, again.

The case $(t', \phi'(0)) \in M^j$ for some $j = 1,..., q$ is dealt with in a completely analogous manner and Theorem 4.3 is proved completely.

Remarks. I. Condition (AC) is trivially satisfied if $r = 0$, i.e. in the ODE case. (AC) is also satisfied if ω is of the form: $\omega = \mathbb{R} \times \omega'$, where ω' is an open region in $\mathbb{R}^n$ enjoying the following property (S):

(S) For every $\epsilon > 0$, ω' is the union of a finite number of connected sets each of diameter $< \epsilon$. (Cf. Whyburn [10], p. 16).

In fact, if $(t, x) \in \partial\omega$, then $x \in \partial\omega'$, and by Theorem II, 4.3, of Whyburn [10], there is a $\phi \in C$, such that $\phi(0) = x$ and $\phi(\theta) \in \omega$, for all $\theta \in [-r, 0)$. Hence (AC) is satisfied.

Conjecture. The condition (AC) cannot, in general be dispensed with.

II. The approximating mappings F_ϵ in the above theorems are locally Lipschitzian. Can F_ϵ be chosen to be smoother, e.g., C^1-functions? The answer is, in general, no. See e.g. Kurzweil [4], p. 223, where it is proved that there is no sequence F_ν of C^1-functions on $C([-r, 0], \mathbb{R}^n)$ such that $F_\nu(\phi) \to \|\phi\|$ uniformly for $\phi \in \{\phi \in C \mid \|\phi\| < 1\}$.

Note, however, the following: Let F_0 be such that $F_0(t, \phi) = f_0(t, \phi(0), \phi(-r_1(t)),..., \phi(-r_k(t)))$ where $f_0 : U \to \mathbb{R}^n$, U open in $\mathbb{R} \times \mathbb{R}^{k+1}$, f_0 continuous, $r_j : \mathbb{R} \to \mathbb{R}$, $0 \leqslant r_j(t) \leqslant r$, r_j continuous. Then, by a similar argument to the one used in the proof of Theorem 4.1, it follows that there is a $\tilde{U} \subset U$, $\tilde{U}$ open in $\mathbb{R} \times \mathbb{R}^{k+1}$, and a sequence (f_ν) of C^∞*-functions* defined on $\tilde{U}$ such that $F_\nu \to F_0$ uniformly on $\tilde{\Omega}$, where

$$\tilde{\Omega} = \{(t, \phi) \in \mathbb{R} \times C \mid (t, \phi(0), \phi(-r_1(t)),..., \phi(-r_k(t))) \in \tilde{U}\}$$

and

$$F_\nu(t, \phi) = f_\nu(t, \phi(0), \phi(-r_1(t)),..., \phi(-r_k(t))) \qquad (\text{for } \nu \geqslant 0).$$

$\tilde{\Omega}$ is easily seen to be open. If the additional condition (AC) is satisfied, then $\tilde{U}$ can be chosen to be equal to U, hence $\tilde{\Omega} = \Omega$.

III. As an application of Theorem 4.1, let us note that the assumption of uniqueness of solutions in Examples 3.2 through 3.4 can be omitted.

Acknowledgment

This paper is a part of my doctoral thesis, written at Brown University under the direction of Professor Jack K. Hale. I wish to express my deepest gratitude to Professor Hale for his advice, encouragement and inspiration.

References

1. K. Borsuk, "Theory of Retracts," PWN Polish Scientific Publishers, Warsaw, 1967.
2. C. Conley, "Isolated Invariant Sets and the Morse Index," CBMS 38, Amer. Math. *Soc.*, Providence, R.I.
3. J. K. Hale, "Theory of Functional Differential Equations," Springer-Verlag, New York/Berlin, 1977.
4. J. Kurzweil, On Approximation in Real Banach Spaces, *Studia Math.* **14** (1953), 214–231.
5. A. Lasota and J. A. Yorke, The generic property of existence of solutions of differential equations in Banach spaces, *J. Differential Equations* **13** (1973), 1–12.
6. Z. Mikołajska, Une remarque sur les solutions bornées d'une équation differo-différentielle non linéaire, *Ann. Polon. Math.* **15** (1954), 23–32.
7. Z. Mikołajska, Sur le pilotage dans un champ de forces répulsives, *Ann. Polon. Math.* **31** (1976), 234–240.
8. N. Onuchic, On the Asymptotic Behavior of the Solutions of Functional Differential Equations, *in* "Differential Equations and Dynamical Systems"](J. K. Hale and J. P. LaSalle, Eds.), Academic Press, New York, 1967.
9. T. Ważewski, Sur un principe topologique de l'examen de l'allure asymptotique des intégrales des équations différentielles ordinaires, *Ann. Soc. Polon. Math.* **20** (1947), 279–313.
10. G. T. Whyburn, "Topological Analysis," Princeton Univ. Press, Princeton, N.J., 1958.