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On planar gluon amplitudes/Wilson loops duality

J.M. Drummond^a, J. Henn^a, G.P. Korchemsky^{b,*}, E. Sokatchev^a^a *Laboratoire d'Annecy-le-Vieux de Physique Théorique LAPTH, ¹ B.P. 110, F-74941 Annecy-le-Vieux, France*^b *Laboratoire de Physique Théorique, ² Université de Paris XI, F-91405 Orsay Cedex, France*

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Abstract

There is growing evidence that on-shell gluon scattering amplitudes in planar $\mathcal{N} = 4$ SYM theory are equivalent to Wilson loops evaluated over contours consisting of straight, light-like segments defined by the momenta of the external gluons. This equivalence was first suggested at strong coupling using the AdS/CFT correspondence and has since been verified at weak coupling to one loop in perturbation theory. Here we perform an explicit two-loop calculation of the Wilson loop dual to the four-gluon scattering amplitude and demonstrate that the relation holds beyond one loop. We also propose an anomalous conformal Ward identity which uniquely fixes the form of the finite part (up to an additive constant) of the Wilson loop dual to four- and five-gluon amplitudes, in complete agreement with the BDS conjecture for the multi-gluon MHV amplitudes.

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1. Introduction

Recent studies of scattering amplitudes in $\mathcal{N} = 4$ SYM theory have led to a very interesting all-loop iteration conjecture [1] about the IR finite part of the color-ordered planar gluon amplitudes. In particular, the four-gluon amplitude is expected to take the surprisingly simple

* Corresponding author.

E-mail address: korchems@th.u-psud.fr (G.P. Korchemsky).¹ UMR 5108 associée à l'Université de Savoie.² Unité Mixte de Recherche du CNRS (UMR 8627).

form

$$\ln \mathcal{M}_4 = [\text{IR divergences}] + \frac{\Gamma_{\text{cusp}}(a)}{4} \ln^2 \frac{s}{t} + \text{const}, \tag{1}$$

where $a = g^2 N / (8\pi^2)$ is the coupling constant and s and t are the Mandelstam kinematic variables.³ The relation (1) suggests that finite corrections to the amplitude \mathcal{M}_4 exponentiate and the coefficient of $\ln^2(s/t)$ in the exponent is determined by the universal cusp anomalous dimension $\Gamma_{\text{cusp}}(a)$ [2,3]. The conjecture (1) has been verified up to three loops in [1]. Also, a similar conjecture has been put forward for maximal helicity violating (MHV) n -gluon amplitudes [1] and it has been confirmed for $n = 5$ at two loops in [4].

Recently, another very interesting proposal has been made [5] for studying gluon scattering amplitudes at strong coupling. Within the context of the AdS/CFT correspondence [6], and making use of the T-duality transformation [7], $\ln \mathcal{M}_4$ is identified with the area of the world-sheet of a classical string in AdS space, whose boundary conditions are determined by the gluon momenta p_i^μ (with $i = 1, \dots, 4$). As was noticed in [5], their calculation is “mathematically similar” to that of the expectation value of a Wilson loop made out of four light-like segments (x_i, x_{i+1}) in $\mathcal{N} = 4$ SYM at strong coupling [8,9] with dual coordinates x_i^μ related to the on-shell gluon momenta by

$$x_{i,i+1}^\mu \equiv x_i^\mu - x_{i+1}^\mu := p_i^\mu. \tag{2}$$

Quite remarkably, the resulting stringy expression for $\ln \mathcal{M}_4$ takes exactly the same form as in (1), with the strong-coupling value of $\Gamma_{\text{cusp}}(a)$ obtained from the semiclassical analysis of [10,11]. This result constitutes a non-trivial test of the AdS/CFT correspondence.

In the paper [12] by three of us, we argued that the duality observed in [5] between the on-shell gluon amplitudes and light-like Wilson loops is also valid in planar $\mathcal{N} = 4$ SYM at weak coupling,

$$\ln \mathcal{M}_4 = \ln W(C_4) + \mathcal{O}(1/N^2). \tag{3}$$

The Wilson loop expectation value on the right-hand side of this relation is defined as

$$W(C_4) = \frac{1}{N} \langle 0 | \text{Tr P exp} \left(ig \oint_{C_4} dx^\mu A_\mu(x) \right) | 0 \rangle, \tag{4}$$

where $A_\mu(x) = A_\mu^a(x)t^a$ is a gauge field, t^a are the $SU(N)$ generators in the fundamental representation normalized as $\text{tr}(t^a t^b) = \frac{1}{2} \delta^{ab}$ and P indicates the ordering of the $SU(N)$ indices along the integration contour. As in [5], the integration contour C_4 consists of four light-like segments joining the points x_i^μ (with $i = 1, 2, 3, 4$) such that $x_i - x_{i+1} = p_i$ coincide with the external on-shell momenta of the four-gluon scattering amplitude, Eq. (2). Going through the calculation of the light-like Wilson loop (4) in the weak coupling limit, we established the correspondence between the IR singularities of the four-gluon amplitude and the UV singularities of the Wilson loop and extracted the finite part of the latter at one loop. Remarkably, our result turned out to be again of the form (1). The relation (3) admits a natural generalization to multi-gluon amplitudes. As was later shown in [13], the one-loop MHV amplitudes with an arbitrary number $n \geq 5$ of

³ To avoid the appearance of an imaginary part in $\ln \mathcal{M}_4$, it is convenient to choose s and t negative.

external legs coincides with the expectation value of the Wilson loop

$$\ln \mathcal{M}_n^{(\text{MHV})} = \ln W(C_n) + O(1/N^2) \quad (5)$$

evaluated along a polygonal loop consisting of n light-like segments.

A natural question arises whether the above mentioned correspondence between gluon scattering amplitudes and light-like Wilson loops holds true to higher loops. In the present letter, we perform an explicit two-loop calculation of the light-like Wilson loop $W(C_4)$ entering the right-hand side of (3) and match the result into the known expression for the four-gluon amplitude [14]. We demonstrate that the two expressions are in a perfect agreement with each other, thus providing an additional support to the scattering amplitude/Wilson loop correspondence (3).

Finally, in Section 3 we discuss the possible consequences of the conformal symmetry of the Wilson loop on the form of its finite part. Since the presence of cusps leads to divergences, we should expect that conformal invariance manifests itself in the form of anomalous Ward identities. We propose a very simple anomalous conformal-boost Ward identity, which we conjecture to be valid to all orders. We then show that it uniquely fixes the form of the finite part (up to an additive constant) of the Wilson loop dual to four- and five-gluon amplitudes, in complete agreement with the BDS ansatz [1] for the n -gluon MHV amplitudes. Furthermore, the n -point ansatz of [1] also satisfies this Ward identity for arbitrary n . However, starting with six points, conformal symmetry leaves room for an arbitrary function of the conformally invariant cross-ratios.

2. Light-like Wilson loops

2.1. One-loop calculation

In this section, we summarize some general properties of Wilson loops and review the one-loop calculation of [12].

The Wilson loop (4) is a gauge-invariant functional of the integration contour C_4 . We would like to stress that it is defined in configuration space. The contour C_4 consists of four light-cone segments in Minkowski space-time between the points x_i^μ (with $i = 1, 2, 3, 4$). It can be parameterized as

$$C_4 = \ell_1 \cup \ell_2 \cup \ell_3 \cup \ell_4, \quad \ell_i = \{(1 - \tau_i)x_i^\mu + \tau_i x_{i+1}^\mu | 0 \leq \tau_i \leq 1\}, \quad (6)$$

with $x_{i,i+1}^2 \equiv (x_i - x_{i+1})^2 = 0$. Then, to lowest order in the coupling constant, $W(C_4)$ is given by a sum over double contour integrals

$$W(C_4) = 1 + \frac{1}{2}(ig)^2 C_F \sum_{1 \leq j, k \leq 4} \int_{\ell_j} dx^\mu \int_{\ell_k} dy^\nu G_{\mu\nu}(x - y) + O(g^2), \quad (7)$$

where $C_F = t^a t^a = (N^2 - 1)/(2N)$ is the quadratic Casimir of $SU(N)$ in the fundamental representation and $G_{\mu\nu}(x - y)$ is the gluon propagator in the coordinate representation. In our calculation we employ the Feynman gauge and introduce dimensional regularization, $D = 4 - 2\epsilon$ (with $\epsilon > 0$), in which case the gluon propagator is given by

$$G_{\mu\nu}(x) = -g_{\mu\nu} \frac{\Gamma(1 - \epsilon)}{4\pi^2} (-x^2 + i0)^{-1+\epsilon} (\mu^2 \pi)^\epsilon. \quad (8)$$

It proves convenient to redefine the dimensional regularization scale as

$$\mu^2 \pi e^\gamma \mapsto \mu^2, \quad (9)$$

with the Euler constant γ originating from $\Gamma(1 - \epsilon) = \exp(\gamma\epsilon + O(\epsilon^2))$.

Let us represent each term in the double sum (7) as a Feynman diagram in which the integration contour is depicted by a double line and the gluon is attached to the j th and the k th segments (see Figs. 1(a) and 1(b)). Then it is easy to see that if both ends of the gluon are attached to the same segment, $k = j$, the diagram vanishes due to $G_{\mu\nu}(x - y)x_{j,j+1}^\mu x_{j,j+1}^\nu \sim x_{j,j+1}^2 = 0$. If the gluon is attached to two adjacent segments, $k = j + 1$, the corresponding diagram develops a double pole in ϵ . Finally, for $k = j + 2$, the diagram remains finite as $\epsilon \rightarrow 0$.

After a back-of-the-envelope calculation of the one-loop diagrams shown in Figs. 1(a) and 1(b), we obtain the one-loop expression for the light-like Wilson loop,

$$\begin{aligned} \ln W(C_4) = & \frac{g^2}{4\pi^2} C_F \left\{ -\frac{1}{\epsilon^2} [(-x_{13}^2 \mu^2)^\epsilon + (-x_{24}^2 \mu^2)^\epsilon] \right. \\ & \left. + \frac{1}{2} \ln^2 \left(\frac{x_{13}^2}{x_{24}^2} \right) + \frac{\pi^2}{3} + O(\epsilon) \right\} + O(g^4), \end{aligned} \tag{10}$$

where

$$x_{jk}^2 \equiv (x_j - x_k)^2 \tag{11}$$

denotes the distances between the vertices of C_4 . The double poles in ϵ on the right-hand side of (10) originate from the vertex-type Feynman diagram shown in Fig. 1(a). They come from integration over the position of the gluon in the vicinity of the cusp and have a clear ultraviolet meaning.

Let us substitute (10) into the duality relation (3) and apply (2) to identify the coordinates $x_{i,i+1}^\mu$ with the on-shell gluon momenta p_i^μ . This leads to

$$x_{13}^2 := (p_1 + p_2)^2 = s, \quad x_{24}^2 := (p_2 + p_3)^2 = t, \tag{12}$$

where s and t stand for Mandelstam invariants corresponding to the four-gluon amplitude. Then, we take into account the relation $C_F = N/2 + O(1/N^2)$ and use the well-known expression for the one-loop cusp anomalous dimension [2,3], $\Gamma_{\text{cusp}} = 2a + O(a^2)$, to observe that, firstly, upon identification of the dimensional regularization parameters

$$x_{13}^2 \mu^2 := s/\mu_{\text{IR}}^2, \quad x_{24}^2 \mu^2 := t/\mu_{\text{IR}}^2, \quad x_{13}^2/x_{24}^2 := s/t \tag{13}$$

the UV divergencies of the light-like Wilson loop match the IR divergent part of the four-gluon scattering amplitude (1) and, secondly, the finite ($\sim \ln^2(s/t)$) corrections to these two seemingly different objects coincide to one loop.

2.2. Non-Abelian exponentiation

Let us extend the analysis beyond one loop and examine the perturbative expansion of the light-like Wilson loop (4) to order g^4 . The corresponding perturbative corrections to $\ln W(C_4)$ come both from subleading terms in the expansion of the path-ordered exponential (4) in powers of the gauge field and from interaction vertices in the Lagrangian of $\mathcal{N} = 4$ SYM theory. As before, it is convenient to represent $O(g^4)$ corrections as Feynman diagrams.

Let us classify all possible $O(g^4)$ diagrams according to the order in the expansion of the path-ordered exponential in (4) in the gauge field, or equivalently, to the number of gluons $n_g = 2, 3, 4$ attached to the integration contour (depicted by a double line in Fig. 1). It is easy to see that for $n_g = 4$ the relevant diagrams cannot contain interaction vertices (see Figs. 1(d), 1(e), 1(h), 1(i) and 1(j)). Moreover, if one of the gluons is attached by both legs to the same light-like segment,

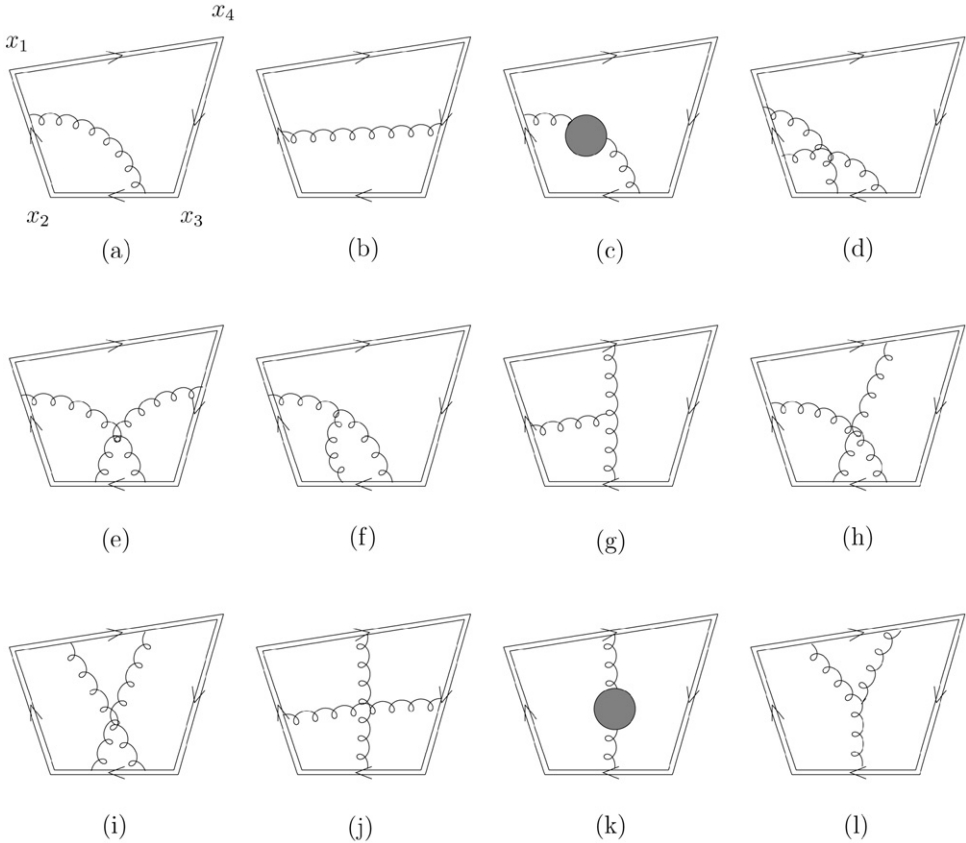


Fig. 1. The Feynman diagrams contributing to $\ln W(C_4)$ to two loops. The double line depicts the integration contour C_4 , the wiggly line the gluon propagator and the blob the one-loop polarization operator.

the diagram vanishes in the Feynman gauge for the same reason as at one loop. For $n_g = 3$, three gluons attached to the integration contour can be only joined together through a three-gluon vertex $V_{\mu_1\mu_2\mu_3}$ (see Figs. 1(f), 1(g) and 1(l)). In addition, if all three gluons are attached to the same light-like segment, the diagram vanishes by virtue of $V_{\mu_1\mu_2\mu_3}x_{j,j+1}^{\mu_1}x_{j,j+1}^{\mu_2}x_{j,j+1}^{\mu_3} \sim x_{j,j+1}^2 = 0$. Finally, for $n_g = 2$, the corresponding diagrams take the same form as the one-loop diagrams with the only difference being that the gluon propagator gets “dressed” by $O(g^2)$ corrections (see Figs. 1(c) and 1(k)). The latter corrections come both from tadpole diagrams which vanish in dimensional regularization and from gauge fields/gauginos/scalars/ghosts propagating inside the loop.

Another dramatic simplification occurs after we take into account the non-Abelian exponentiation property of Wilson loops [15]. In application to the light-like Wilson loop (4) in $\mathcal{N} = 4$ SYM theory, it can be formulated as follows,

$$W(C_4) = 1 + \sum_{n=1}^{\infty} \left(\frac{g^2}{4\pi^2}\right)^n W^{(n)} = \exp \left[\sum_{n=1}^{\infty} \left(\frac{g^2}{4\pi^2}\right)^n c^{(n)} w^{(n)} \right]. \tag{14}$$

Here $W^{(n)}$ denote perturbative corrections to the Wilson loop, while $c^{(n)} w^{(n)}$ are given by the contribution to $W^{(n)}$ with the “maximally nonabelian” color factor $c^{(n)}$. To the first few orders

in $n = 1, 2, 3$ the maximally non-Abelian color factor takes the form $c^{(n)} = C_F N^{n-1}$ but starting from $n = 4$ loops it is not expressible in terms of simple Casimir operators. We deduce from (14) that

$$W^{(1)} = C_F w^{(1)}, \quad W^{(2)} = C_F N w^{(2)} + \frac{1}{2} C_F^2 (w^{(1)})^2, \quad \dots \tag{15}$$

Then it follows from (15) that the correction to $W^{(2)}$ proportional to C_F^2 is uniquely determined by the one-loop correction $W^{(1)}$. This property allows us to reduce significantly the number of relevant two-loop diagrams.

Let us examine the color factors corresponding to the various two-loop Feynman diagrams. Following the classification of diagrams presented at the beginning of this section, we find that nonvanishing diagrams with $n_g = 2$ and $n_g = 3$ are accompanied by the same color factor $C_F N$ and, therefore, contribute to $w^{(2)}$. For $n_g = 4$, that is for diagrams without interaction vertices (Abelian-like diagrams), the color factor equals $t^a t^a t^b t^b = C_F^2$ and $t^a t^b t^a t^b = C_F(C_F - N/2)$ for planar and nonplanar diagrams, respectively. Applying non-Abelian exponentiation (15) we obtain that the planar $n_g = 4$ diagrams do not contribute to $w^{(2)}$ and, therefore, they can be safely neglected. At the same time, to define the contribution of nonplanar $n_g = 4$ diagrams to $w^{(2)}$, we have to retain the maximally non-Abelian contribution only. This can be easily done by replacing the color factor of the diagram by $C_F(C_F - N/2) \rightarrow -C_F N/2$.⁴

To summarize, we list in Fig. 1 all nonvanishing, two-loop diagrams of different topology contributing to $w^{(2)}$.

2.3. Two-loop calculation

In order to compute the two-loop Feynman diagrams shown in Fig. 1 we employ the technique described in detail in Refs. [16,17]. Furthermore, the rectangular light-like Wilson loop under consideration has been already calculated to two loops in [17] in the so-called forward limit $x_{12}^\mu = -x_{34}^\mu$, or equivalently $x_{13}^2 = -x_{24}^2$, in which the contour C_4 takes the form of a rhombus. The vertex-like diagrams shown in Figs. 1(c), 1(d) and 1(f) only depend on the distances x_{13}^2 and we can take the results from [17]. The Feynman diagrams shown in Figs. 1(h)–1(l) are proportional to scalar products $(x_{12} \cdot x_{34})$ and/or $(x_{23} \cdot x_{14})$ and, therefore, they vanish in the limit $x_{12}^\mu = -x_{34}^\mu$ due to $x_{i,i+1}^2 = 0$. For generic forms of C_4 , these diagrams have to be calculated anew. For the same reason, we have also to reexamine the contribution of diagrams shown in Figs. 1(e) and 1(g). Finally, the calculation in Ref. [17] was performed within conventional dimensional regularization (DREG) scheme. To preserve supersymmetry, we have to use instead the dimensional reduction (DRED) scheme. The change of scheme only affects the diagrams shown in Figs. 1(c) and 1(k) which involve an internal gluon loop [18].

The results of our calculation can be summarized as follows. Thanks to non-Abelian exponentiation, the two-loop expression for the (unrenormalized) light-like Wilson loop can be represented as

$$\ln W(C_4) = \frac{g^2}{4\pi^2} C_F w^{(1)} + \left(\frac{g^2}{4\pi^2} \right)^2 C_F N w^{(2)} + O(g^6). \tag{16}$$

⁴ Notice that $C_F(C_F - N/2)$ is subleading in the multi-color limit. It is amusing that the planar expression for the Wilson loop $W(C_4)$ is given by an exponential which receives corrections from nonplanar diagrams.

According to (10) the one-loop correction $w^{(1)}$ is given by

$$w^{(1)} = -\frac{1}{\epsilon^2} [(-x_{13}^2 \mu^2)^\epsilon + (-x_{24}^2 \mu^2)^\epsilon] + \frac{1}{2} \ln^2 \left(\frac{x_{13}^2}{x_{24}^2} \right) + \frac{\pi^2}{3} + O(\epsilon). \quad (17)$$

The two-loop correction $w^{(2)}$ is given by a sum over the individual diagrams shown in Fig. 1 plus crossing symmetric diagrams. It is convenient to expand their contributions in powers of $1/\epsilon$ and separate the UV divergent and finite parts as follows

$$w^{(2)} = \sum_{\alpha} [(-x_{13}^2 \mu^2)^{2\epsilon} + (-x_{24}^2 \mu^2)^{2\epsilon}] \left\{ \frac{1}{\epsilon^4} A_{-4}^{(\alpha)} + \frac{1}{\epsilon^3} A_{-3}^{(\alpha)} + \frac{1}{\epsilon^2} A_{-2}^{(\alpha)} + \frac{1}{\epsilon} A_{-1}^{(\alpha)} \right\} + A_0^{(\alpha)} + O(\epsilon), \quad (18)$$

where the sum goes over the two-loop Feynman diagrams shown in Fig. 1(c)–(l). Here $A_{-n}^{(\alpha)}$ (with $0 \leq n \leq 4$) are dimensionless functions of the ratio of distances x_{13}^2/x_{24}^2 . Making use of (18), we can parameterize the contribution of each individual diagram to the Wilson loop by the set of coefficient functions $A_{-n}^{(\alpha)}$:

- UV divergent $O(1/\epsilon^4)$ terms only come from the two Feynman diagrams shown in Figs. 1(d) and 1(f)

$$A_{-4}^{(d)} = -\frac{1}{16}, \quad A_{-4}^{(f)} = \frac{1}{16}. \quad (19)$$

- UV divergent $O(1/\epsilon^3)$ terms only come from the two Feynman diagrams shown in Figs. 1(c) and 1(f)

$$A_{-3}^{(c)} = \frac{1}{8}, \quad A_{-3}^{(f)} = -\frac{1}{8}. \quad (20)$$

- UV divergent $O(1/\epsilon^2)$ terms only come from the Feynman diagrams shown in Figs. 1(c)–1(g)

$$A_{-2}^{(c)} = \frac{1}{4}, \quad A_{-2}^{(d)} = -\frac{\pi^2}{96}, \quad A_{-2}^{(e)} = -\frac{\pi^2}{24}, \\ A_{-2}^{(f)} = -\frac{1}{4} + \frac{5}{96}\pi^2, \quad A_{-2}^{(g)} = \frac{\pi^2}{48}. \quad (21)$$

- UV divergent $O(1/\epsilon^1)$ terms come from the Feynman diagrams shown in Figs. 1(c)–1(h), 1(k) and 1(l)

$$A_{-1}^{(c)} = \frac{1}{2} + \frac{\pi^2}{48}, \quad A_{-1}^{(d)} = -\frac{1}{24}\zeta_3, \\ A_{-1}^{(e)} = \frac{1}{2}\zeta_3, \quad A_{-1}^{(f)} = -\frac{1}{2} - \frac{\pi^2}{48} + \frac{7}{24}\zeta_3, \\ A_{-1}^{(g)} = -\frac{1}{8}M_2 + \frac{1}{8}\zeta_3, \quad A_{-1}^{(h)} = \frac{1}{4}M_2, \\ A_{-1}^{(k)} = \frac{1}{4}M_1, \quad A_{-1}^{(l)} = -\frac{1}{4}M_1 - \frac{1}{8}M_2. \quad (22)$$

- Finite $O(\epsilon^0)$ terms come from all Feynman diagrams shown in Figs. 1(c)–1(l)

$$\begin{aligned}
 A_0^{(c)} &= 2 + \frac{\pi^2}{12} + \frac{1}{6}\zeta_3, & A_0^{(d)} &= -\frac{7}{2880}\pi^4, \\
 A_0^{(e)} &= -\frac{\pi^2}{12}M_1 - \frac{49}{720}\pi^4, & A_0^{(f)} &= -2 - \frac{\pi^2}{12} + \frac{119}{2880}\pi^4 - \frac{1}{6}\zeta_3, \\
 A_0^{(g)} &= \frac{1}{24}M_1^2 - \frac{1}{4}M_3 + \frac{7}{360}\pi^4, & A_0^{(h)} &= \frac{1}{8}M_1^2 + \frac{3}{8}M_3 + \frac{\pi^2}{8}M_1, \\
 A_0^{(i)} &= -\frac{1}{24}M_1^2, & A_0^{(j)} &= -\frac{1}{8}M_1^2, \\
 A_0^{(k)} &= M_1 + \frac{1}{2}M_2, & A_0^{(l)} &= -M_1 + \frac{\pi^2}{24}M_1 - \frac{1}{2}M_2 - \frac{1}{8}M_3.
 \end{aligned} \tag{23}$$

Here the notation was introduced for the integrals $M_i = M_i(x_{13}^2/x_{24}^2)$

$$\begin{aligned}
 M_1 &= \int_0^1 \frac{d\beta}{\beta - \bar{\alpha}} \ln\left(\frac{\bar{\alpha}\bar{\beta}}{\alpha\beta}\right) = -\frac{1}{2}\left[\pi^2 + \ln^2\left(\frac{x_{13}^2}{x_{24}^2}\right)\right], \\
 M_2 &= \int_0^1 \frac{d\beta}{\beta - \bar{\alpha}} \ln\left(\frac{\bar{\alpha}\bar{\beta}}{\alpha\beta}\right) \ln(\beta\bar{\beta}), \\
 M_3 &= \int_0^1 \frac{d\beta}{\beta - \bar{\alpha}} \ln\left(\frac{\bar{\alpha}\bar{\beta}}{\alpha\beta}\right) \ln^2(\beta\bar{\beta}),
 \end{aligned} \tag{24}$$

with $\bar{\beta} = 1 - \beta$, $\bar{\alpha} = 1 - \alpha$ and $\bar{\alpha}/\alpha = x_{13}^2/x_{24}^2$. We do not need the explicit expressions for the integrals M_2 and M_3 since, as we will see shortly, the contributions proportional to M_2 and M_3 cancel in the sum of all diagrams (for completeness, they can be found in Appendix A). In what follows it will be only important that the integrals (24) vanish in the forward limit $x_{13}^2 = -x_{24}^2$.

We would like to stress that the above results were obtained in Feynman gauge. Despite the fact that the contribution of each individual Feynman diagram to the light-like Wilson loop (or equivalently, the $A_{-n}^{(\alpha)}$ -functions) is gauge-dependent, their sum is gauge-invariant. Then, we substitute the obtained expressions for the coefficient functions, Eqs. (19)–(23), into (18) and finally arrive at the following remarkably simple expression for the two-loop correction,

$$\begin{aligned}
 w^{(2)} &= [(-x_{13}^2\mu^2)^{2\epsilon} + (-x_{24}^2\mu^2)^{2\epsilon}] \left\{ \epsilon^{-2} \frac{\pi^2}{48} + \epsilon^{-1} \frac{7}{8}\zeta_3 \right\} \\
 &\quad - \frac{\pi^2}{24} \ln^2\left(\frac{x_{13}^2}{x_{24}^2}\right) - \frac{37}{720}\pi^4 + O(\epsilon).
 \end{aligned} \tag{25}$$

This relation is one of the main results of the paper. We verify that in the forward limit, for $x_{13}^2 = -x_{24}^2$, this relation is in agreement with the previous calculations of Refs. [17].

The following comments are in order. Arriving at (25) we notice that the leading UV divergent $O(1/\epsilon^4)$ and $O(1/\epsilon^3)$ terms cancel in the sum of all diagrams. According to (21), the coefficients in front of $1/\epsilon^2$ are given by a sum of a rational number and π^2 -term. The rational terms cancel in the sum of all diagrams. As a consequence, the residue of the double pole in ϵ of $w^{(2)}$ in

Eq. (25) is proportional to ζ_2 . In a similar manner, the residue of $w^{(2)}$ at the single pole in ϵ is proportional to ζ_3 and this comes about as the result of a cancelation between various terms in (22) containing rational numbers, π^2 -terms as well as the integrals M_1 and M_2 . The most striking simplifications occur in the sum of finite $O(\epsilon^0)$ terms (23). We find that the integrals M_2, M_3, M_1^2 as well as the rational corrections and the terms proportional to π^2 and ζ_3 cancel in the sum of all diagrams leading to $-\frac{7}{720}\pi^4 + \frac{1}{12}\pi^2 M_1$.

We would like to stress that the two-loop expression (25) verifies the “maximal transcendentality principle” in $\mathcal{N} = 4$ SYM [19]. Let us assign transcendentality 1 to a single pole $1/\epsilon$. Then, it is easy to see from (25) that the coefficient in front of $1/\epsilon^n$ (including the finite $O(\epsilon^0)$ term!) has transcendentality equal to $4 - n$. In this way, each term in the two-loop expression (25) has transcendentality 4. For the same reason, the one-loop correction to the Wilson loop, Eq. (17) has transcendentality 2. Generalizing this remarkable property to higher loops in planar $\mathcal{N} = 4$ SYM, we expect that the perturbative correction to the Wilson loop (4) to order $O(g^{2n})$ should have transcendentality $2n$.

Notice that in our calculation of the two-loop Wilson loop we did not rely on the multi-color limit. In fact, due to the special form of the maximally non-Abelian color factors, $c^{(n)} = C_F N^{n-1}$ the relation (16) is exact for arbitrary N . As was already mentioned, these color factors become more complicated starting from $n = 4$ loops, where we should expect terms subleading in N .

3. Duality relation to two loops

We combine the one-loop and two-loop corrections to the Wilson loop, Eqs. (17) and (25), respectively, and rewrite the relation (16) in the multi-color limit by splitting it into UV divergent and finite parts as

$$\ln W(C_4) = D_4(-x_{13}^2\mu^2) + D_4(-x_{24}^2\mu^2) + \mathcal{F}_4(x_{13}^2/x_{24}^2). \quad (26)$$

The divergent part is given by the sum over poles

$$D_4(-x_{13}^2\mu^2) = -\frac{a}{\epsilon^2}(-x_{13}^2\mu^2)^\epsilon + a^2(-x_{13}^2\mu^2)^{2\epsilon} \left\{ \frac{1}{\epsilon^2} \frac{\pi^2}{24} + \frac{1}{\epsilon} \frac{7}{4} \zeta_3 \right\} + O(a^3) \quad (27)$$

and similarly for $D_4(-x_{24}^2\mu^2)$, while the finite part is given by

$$\begin{aligned} \mathcal{F}_4(x_{13}^2/x_{24}^2) &= \frac{1}{4} \ln^2 \left(\frac{x_{13}^2}{x_{24}^2} \right) \left[2a - \frac{\pi^2}{3} a^2 + O(a^3) \right] \\ &+ \left(\frac{\pi^2}{3} a - \frac{37\pi^4}{360} a^2 + O(a^3) \right), \end{aligned} \quad (28)$$

with the coupling constant $a = g^2 N / (8\pi^2)$.

We remind the reader that the poles in ϵ in the expressions for $D_4(-x_{13}^2\mu^2)$ and $D_4(-x_{24}^2\mu^2)$ have an UV origin and they appear due to the fact that the integration contour C_4 in the definition of the Wilson loop (4) has light-like cusps. This implies that the dependence of $W(C_4)$ on the UV cut-off μ^2 should be described by a renormalization group equation. For the light-like Wilson loop under consideration, such an equation was derived in Ref. [17]:

$$\begin{aligned} \frac{\partial}{\partial \ln \mu^2} \ln W(C_4) = & -\frac{1}{2} \Gamma_{\text{cusp}}(a) \ln(x_{13}^2 x_{24}^2 \mu^4) - \Gamma(a) \\ & - \frac{1}{\epsilon} \int_0^a \frac{da'}{a'} \Gamma_{\text{cusp}}(a') + O(\epsilon), \end{aligned} \tag{29}$$

where $\Gamma_{\text{cusp}}(a)$ is the cusp anomalous dimension and $\Gamma(a)$ is the so-called collinear anomalous dimension. As a nontrivial check of our calculation, we verify that the obtained two-loop expression for $\ln W(C_4)$, Eq. (26), satisfies the evolution equation (29). Furthermore, we find the following two-loop expressions for the anomalous dimensions:

$$\Gamma_{\text{cusp}}(a) = 2a - \frac{\pi^2}{3} a^2 + O(a^3), \quad \Gamma(a) = -7\zeta_3 a^2 + O(a^3). \tag{30}$$

These relations are in agreement with the known results [3,12,18]. Given the fact that $\ln W(C_4)$ obeys the maximal transcendentality principle, it is not surprising that the two anomalous dimensions have the same property.

We are now in a position to test the duality relation (3). According to (3), the two-loop expression for the light-like Wilson loop (26) should match the two-loop expression for the four-gluon scattering amplitude (1) upon appropriate identification of the kinematic variables and dimensional regularization parameter. As was explained in [12], this can be done in two steps. Firstly, we identify the dual coordinates x_i^μ with the on-shell gluon momenta according to the rule (2). Secondly, we replace the dimensionless combinations $x_{jk}^2 \mu^2$ of the Wilson loop by the following kinematic invariants of the gluon amplitude:

$$x_{13}^2 \mu^2 := s / \mu_{\text{IR}}^2 e^{\gamma(a)}, \quad x_{24}^2 \mu^2 := t / \mu_{\text{IR}}^2 e^{\gamma(a)}, \quad x_{13}^2 / x_{24}^2 := s / t. \tag{31}$$

Here μ_{IR}^2 is the parameter of dimensional regularization which plays the rôle of an IR cut-off in the four-gluon amplitude while s and t are the Mandelstam variables. Compared to the corresponding one-loop relation (13), Eq. (31) involves an additional function of the coupling constant $\gamma(a)$. Its expression to one loop can be found in [12].

Let us start with the divergent part of the Wilson loop defined in (27) and apply the relations (31). Making use of the evolution equation (29) and following the analysis of [12], it is straightforward to verify that, for arbitrary values of the kinematic invariants s and t , the UV divergent part of the Wilson loop coincides with the IR divergent part of the four-gluon scattering amplitude. Finally, we compare the finite part of the Wilson loop, Eq. (28), with the finite part of the four-gluon amplitude, Eq. (1). Substituting the cusp anomalous dimension in (1) by its two-loop expression (30), we find that the finite $\sim \ln^2(s/t)$ parts in the two relations coincide indeed! This constitutes yet another confirmation that the duality between gluon amplitudes and Wilson loops holds true in planar $\mathcal{N} = 4$ SYM both at weak and strong coupling.

In our analysis, we restricted ourselves to four-gluon scattering amplitudes. It is straightforward to extend these considerations to multi-gluon MHV amplitudes. According to (5), the duality relation in that case involves a light-like Wilson loop evaluated along an n -sided polygon, where n matches the number of external gluons. To one-loop level, the relation has been established in [13]. To two loops, the corresponding Wilson loop can be calculated by the method described above. It would be interesting to check whether the correspondence works in this case as well.

4. A conformal Ward identity for the Wilson loop

In this section, we examine the possible consequences of the conformal invariance of $\mathcal{N} = 4$ SYM on the functional form of the Wilson loop $W(C_n)$ entering the duality relations (3) and (5). First, we remark that a straight light-like segment ℓ_j of the type (6) maps to another straight light-like segment under the $SO(2, 4)$ conformal transformations (most easily seen by performing conformal inversion). This means that the entire n -sided polygonal Wilson loop C_n with light-like edges maps into a similar contour under such transformations. Furthermore, the gauge field $A_\mu(x)$ transforms with conformal weight one and hence the Wilson loop $W(C_n)$ is invariant but for the change of the contour.

If we were dealing with a finite object, we would then apply the usual procedure of deriving Ward identities (exploiting the conformal invariance of the classical action) to deduce that the dilatation and special conformal generators annihilate the Wilson loop expectation value (see e.g. [20] and references therein). However, the Wilson loop $W(C_n)$ is in fact divergent because of the presence of cusps on the contour. Therefore we need to introduce a regulator, e.g., dimensional, which breaks conformal invariance. As a consequence, we have to expect that the conformal Ward identity will receive an anomalous contribution [21]. We have investigated in detail how this works at the one-loop perturbative level. The resulting anomalous conformal Ward identity has a very simple and suggestive structure, which allows us to make a conjecture about its all-loop form. Leaving the detailed study and proof of this identity to a further publication, here we examine its consequences. Quite remarkably, the proposed Ward identity fixes uniquely the four- and five-point Wilson loops (up to an additive constant), in perfect agreement with the all-order ansatz of [1] and the conjectured equivalence with MHV amplitudes. We also show that the conjectured form of the n -point amplitude of [1] does indeed verify the identity for any n .

In a close analogy with (26), the Wilson loop $W_n \equiv W(C_n)$ can be split into divergent and finite parts

$$\ln W_n = \ln Z_n + \frac{1}{2} \Gamma_{\text{cusp}}(a) F_n + O(\epsilon), \quad (32)$$

where the cusp anomalous dimension is put in front of F_n for later convenience. Here Z_n is a UV divergent factor dependent on the dimensional regularisation scale μ , the regulator ϵ as well as the points x_i describing the contour C_n . It takes the following form [17]

$$\ln Z_n = -\frac{1}{4} \sum_{l=1}^{\infty} a^l \left(\frac{\Gamma_{\text{cusp}}^{(l)}}{(l\epsilon)^2} + \frac{\Gamma^{(l)}}{l\epsilon} \right) \sum_{i=1}^n (-x_{i,i+2}^2 \mu^2)^{l\epsilon}, \quad (33)$$

where $\Gamma_{\text{cusp}}(a) = \sum_{l=1}^{\infty} a^l \Gamma_{\text{cusp}}^{(l)}$ and $\Gamma(a) = \sum_{l=1}^{\infty} a^l \Gamma^{(l)}$. The term F_n is the finite contribution dependent only on the points x_i , as we argue below.

What can we say about the finite part F_n on general grounds? First of all, it is manifestly rotation and translation invariant, i.e., it is a function of the variables x_{ij}^2 . Furthermore, due to the special choice of the divergent part Z_n (33), where the scale μ appears in combination with the coupling, $a\mu^{2\epsilon}$, we expect that F_n at $\epsilon = 0$ should be independent of μ . Then it is easy to show that the dilatation Ward identity for the Wilson loop does not produce an anomalous dimension term for F_n and it is reduced to the trivial form

$$DF_n \equiv \sum_{i=1}^n (x_i \cdot \partial_{x_i}) F_n = 0. \quad (34)$$

This just has the consequence that F_n is a function of the dilatation invariant ratios of the x_{ij}^2 . However, an anomaly does appear in the special conformal Ward identity for F_n . As mentioned earlier, based on our experience with one loop, we propose the following general form,

$$K^\mu F_n = \sum_{i=1}^n (x_i^\mu + x_{i+2}^\mu - 2x_{i+1}^\mu) \ln x_{i,i+2}^2 = \sum_{i=1}^n x_{i,i+1}^\mu \ln \left(\frac{x_{i,i+2}^2}{x_{i-1,i+1}^2} \right), \tag{35}$$

where the special conformal generators (“boosts”) are given by the standard expression

$$K^\mu = \sum_{i=1}^n 2x_i^\mu (x_i \cdot \partial_{x_i}) - x_i^2 \partial_{x_i}^\mu \tag{36}$$

and a periodicity condition $x_i = x_{i+n}$ is assumed.

Let us now examine the consequences of the conformal Ward identity (35) for the finite part of the Wilson loop W_n . We find that the cases of $n = 4$ and $n = 5$ are special because here the Ward identity (35) has a unique solution up to an additive constant. The solutions are, respectively,

$$F_4 = \frac{1}{2} \ln^2 \left(\frac{x_{13}^2}{x_{24}^2} \right) + \text{const},$$

$$F_5 = -\frac{1}{4} \sum_{i=1}^5 \ln \left(\frac{x_{i,i+2}^2}{x_{i,i+3}^2} \right) \ln \left(\frac{x_{i+1,i+3}^2}{x_{i+2,i+4}^2} \right) + \text{const}. \tag{37}$$

This is easy to check by making use of the action of the special conformal generators (36) on the logarithmic terms,

$$K^\mu \ln \left(\frac{x_{i,j}^2}{x_{k,l}^2} \right) = 2(x_i^\mu + x_j^\mu - x_k^\mu - x_l^\mu). \tag{38}$$

We find that, upon identification of the kinematic invariants

$$x_{k,k+r}^2 := (p_k + \dots + p_{k+r-1})^2, \tag{39}$$

the relations (37) are exactly the functional forms of the ansatz of [1] for the finite parts of the four- and five-point MHV amplitudes.

The reason that the functional form of F_4 and F_5 is fixed up to an additive constant is that there are no conformal invariants one can build from four or five points x_i with light-like separations $x_{i,i+1}^2 = 0$. Starting from six points there are conformal invariants in the form of cross-ratios,

$$K^\mu \left(\frac{x_{i,j}^2 x_{k,l}^2}{x_{i,l}^2 x_{j,k}^2} \right) = 0. \tag{40}$$

For example, at six points there are three of them,

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}, \quad u_2 = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2}, \quad u_3 = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2}. \tag{41}$$

Hence the general solution of the Ward identity at six points and higher will contain an arbitrary function of the conformal cross-ratios.

We now wish to show that the ansatz of [1] for the finite part of the n -point MHV gluon amplitudes in $\mathcal{N} = 4$ SYM does satisfy our proposed conformal Ward identity (35). The ansatz of [1] for the logarithm of the ratio of the amplitude to the tree amplitude reads (cf. (32))

$$\ln \mathcal{M}_n^{(\text{MHV})} = \ln Z_n + \frac{1}{2} \Gamma_{\text{cusp}}(a) F_n + C_n + \mathcal{O}(\epsilon), \tag{42}$$

where Z_n is the IR divergent part, F_n is the finite part depending on the Mandelstam variables and C_n is the constant term. At four points the proposed form of the finite part is (using the notation (39))

$$F_4 = \frac{1}{2} \ln^2 \left(\frac{x_{13}^2}{x_{24}^2} \right) + 4\zeta_2, \tag{43}$$

while for $n \geq 5$ it is

$$F_n = \frac{1}{2} \sum_{i=1}^n g_{n,i},$$

$$g_{n,i} = - \sum_{r=2}^{\lfloor \frac{n}{2} \rfloor - 1} \ln \left(\frac{x_{i,i+r}^2}{x_{i,i+r+1}^2} \right) \ln \left(\frac{x_{i+1,i+r+1}^2}{x_{i,i+r+1}^2} \right) + D_{n,i} + L_{n,i} + \frac{3}{2} \zeta_2. \tag{44}$$

The functions $D_{n,i}$ and $L_{n,i}$ depend on whether n is odd or even. For n odd, $n = 2m + 1$ they are

$$D_{n,i} = - \sum_{r=2}^{m-1} \text{Li}_2 \left(1 - \frac{x_{i,i+r}^2 x_{i-1,i+r+1}^2}{x_{i,i+r+1}^2 x_{i-1,i+r}^2} \right),$$

$$L_{n,i} = - \frac{1}{2} \ln \left(\frac{x_{i,i+m}^2}{x_{i,i+m+1}^2} \right) \ln \left(\frac{x_{i+1,i+m+1}^2}{x_{i+m,i+2m}^2} \right). \tag{45}$$

For n even, $n = 2m$ they are

$$D_{n,i} = - \sum_{r=2}^{m-2} \text{Li}_2 \left(1 - \frac{x_{i,i+r}^2 x_{i-1,i+r+1}^2}{x_{i,i+r+1}^2 x_{i-1,i+r}^2} \right) - \frac{1}{2} \text{Li}_2 \left(1 - \frac{x_{i,i+m-1}^2 x_{i-1,i+m}^2}{x_{i,i+m}^2 x_{i-1,i+m-1}^2} \right),$$

$$L_{n,i} = \frac{1}{4} \ln^2 \left(\frac{x_{i,i+m}^2}{x_{i+1,i+m+1}^2} \right). \tag{46}$$

We have already seen that at four points and five points the general solution to the Ward identity coincides with (42). We now show that the ansatz (42) is a solution of the Ward identity for arbitrary n .

First we observe that the dilogarithmic contributions in (45) and (46) are functions of conformal cross-ratios of the form (40). They are therefore invariant under conformal transformations and we have immediately

$$K^\mu D_{n,i} = 0. \tag{47}$$

For the logarithmic contributions we use the identity (38). When n is odd we then find

$$\begin{aligned}
 K^\mu g_{n,i} = & -2 \sum_{r=2}^{m-1} \left[x_{i+r,i+r+1}^\mu (\ln x_{i+1,i+r+1}^2 - \ln x_{i,i+r+1}^2) \right. \\
 & - x_{i,i+1}^\mu (\ln x_{i,i+r}^2 - \ln x_{i,i+r,2}^2) \left. \right] - x_{i+m,i+m+1}^\mu (\ln x_{i+1,i+m+1}^2 - \ln x_{i+m,i+2m}^2) \\
 & - (x_{i+1,i+2m}^\mu - x_{i+m,i+m+1}^\mu) (\ln x_{i,i+m}^2 - \ln x_{i+m+1,i}^2). \tag{48}
 \end{aligned}$$

Changing variables term by term in the sum over i one finds that only the $\ln x_{i,i+2}^2$ terms remain and indeed (35) is satisfied. The proof for n even goes exactly the same way except that one obtains

$$\begin{aligned}
 K^\mu F_n = & \sum_{i=1}^n \left[\ln x_{i,i+2}^2 (x_i^\mu + x_{i+2}^\mu - 2x_{i+1}^\mu) \right. \\
 & \left. + \frac{1}{2} \ln x_{i,i+m}^2 (x_{i+m-1,i+m+1}^\mu - x_{i-1,i+1}^\mu) \right] \tag{49}
 \end{aligned}$$

and one has to use the fact that $n = 2m$ to see that the $\ln x_{i,i+m}^2$ term vanishes under the sum.

Thus we have seen that the BDS ansatz for the MHV amplitudes satisfies the proposed conformal Ward identity for the Wilson loop. It would be interesting to find out if the precise form of the conformally invariant dilogarithmic contributions can be fixed by some other general properties of the Wilson loop. The collinear behavior of the MHV amplitude discussed in Ref. [22] indicates that such a guiding principle could come from the reduction from $n + 1$ points to n points. The requirement that the Wilson loop has a certain analytic behavior, together with some mild assumption about the class of functions involved, might be sufficient to fix this form uniquely.

5. Conclusions

In this paper we have demonstrated by explicit calculation that the duality relation between light-like Wilson loop and four-gluon planar scattering amplitude holds true in $\mathcal{N} = 4$ SYM at weak coupling beyond one loop. Making use of the non-Abelian exponentiation theorem, we have shown that the finite two-loop corrections to the Wilson loop dual to four-gluon amplitude exponentiate with the prefactor given by the universal cusp anomalous dimension. At first glance, this may be surprising since the same anomalous dimension controls ultraviolet divergences that appear in the Wilson loop due to the presence of the cusps on the integration contour. We have argued that this property can be understood with the help of conformal symmetry. The cusp singularities alter transformation properties of the Wilson loop under the conformal $SO(2, 4)$ transformations and produce anomalous contribution to the conformal Ward identities. We proposed the all-loop form of the anomalous conformal Ward identities and demonstrated that they uniquely fix the form of the finite part (up to an additive constant) of the Wilson loop dual to four- and five-gluon amplitudes, in complete agreement with the BDS conjecture for the multi-gluon MHV amplitudes. Starting from six-gluon amplitudes, the conformal symmetry is not sufficient to fix the finite part of the dual Wilson loops and the conformal Ward identities should be supplemented by additional constraints on analytical properties of Wilson loops. One may speculate on their possible relation with the remarkable integrability symmetry of $\mathcal{N} = 4$ SYM.

The conjectured planar gluon amplitudes/Wilson loop duality offers a very clear context in which to understand the appearance of conformal integrals in the on-shell four-point MHV amplitude. These integrals were shown to be conformal up to three loops in [23] by writing the momenta in terms of dual coordinates. It was found that the conformal pattern continues at four

loops in [24] and at five loops [25] was used as a guiding principle in order to conjecture an amplitude whose consistency was checked with various unitarity cuts. In [12] it was shown that certain integrals whose coefficients vanish in the final expressions for the four-loop and conjectured five-loop amplitudes are precisely those which do not have well-defined conformal properties in four dimensions. Thus there is considerable evidence for a conformal structure behind the perturbative four-point gluon amplitude. In [12] it was argued that if the *off-shell* regulated amplitude also possessed such a conformal structure then basic properties of factorisation and exponentiation of infrared divergences would fix the functional form of the finite part (again up to an additive constant).

Since the conformal structure is best revealed in terms of the dual coordinate variables x_i , it was referred to in [12] as ‘dual conformal invariance’. In the on-shell four-point MHV amplitude this dual conformal invariance is broken by the presence of a dimensional regulator. Similarly, for the Wilson loop the ordinary conformal invariance of $\mathcal{N} = 4$ SYM is broken by the presence of a dimensional regulator. Thus the broken dual conformal structure of the on-shell amplitude is mapped directly to the broken conformal structure of the Wilson loop. The approach outlined here thus suggests the possibility of directly exploiting the dual conformal structure of the on-shell amplitude to understand the origin of its functional form. In this case one would expect that also the form of the five-point amplitude would be fixed by dual conformal symmetry. It would be very interesting to try to understand this in terms of the five point integrals appearing there.

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Appendix A

For completeness, we give the explicit expressions for the integrals (24) that appeared during the two-loop calculation

$$\begin{aligned}
 M_2 &= -\frac{\pi^2}{2} \ln(\alpha\bar{\alpha}) + 2\text{Li}_3(1) - \text{Li}_3\left(-\frac{\alpha}{\bar{\alpha}}\right) - \text{Li}_3\left(-\frac{\bar{\alpha}}{\alpha}\right) - \ln\left(\frac{\alpha}{\bar{\alpha}}\right) [\text{Li}_2(\alpha) - \text{Li}_2(\bar{\alpha})], \\
 M_3 &= -\frac{49}{180}\pi^4 - \frac{1}{3}\pi^2 [\ln^2(\alpha) + 6\ln(\alpha)\ln(\bar{\alpha}) + \ln^2(\bar{\alpha})] \\
 &\quad - \frac{1}{12} [\ln^4(\alpha) + \ln^4(\bar{\alpha}) + 4\ln(\bar{\alpha})\ln^3(\alpha) - 18\ln^2(\alpha)\ln^2(\bar{\alpha}) + 4\ln(\alpha)\ln^3(\bar{\alpha})] \\
 &\quad - 4\ln\left(\frac{\alpha}{\bar{\alpha}}\right) [\text{Li}_3(\alpha) - \text{Li}_3(\bar{\alpha})] + 8[\text{Li}_4(\alpha) + \text{Li}_4(\bar{\alpha})],
 \end{aligned}$$

where $\bar{\alpha} = 1 - \alpha$ and $\text{Li}_n(z)$ (with $n = 2, 3, 4$) are polylogarithms [26].

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