A Note on the Extremal Properties of the Discrete-Time Lyapunov Matrix Equation

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The discrete-time Lyapunov matrix equation

\[ A'QA - Q = -R \]

is considered. Fundamental inequalities declaring some extremal properties of the solution are proposed. Lower bounds for the minimum and maximum eigenvalues of \( Q \) in terms of \( \{A, R\} \) are established. Upper bounds are also attained under some conditions.

INTRODUCTION

It is well known that the \( n \)th-order linear time-invariant discrete-time system

\[ x(j+1) = Ax(j) \]  \hspace{1cm} (1)

is asymptotically stable if and only if, for an arbitrary symmetric positive-definite matrix \( R \), the matrix equation

\[ A'QA - Q = -R \]  \hspace{1cm} (2)

yields a symmetric positive-definite solution for \( Q \). In this case the quadratic form \( x'(j)Qx(j) \) is a Lyapunov function for (1) and, accordingly, (2) is often termed the discrete-time Lyapunov matrix equation.

In many practical problems, one is interested in recognizing lower and/or upper bounds for the solution \( Q \) of (2). Kwon and Pearson (1977) in a recent paper arrived at lower bounds for the minimum and maximum eigenvalues of \( Q, \lambda_{\text{min}}(Q) \) and \( \lambda_{\text{max}}(Q) \). However, their conclusion is restricted to the case of nonsingular matrix \( A \). Moreover, the bound suggested for \( \lambda_{\text{min}}(Q) \) is not generally valid (Fahmy and Hanafy, submitted for publication).
On the following pages, we shall develop four propositions dealing with the extremal properties of $Q$. Propositions 1 and 2 specify lower bounds for $\lambda_{\text{min}}(Q)$ and $\lambda_{\text{max}}(Q)$ in terms of $\{A, R\}$. Propositions 4 and 5, on the other hand, render upper bounds for $\|Q\|_s$ (some norm of $Q$, defined later) and for $\lambda_{\text{max}}(Q)$ under some conditions.

We recall here that the solution $Q_1$ of the continuous-time Lyapunov matrix equation

$$A_1^TQ_1 + Q_1A_1 = -R_1$$

possesses the lower bounds (Shapiro, 1974, and Kwon and Pearson, 1977)

$$\lambda_{\text{min}}(Q_1) \geq \lambda_{\text{min}}(R_1) 2\lambda_{\text{max}}^{1/2}(A_1^T A_1)$$

$$\lambda_{\text{max}}(Q_1) \geq \lambda_{\text{max}}(R_1) 2\lambda_{\text{max}}^{1/2}(A_1^T A_1).$$

These will be used for the sake of comparison with some of our main results.

**MAIN RESULTS**

We start with introducing three basic definitions. In what follows, $R^n$ is the $n$-dimensional Euclidean space and $R^{m \times n}$ is the linear vector space consisting of all $m \times n$ matrices defined over a real scalar field.

**Definition 1.** A compact set $U$ in $R^n$ is defined as

$$U = \{x \in U \mid x^T x = 1\}.$$  

This is the surface of the unit hypersphere in $R^n$.

**Definition 2.** The Euclidean norm of a vector $x \in R^n$, written $\|x\|_2$, is given by

$$\|x\|_2 = (x^T x)^{1/2}. \quad (7)$$

**Definition 3.** The spectral norm of a matrix $M \in R^{m \times n}$, written $\|M\|_2$, is given by

$$\|M\|_2 = \sup_{x \in U} \|Mx\|_2 = \lambda_{\text{max}}^{1/2}(M'M). \quad (8)$$

This is the matrix norm induced by the Euclidean vector norm. If $M \in R^{n \times n}$ is symmetric nonnegative-definite, then

$$\|M\|_2 = \sup_{x \in U} (x'Mx) = \lambda_{\text{max}}(M). \quad (9)$$

Now we derive the main results in succession.
Proposition 1. Consider the discrete-time Lyapunov matrix equation
\[ A'QA - Q = -R, \tag{10} \]
where \( A \) is a discrete-time stability matrix, i.e., all its eigenvalues lie within the unit circle:
\[ |\lambda_i(A)| < 1, \quad i = 1, 2, \ldots, n \tag{11} \]
and \( R \) is a symmetric positive-definite matrix.

It then follows that \( \lambda_{\text{min}}(Q) \) is bounded from below as
\[ \lambda_{\text{min}}(Q) \geq \lambda_{\text{min}}(R)/(1 - \lambda_{\text{min}}(A'A)). \tag{12} \]

Proof. Let \( x \in U \). Pre- and post-multiplying (10) by \( x' \) and \( x \), respectively, we obtain the scalar equation
\[ x'Qx = x'Rx + x'A'QAx. \tag{13} \]
Consider the quadratic form \( x'A'QAx \) and put \( y = Ax \). For the form \( y'Qy \) we can write
\[ y'Qy \geq \lambda_{\text{min}}(Q) \cdot (y'y) \tag{14} \]
and
\[ y'y = x'A'Ax \geq \lambda_{\text{min}}(A'A). \tag{15} \]
Combining (14) and (15),
\[ x'A'QAx \geq \lambda_{\text{min}}(Q) \cdot \lambda_{\text{min}}(A'A). \tag{16} \]
Equation (13), along with (16), gives
\[ x'Qx \geq x'Rx + \lambda_{\text{min}}(Q) \cdot \lambda_{\text{min}}(A'A). \tag{17} \]
Therefore
\[ \inf_{x \in U} (x'Qx) \geq \inf_{x \in U} (x'Rx) + \lambda_{\text{min}}(Q) \cdot \lambda_{\text{min}}(A'A) \tag{18} \]
Since
\[ \inf_{x \in U} (x'Qx) = \lambda_{\text{min}}(Q), \tag{19} \]
\[ \inf_{x \in U} (x'Rx) = \lambda_{\text{min}}(R), \tag{20} \]
it follows from (18) that
\[ \lambda_{\text{min}}(Q)[1 - \lambda_{\text{min}}(A'A)] \geq \lambda_{\text{min}}(R). \tag{21} \]
From (11) it is seen that \( \det(A) < 1 \). Hence \( \det(A' A) < 1 \) and \( \lambda_{\min}(A' A) < 1 \). Relation (21) thus yields
\[
\lambda_{\min}(Q) \geq \lambda_{\min}(R)/(1 - \lambda_{\min}(A' A)).
\]
Q.E.D.

It is to be noted that if \( A \) is singular, then (12) reduces to
\[
\lambda_{\min}(Q) \geq \lambda_{\min}(R).
\]

The estimation in (12) is exact in certain cases. For example, let \( A = \frac{1}{2}I \) and \( R = I \). From (12), \( \lambda_{\min}(Q) \geq \frac{3}{2} \). The true value is \( \lambda_{\min}(Q) = \frac{3}{2} \).

**COROLLARY.** For the discrete-time Lyapunov matrix equation (10), we have
\[
Q \geq \frac{\lambda_{\min}(R)}{1 - \lambda_{\min}(A' A)} I.
\]

**Proof.** It is evident, by virtue of (12), that the eigenvalues of matrix
\[
\left[ Q - \frac{\lambda_{\min}(R)}{1 - \lambda_{\min}(A' A)} I \right]
\]
are all nonnegative. This proves the corollary.

**PROPOSITION 2.** Consider the discrete-time Lyapunov matrix equation (10). For all possible \( A \) it holds that
\[
\lambda_{\max}(Q) \geq \lambda_{\max}(R).
\]

**Proof.** As in the proof of Proposition 1 we obtain (13). Since the quadratic form \( x' A' Q A x \) is nonnegative-definite, we can write
\[
x' Q x \geq x' R x.
\]

Therefore
\[
\sup_{x \in U} (x' Q x) \geq \sup_{x \in U} (x' R x).
\]

That is,
\[
\lambda_{\max}(Q) \geq \lambda_{\max}(R). \quad \text{Q.E.D.}
\]

The simple bound of (24) is better, in some cases, than the bound obtained by Kwon and Pearson (1977). As an example, let
\[
A = \begin{bmatrix} 0.5 & 0 \\ 0 & -0.5 \end{bmatrix}, \quad R = \begin{bmatrix} 15 & 10 \\ 10 & 12 \end{bmatrix}
\]
Then (24) gives \( \lambda_{\max}(Q) \geq 23.6 \). Kwon and Pearson suggest that \( \lambda_{\max}(Q) \geq 20 \). The exact value is \( \lambda_{\max}(Q) = 26.3 \).
Before establishing Proposition 3 we interpolate the following assertion.

**Assertion.** For any discrete-time stability matrix there is some norm which is less than unity (Broyden, 1975). Such a norm is designated here as the $s$-norm, written $\| \cdot \|_{s}$, and we have

$$\| A \|_{s} < 1$$  \hspace{1cm} (27)

**Proposition 3.** Consider the discrete-time Lyapunov matrix equation (10). It then follows that $\| Q \|_{s}$ is bounded from above as

$$\| Q \|_{s} \leq \| R \|_{s}(1 - \| A \|_{s} \| A' \|_{s}).$$ \hspace{1cm} (28)

**Proof.** The solution of (10) can be expressed in the form of an infinite (convergent) matrix series as

$$Q = R + A'RA + A^2'RA^2 + \cdots$$ \hspace{1cm} (29)

Taking the $s$-norms and applying the triangle inequalities, we can write

$$\| Q \|_{s} \leq \| R \|_{s}(1 + \| A \|_{s} \| A' \|_{s} + \| A \|_{s}^2 \| A' \|_{s}^2 + \cdots).$$ \hspace{1cm} (30)

Since $\| A \|_{s} \| A' \|_{s} < 1$, and using the familiar sum of the infinite geometric progression, we finally conclude

$$\| Q \|_{s} \leq \| R \|_{s}(1 - \| A \|_{s} \| A' \|_{s}).$$ \hspace{1cm} Q.E.D.

**Corollary.** Consider the discrete-time Lyapunov matrix equation (10). If we have

$$\lambda_{\text{max}}(A'A) < 1$$ \hspace{1cm} (31)

then $\lambda_{\text{max}}(Q)$ is bounded from above as

$$\lambda_{\text{max}}(Q) \leq \lambda_{\text{max}}(R)/(1 - \lambda_{\text{max}}(A'A)).$$ \hspace{1cm} (32)

**Proof.** Relation (32), under condition (31), follows directly from (28) by replacing the $s$-norms by the spectral norms.

Note that one form of $A$, among other forms, for which (31) is satisfied, is the symmetrical form.

**Proposition 4.** Consider the discrete-time Lyapunov matrix equation (10). If $A$ is a nilpotent matrix of degree $k$, then $\lambda_{\text{max}}(Q)$ is bounded from above as

$$\lambda_{\text{max}}(Q) \leq \lambda_{\text{max}}(R) \cdot \sum_{i=0}^{k-1} \lambda_{\text{max}}(A^i'A^i).$$ \hspace{1cm} (33)
Proof. Again expressing the solution of (10) as a matrix series and using the property of nilpotency, we can write

$$Q = R + A'RA + A^2RA^2 + \cdots + A^{(k-1)}RA^{(k-1)}.$$  \hfill (34)

Taking the spectral norms and using the triangle inequalities, we obtain

$$\lambda_{\text{max}}(Q) \leq \lambda_{\text{max}}(R)[1 + \lambda_{\text{max}}(A'A) + \lambda_{\text{max}}(A^2'A^2) + \cdots + \lambda_{\text{max}}(A^{(k-1)}'A^{(k-1)})].$$

Q.E.D.

It can be shown also that the upper bounds of (28), (32), and (33) render the exact results in certain cases.

**CONCLUSION**

Fundamental inequalities, illuminating some extremal properties for the solution $Q$ of the discrete-time Lyapunov matrix equation (10), have been established. Lower bounds for $\lambda_{\text{min}}(Q)$ and $\lambda_{\text{max}}(Q)$ are specified in Propositions 1 and 2. Propositions 3 and 4 render upper bounds for $\|Q\|_\infty$ and $\lambda_{\text{max}}(Q)$ under some conditions. The estimations obtained are simple in form and amenable to manipulation. In addition, they are sharp in the sense that the bounds coincide with the exact results in certain cases.

In comparison with the continuous-time version, it is observed that the lower bounds in (4) and (5) may be reduced indefinitely by increasing $\lambda_{\text{max}}(A_1'A_1)$. Yet the lower bounds in (12) and (24), arising for the discrete-time case, are themselves bounded; they cannot be reduced, for all possible $A$, below $\lambda_{\text{min}}(R)$ and $\lambda_{\text{max}}(R)$, respectively.

**REFERENCES**


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