A Generalization of a Theorem of Diderrich in Additive Group Theory to Vertex-transitive Graphs

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Let $X_1, X_2, \ldots, X_h$ be a family of vertex-transitive graphs without triangles on the same vertex-set. We consider the set $\Lambda$ of vertices which can be reached from a given vertex $a$ by a path $e_1 \cdots e_t$, where $e_t$ is an edge of $X_{i(t)}$ with $1 \leq j(1) < j(2) < \cdots < j(t) \leq k$ and $0 \leq t \leq k$ (for $t = 0$, $\Lambda$ is equal to the singleton $(a)$).

We prove the following:

Let $k$ be a positive integer and let $(X_i = (V, E_i))_{1 \leq i \leq k}$ be a family of vertex-transitive graphs without triangles. Let $a$ be an element of $V$. We denote by $\Lambda_k$ the set of vertices of

$\{a\} \cup \bigcup_{i=1}^{k} \bigcup_{1 \leq j_1 < j_2 < \cdots < j_t \leq k} \Gamma_{j_t}^+(\Gamma_{j_{t-1}}^+(\cdots \Gamma_{j_1}^+(a)))$,

where $\Gamma_{i}^+(A) = \Gamma_{X_i}(A)$ for any subset $A$ of $V$.

Then, either $|\Lambda_k| \geq 1 + \sum_{t=1}^{k} d^+(X_t)$ or $\Lambda_k$ contains one of the connected components $C_i(a)$ of $a$ in $X_i$. 

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As a corollary, we obtain the following result in additive group theory:

Let $G$ be an arbitrary group written multiplicatively. Let $A$ and $B$ be two non-empty subsets of $G$. We denote by $AB$ the set of products $ab$, where $a \in A$ and $b \in B$, and by $\langle A \rangle$ the subgroup generated by $A$. A subset $S$ of a group $G$ is said to be product-free if $SS \cap S = \emptyset$.

We prove the following result:

Let $G$ be an arbitrary finite group. Let $k$ be a positive integer and let $S_1, S_2, \ldots, S_k$ be non-empty product-free subsets of $G$. We denote by $S_k$ the set of products $a_1a_2\cdots a_t$, where $1 \leq i_1 < i_2 < \cdots < i_t = k$, $1 \leq t \leq k$, and $a_i \in S_i$. Then, only two cases are possible:

- either there exists $i$ with $\langle S_i \rangle \subseteq S_k$ or $|S_k| + 1 > \sum_{i=1}^{k} |S_i|$.

This is Diderrich’s theorem [2] when all the subsets $S_i$’s are singletons. However, our result is more precise, because we know the form of the subgroup. Moreover, our proof is easier.

2. Proof of the Theorem

**Lemma 2.1.** Let $X$ be a finite vertex-transitive connected graph without triangles. Then, $\kappa(X) = d^+(X)$.

**Proof.** It is well-known that a finite vertex-transitive connected graph is strongly connected. Therefore $\kappa(X) > 0$. Let $A$ be an atom of $X$ (a graph has always an atom which is positive of negative): $A \neq \emptyset$ because $\kappa(X) > 0$. We suppose $A$ to be positive (if not, we work on the inverse graph of $X$).

Let $x$ be a vertex of $A$.

- **Case 1.** $N^+(x) \subseteq N^+(A)$. In this case, $\kappa(X) = |N^+(A)| \geq d^+(x) = d^+(X)$.

- **Case 2.** There exists $y$ in $N^+(x) \cap A$.

We have $\Gamma^+(x) \cup \Gamma^+(y) \subseteq A \cup N^+(A)$. As $X$ has no triangle, we have $\Gamma^+(x) \cap \Gamma^+(y) = \emptyset$. Therefore, $2d^+(X) \leq |A| + \kappa(X)$. Moreover, $|A| \leq \kappa(X)$ [4, 5], and thus $d^+(X) \leq \kappa(X)$.

As we trivially have $\kappa(X) = d^+(X)$, we obtain finally $d^+(X) = \kappa(X)$. \qed

The following lemma is well-known:

**Lemma 2.2.** A connected component of a finite vertex-transitive graph is a vertex-transitive graph.

**Theorem 2.3.** Let $k$ be a positive integer and let $(X_i = (V, E_i))_{1 \leq i \leq k}$ be a family of finite vertex-transitive graphs without triangles. Let $a$ be an element of $V$. Then, either $|\Lambda_k| \geq 1 + \sum_{i=1}^{k} d^+(X_i)$ or $\Lambda_k$ contains one of the connected components $C_i(a)$ of $a$ in $X_i$.

**Proof.** We prove the theorem by induction on $k$. Suppose that $\Lambda_{k+1}$ does not contain any of the $C_i(a)$’s. By the induction hypothesis, $|\Lambda_k| \geq 1 + \sum_{i=1}^{k} d^+(X_i)$.

We will apply Lemma 2.1 on $C_{k+1}(a)$, which is vertex-transitive by Lemma 2.2.

We set $A_k = \Lambda_k \cap C_{k+1}(a)$.

$A_k \neq \emptyset$ because $a \in A_k$. We have $A_k \neq C_{k+1}(a)$ because $\Lambda_{k+1}$ does not contain any
of the $C_i(a)$’s. For the same reason, $\Gamma_k^+(A_k) \cup \Lambda_k \neq C_{k+1}(a)$. Therefore, by Lemma 2.1 applied on $C_{k+1}(a)$,

$$|\Gamma_k^+(A_k) - A_k| \geq d^+(C_{k+1}(a)) = d^+(X_{k+1}).$$

As $\Lambda_k \cap (\Gamma_k^+(A_k) - A_k) = \emptyset$, we obtain $|\Lambda_{k+1}| \geq 1 + \sum_{i=1}^{k+1} d^+(X_i)$. □

**Corollary 2.4.** Let $G$ be an arbitrary finite group. Let $k$ be a positive integer and let $S_1, S_2, \ldots, S_k$ be non-empty product-free subsets of $G$. Then, either there exists $i$ with $\langle S_i \rangle \leq S_k$ or $|S_k| \geq 1 + \sum_{i=1}^{k} |S_i|$.  

**Proof.** We apply Theorem 2.3 on the family $X_i = \text{Cay}(\langle S_i \rangle, S_i) = \langle \langle S_i \rangle, E_i \rangle$, where $E_i = \{(x, y) : x^{-1}y \in S_i\}$ with $a = 1$. It is easy to see that the graphs are vertex-transitive and that $C_i(1) = \langle S_i \rangle$. □

**Corollary 2.5.** Let $G$ be a finite group of order $n$ ($n > 1$) and let $a_1, a_2, \ldots, a_n$ be a sequence of non-unit elements of $G$. The set $S$ of products $a_{i_1} \times \cdots \times a_{i_t}$, where $1 \leq i_1 < \cdots < i_t \leq n$ and $1 \leq t \leq n$ must contain a non-trivial subgroup $H$ of $G$ of the form $\langle a_i \rangle$.  

**Proof.** This is Corollary 2.4 for $S_i = \{a_i\}$. This is Theorem B of Diderrich [2], but our result is more precise, because we know that $H$ is one of the $\langle a_i \rangle$’s. □

**References**


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