A Direct Interpolation Method for Irregular Sampling

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In this paper, band-limited functions are reconstructed from their values taken at a sequence of irregularly spaced sample points. We use a modified Lagrange formula, which is attributed to Boas and Bernstein. The formula used in this paper differs from the classical Boas–Bernstein formula in the following way. Instead of using infinite canonical products with respect to the whole sequence of sample points, we use canonical products with respect to sequences of sample points which are irregularly spaced only on finite intervals. Estimates for the truncation error of this reconstruction method are given.

Key Words: irregular sampling; Lagrange interpolation; Boas–Bernstein formula; windowed canonical products

1. INTRODUCTION

The problem of reconstructing a band-limited function from its values taken at a sequence of irregularly spaced sample points has attracted much attention in both the mathematical and the signal processing literature. A number of computational methods have been proposed; see [1, 4, 6, 8, 12, 13] for different aspects of the subject.

A key result in this field is the theorem of Duffin and Schaeffer [5] which says that if the density of the sequence of sample points is strictly larger than the Nyquist density, then stable reconstruction is possible. An easy proof of this result is obtained from a classical interpolation method known as the Boas–Bernstein formula (see, e.g., [15]). Although this formula yields an explicit reconstruction of the function, it is usually considered to be of theoretical interest only, because it requires numerically intractable computations of infinite products. Instead, other approaches seem to be preferred in practice, for instance, frame or iterative methods [1, 6].

The purpose of this paper is to modify the Boas–Bernstein formula, so that we obtain a direct and computationally efficient interpolation method. The idea is simple: Since only a finite number of sampled values are used in the reconstruction, we may assume that the sequence of sample points has a regular structure outside a finite interval. This allows us to replace the infinite products by finite products, which we will call windowed canonical products, since they will depend on the location of the reconstruction. The size of the
windows is determined by the relative error accepted in the reconstruction. Thus the error analysis plays a crucial role in setting up the interpolation formula. We present estimates of the truncation error and consider also a numerical example.

2. PRELIMINARIES

For each \( \tau > 0 \) we denote by \( L^2_\tau \) the Paley–Wiener space of all \( L^2(\mathbb{R}) \)-functions whose Fourier transforms are supported by \( [-\tau, \tau] \). Our starting point will be the classical (Whittaker–Kotel’nikov–Shannon) sampling theorem, which says that an arbitrary function \( f \in L^2_\tau \) can be reconstructed from its values at the integers

\[
f(x) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi x}{(-1)^n \pi(x-n)},
\]

with convergence both in \( L^2 \) and uniformly on \( \mathbb{R} \), and also

\[
\|f\|_{L^2}^2 = \sum_n |f(n)|^2.
\]

A simple extension of the sampling theorem is obtained if we replace a finite number, say \( N \), of the integers by \( N \) distinct noninteger real numbers. If we call such a sequence \( \Lambda = \{\lambda_n\} \subset \mathbb{R} \), i.e., \( \lambda_n = n \) except for finitely many \( n \)'s, and denote by \( 1 \) \( G(t) = G(\Lambda; t) = \lim_{R \to \infty} \prod_{|\lambda_n|<R} \left( 1 - \frac{t}{\lambda_n} \right) = \frac{\sin \pi t}{\pi} \prod_{\lambda_n \neq n} \frac{t - \lambda_n}{t - n} \),

we have

\[
f(x) = \sum_{n=-\infty}^{\infty} f(\lambda_n) \frac{G(x)}{G(\lambda_n)(x-\lambda_n)},
\]

for every \( f \in L^2_\tau \), with convergence both in \( L^2 \) and uniformly on \( \mathbb{R} \), and also

\[
A\|f\|_{L^2}^2 \leq \sum_n |f(\lambda_n)|^2 \leq B\|f\|_{L^2}^2,
\]

A and \( B \) being positive constants independent of \( f \). In order to make the exposition self-contained, we sketch a proof of (3) and (4). The series on the right-hand side of (3) converges to a function \( g \in L^2_\tau \). This follows from the convergence of the series (1) combined with the expression for \( G \) on the right-hand side of (2). In order to prove that \( f(x) = g(x) \), we observe that \( g(\lambda_n) = f(\lambda_n) \), \( n \in \mathbb{Z} \), and hence the function

\[
\phi(x) = \left[ f(x) - g(x) \right] \prod_{n \neq \lambda_n} \frac{x-n}{x-\lambda_n}
\]

belongs to \( L^2_\tau \) and vanishes on \( \mathbb{Z} \). Therefore \( \phi = 0 \), and \( f = g \). Relation (4) now follows from the Banach theorem on inverse operators.

\(^1\) Here and in what follows, if one of the elements \( \lambda_n \) is 0 then the corresponding factor should be replaced by the factor \( t \).
Before proceeding, we would like to mention that the difficult problem of describing all (real) sequences \( \Lambda \) with these properties was solved by Pavlov [14]. We refer the reader to [9, 10] for further background and to [11] for a different approach to this problem.

From now on we shall assume that \( f \in L_1^2 \) with \( \delta = \pi - \tau > 0 \). This means that we require the function to be \textit{oversampled}, with \( \delta \) measuring the redundancy in the sampling process. This redundancy can be used to improve the speed of convergence of (3). A simple way of doing this is as follows. Fix a positive integer \( l \) and define

\[
w(t) = w_{s,l}(t) = \left( \frac{\sin(\delta t/l)}{\delta t/l} \right)^l.
\]

Then the function \( x \mapsto f(x)w(t-x) \in L_1^2 \), and (3), with \( t = x \) yields

\[
f(x) = \sum_{n=-\infty}^{\infty} f(\lambda_n)w_{s,l}(x-\lambda_n) \frac{G(x)}{G(\lambda_n)(x-\lambda_n)}.
\]

This formula, on which we shall base our algorithm, is a version of the classical Boas–Bernstein formula, which deals with sequences \( \Lambda = \{\lambda_n\} \) of the form

\[
\lambda_n = n + h_n \quad \text{with} \quad |h_n| < M,
\]

and satisfying the separation condition \( \inf_{m \neq n} |\lambda_m - \lambda_n| > 0 \). In this more general case, (6) still holds, provided that the integer \( l \) is sufficiently large compared to the positive constant \( M \). (See, e.g., [3, p. 193]. Another version of the Boas–Bernstein formula is Beurling’s linear balayage operator; see [2, pp. 348–350].)

3. CONSTRUCTION OF THE ALGORITHM

We now fix a sequence of the form (7). This means in particular that we consider only sequences of density 1, but the general case can of course be reduced to this one by a suitable scaling. For simplicity we assume that \( M < 1/2 \). In this way, the inequality \( \lambda_n < \lambda_{n+1}, n \in \mathbb{Z} \), is preserved, and we may simplify notations when estimating the error. But the construction remains unchanged for arbitrary values of \( M \).

Note that with no oversampling, i.e., \( f \in L_1^2 \), one needs to require \( M < 1/4 \), in order to have a stable reconstruction of \( f \) from its sampled values \( \{f(\lambda_n)\} \) according to the Kadets 1/4 theorem [9]. In this case, for the usual Lagrange-type interpolation formula, qualitative estimates of the truncation error have been obtained in [8], based on certain precise estimates of infinite products due to Levinson. With minor modifications, our approach may also produce numerical estimates for this case.

We introduce a parameter \( L \) which characterizes the size of a \textit{window}: When evaluating a function \( f \in L_1^2 \) at a point \( x \), we take into account only those sampled values \( f(\lambda_n) \) for which \( |n - x| < L \). Thus we may assume that for a fixed \( x \in \mathbb{R} \), we deal with the sampling sequence

\[
\Lambda_x = \{\lambda_n(x)\}, \quad \lambda_n(x) = \begin{cases} n + h_n & \text{if } |x - n| \leq L; \\ n & \text{otherwise}. \end{cases}
\]
Since the set $\Lambda_x$ differs from $\mathbb{Z}$ only by a finite number of elements, we may consider the windowed canonical product $G(\Lambda_x; t)$ defined by (2) with $\Lambda = \Lambda_x$ and also our reconstruction formula (6) holds for each integer $l > 0$. We rewrite it as

$$f(t) = \sum f(\lambda_n(x))w(t - \lambda_n(x)) \frac{G(\Lambda_x; t)}{G(\Lambda_x; \lambda_n(x))(t - \lambda_n)}$$

$$= \sum_{|n-x| \leq L} f(\lambda_n)w(t - \lambda_n) \frac{G(\Lambda_x; t)}{G(\Lambda_x; \lambda_n)(t - \lambda_n)}$$

$$+ \sum_{|n-x| > L} f(n)w(t - n) \frac{G(\Lambda_x; t)}{G(\Lambda_x; n)(t - n)}$$

for each $t \in \mathbb{R}$.

The numerical formula used for reconstructing $f$ is therefore

$$f(x) \approx S_L(f; x) = \sum_{|n-x| \leq L} f(\lambda_n)w(x - \lambda_n) \frac{G(\Lambda_x; x)}{G(\Lambda_x; \lambda_n)(x - \lambda_n)}$$

$$= \sum_{|n-x| \leq L} f(\lambda_n)w(x - \lambda_n) \left( \frac{n - \lambda_n}{n - t} \right) \frac{\sin \pi t}{\sin \pi \lambda_n} \prod_{|k-x| \leq L} \left( \frac{\lambda_k - t}{\lambda_k - \lambda_n} \right) \left( \frac{k - \lambda_n}{k - t} \right),$$

and the remainder term is

$$R_L(f; x) = \sum_{|n-x| > L} f(n)w(x - n) \frac{G(\Lambda_x; x)}{G(\Lambda_x; n)(x - n)}.$$

In these formulas $w = w_{\lambda, t}$ is defined by (5). There remains now the problem of choosing $l$, which depends on the error accepted in the reconstruction: Given $\epsilon > 0$, find the positive integer $l$ which minimizes a valid estimate of $L$, under the constraint

$$|f(x) - S_L(f; x)| = |R_L(f; x)| \leq \|f\| \epsilon.$$

This is the problem to be considered in the next section.

Let us briefly estimate the complexity of the algorithm. Given $L$ and $x$, one needs $4L + 5$ operations to evaluate each of the $2L + 1$ terms of the sum which represents the approximate value of $f(x)$. When changing $x$ say to $x + 1$, one can use results of the previous calculations and needs $2L + 10$ operations for each of $(2L + 1)$ terms of the sum. Since the values of $L$ are relatively small (see Table 2), the linear and quadratic terms in this estimate are comparable.

4. ERROR ESTIMATES

Set

$$\psi_n(t) = \psi_n(\Lambda_x; t) = \frac{G(\Lambda_x; t)}{G(\Lambda_x; \lambda_n)(t - \lambda_n)}.$$
Then using the Cauchy–Schwarz inequality and the classical sampling theorem, we may estimate \(|R_L|\) as

\[
|R_L(f; x)| = \left| \sum_{|n-x| > L} f(n)w(x-n)\psi_n(x) \right| \leq \sum_{|n-x| > L} |f(n)w(x-n)\psi_n(x)|
\]

\[
\leq \left( \sum_{|n-x| > L} |f(n)|^2 \right)^{1/2} \left( \sum_{|n-x| > L} |w(x-n)|^2 |\psi_n(x)|^2 \right)^{1/2}
\]

\[
\leq \|f\| \left( \sum_{|n-x| > L} |w(x-n)|^2 |\psi_n(x)|^2 \right)^{1/2}.
\]

In order to choose the right \(l\), we need to take a closer look at the behavior of \(\psi_n(\Lambda x; x)\) when \(|n - x| > L\). From the representation

\[
\psi_n(\Lambda x; x) = \frac{(-1)^n \sin \pi x}{n - x} \prod_{|k-x| \leq L} \left( \frac{\lambda_k - x}{\lambda_k - \lambda_n} \right) \left( \frac{k - \lambda_n}{k - x} \right).
\]

we obtain the inequality

\[
|\psi_n(x)| \leq \sqrt{K} \frac{L^{2M-1}(2L + 1)^{M+1}}{|x-n|},
\]

where \(K\) depends on \(M\), and we have assumed \(L \geq 3\) and \(M \leq 1/2\). (The details of this straightforward but somewhat lengthy estimation are given in the Appendix.) The expression for \(\sqrt{K}\) can be obtained explicitly. We do not give it here; we just mention that \(\sqrt{K}\) changes from 0.6 for \(M = 0.1\) to 11 for \(M = 0.5\). For the “critical” value \(M = 0.25\) we get \(\sqrt{K} = 1.8\).

It follows that

\[
\sum_{|n-x| > L} |w(x-n)|^2 |\psi_n(x)|^2 \leq \sum_{|n-x| > L} \frac{\sin[\delta(x-n)/l]|l|^{2l} K L^{4M-2}(2L + 1)^{2M+2}}{|\delta(x-n)/l|^{2l}|x-n|^2}
\]

\[
\leq \frac{K l^{2l} L^{4M-2}(2L + 1)^{2M+2}}{\delta^{2l}} \sum_{|n-x| > L} \left( \frac{1}{|x-n|} \right)^{2l+2}
\]

\[
\leq 2K \frac{l^{2l} L^{4M-2}(2L + 1)^{2M+2}(L-1)^{-2l+1}}{\delta^{2l}(2l+1)}.
\]

Here we used the estimate

\[
\sum_{|n-x| > L} \left( \frac{1}{|x-n|} \right)^{2l+2} \leq 2 \int_{L-1+x}^{\infty} \left( \frac{1}{t-x} \right)^{2l+2} dt = \frac{2(L-1)^{-2l+1}}{2l+1}.
\]

Set

\[
C = 2^{M+3/2} \sqrt{K} \left( 1 + \frac{1}{L-1} \right)^{2M-1} \left( 1 + \frac{3}{2(L-1)} \right)^{M+1}.
\]

Then we get

\[
\left( \sum_{|n-x| > L} |w(x-n)|^2 |\psi_n(x)|^2 \right)^{1/2} \leq C \frac{L^{3M-1/2}}{\sqrt{2l+1}} \left( \frac{l}{\delta(L-1)} \right)^{l}.
\]
<table>
<thead>
<tr>
<th>$M$</th>
<th>$L$</th>
<th>$C$</th>
<th>$\lim_{L \to \infty} C$</th>
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In the case $M = 0$ no deviation from the integers occurs and $\psi_n(x) = \text{sinc}(\pi(x - n))$, $|\psi_n(x)| \leq |\pi(x - n)|^{-1}$. A direct estimate shows that (12) then holds with $C = \sqrt{2}/\pi$. Inequality (12) allows us to obtain an upper bound for the relative error. Given $\epsilon$ we want to pick $l$ and $L$ such that the right-hand side in (12) does not exceed $\epsilon/C$. Taking the logarithm on both sides of the equations and solving with respect to $\log L$, we get

$$\log(L - 1) = \frac{\log(\epsilon/C) + l \log \delta + 1/2 \log(2l + 1) - l \log l}{\rho - l},$$

where $\rho = 3M - 1/2$. The number $L$ gives the number of terms in the reconstruction formula. We want it to be as small as possible. The two factors in $C$ which depend on $L$ are set equal to 1 to simplify calculations. When we differentiate with respect to $L$, $C$ is then treated as a constant. The $l$ which gives the smallest $L$ is the one which solves the equation

$$\rho \log \delta + \frac{\rho - l}{2l + 1} - \rho \log l - \rho + l + \log(\epsilon/C) + 1/2 \log(2l + 1) = 0.$$   (14)

The solution is rounded off to the nearest integer. The value of $C$ is of some significance for the final result. Table 1 shows an upper bound for $C$ for various choices of $M$.

### 5. ERROR BOUNDS AND A NUMERICAL EXAMPLE

Table 2 shows different values of $L$ for different choices of error $\epsilon$, oversampling $\delta$, and deviation from the integers $M$. The pointwise error is

$$|f(x) - S_L(f; x)| \leq \|f\| \epsilon.$$  

It is interesting to see that irregular sampling does not lead to an essential increase of the number of sample points.

**Example 1.** Assume that the signal $f$ which we want to reconstruct is band-limited to $[-\pi/2, \pi/2]$, and let the sample points be of the form (7). The average sampling rate is then twice the Nyquist rate. In this case, the function $w$ can be of type at most $\pi/2$. Let us further assume that $M = 1/6$. Then the parameter $\rho = 3M - 1/2 = 0$, and $|\lambda_{k+1} - \lambda_k| < 4/3$. The acceptable relative error bound is chosen to be $10^{-1}$. The $l$ that gives the minimum number of terms $L$ in the reconstruction formula is the solution to Eq. (14)

$$-l + l + 1/2 \log(2l + 1) + \log 0.1 - \log 6 = 0 \quad \Rightarrow \quad l = 3.$$  

Using Eq. (13) we find that $L \geq 6$. 

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**Table 1**

<table>
<thead>
<tr>
<th>$M$</th>
<th>$L$</th>
<th>$C$</th>
<th>$\lim_{L \to \infty} C$</th>
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### TABLE 2
The Number of Terms Needed in the Boas–Bernstein Formula

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</table>

**Note.** The parameter $\epsilon$ is the relative error, $M$ gives the maximum deviation from the integers for the sampling set, $\delta$ is the type of $w_n$, and indicates thus the oversampling rate, and $l$ is the number found by Eq. (14) which gives the number of terms by Eq. (13).

Figure 1 shows the reconstruction of a signal which is the sum of sinc-functions of type $\pi/2$:

$$f(x) = \sum_{i=1}^{N} a_i \text{sinc}(x - b_i)/2.$$  

The function is reconstructed from the irregular sampling values

$$S_6(x) = \sum_{|\pi - x| \leq 6} f(\lambda_n)w(x - \lambda_n)\frac{G(\Lambda_{x}; x)}{G(\Lambda_{\lambda_n}; \lambda_n)(x - \lambda_n)},$$

where $w(x) = [\text{sinc}(\pi x/6)]^3$. The pointwise error $|R_6(x)| = |f(x) - S_6(x)|$ is shown in Fig. 2. The absolute value of the pointwise error turns out to be smaller than $10^{-3}$. For $L = 6$, $l = 3$, $\delta = \pi/2$, and $M = 1/6$ we have that

$$\frac{l^l (L - 1)^{3M-l-1/2}}{\delta^{l/2} l^l + 1} \approx 0.02.$$
FIG. 1. Approximated signal $S_6(x)$.  

FIG. 2. Pointwise error $|R_6(x)| = |f(x) - S_6(x)|$. 
6. APPENDIX

We now deduce the inequality (9). Denote by \( k \) the integer which is closest to \( x \). Then expression (8) becomes

\[
|\psi_n(\Lambda; x)| = \left| \frac{G(\Lambda; x)}{G(\Lambda; n)(x - n)} \right| = \left| \frac{\sin \pi x}{\pi(x - n)} \right| \prod_{|k - x| \leq L, k \neq k_x} \left| \frac{\lambda_x - x}{\lambda_x - n} \right| \left| \frac{k - n}{k - x} \right|
\]

\[
= \left| \frac{\sin \pi x}{\pi(x - n)} \right| \left( \frac{\lambda_{k_x} - x}{\lambda_{k_x} - n} \right) \left( \frac{k_x - n}{k_x - x} \right) \prod_{|k - x| \leq L, k \neq k_x} \left| \frac{\lambda_k - x}{\lambda_k - n} \right| \left| \frac{k - n}{k - x} \right|
\]

Taking into account that \(|\lambda_k - k| < M\) when \(|k - x| < L\), we obtain

\[
|A| \leq \frac{1}{|x - n|} \frac{n - k_x}{n - \lambda_{k_x}} |x - \lambda_{k_x}| \leq \frac{M + 1/2}{|x - n|} \cdot \frac{2L + 1}{2L}.
\]

and also

\[
B = \prod_{|k - x| \leq L, k \neq k_x} \left| \frac{1 + h_k/(k - x)}{1 - h_k/(k - n)} \right|
\]

\[
\leq \sum_{|k - x| \leq L, k \neq k_x} \left( \log \left( 1 + \frac{M}{|k - x|} \right) - \log \left( 1 - \frac{M}{|k - n|} \right) \right).
\]

We use integrals to estimate the sums. For the first sum, the following holds:

\[
\sum_{0.5 \leq |k - x| \leq L} \log \left( 1 + \frac{M}{|k - x|} \right)
\]

\[
\leq 2 \log(1 + 2M) + \int_{0.5}^{L} \log \left( 1 + \frac{M}{t} \right) dt
\]

\[
= 2 \left( M \log 2 + \left( \frac{1}{2} - M \right) \log(1 + 2M) + (L + M) \log \left( 1 + \frac{M}{L} \right) + M \log L \right)
\]

\[
= 2 \left( M \log 2 + \left( \frac{1}{2} - M \right) \log(1 + 2M) + M \log L + M + \frac{M^2}{L} \right).
\]

To estimate the second sum, we proceed as follows:

\[
\sum_{|n| - L}^{0.5 \leq |k - x| \leq L} - \log \left( 1 - \frac{M}{k} \right)
\]

\[
\leq - \log \left( 1 - \frac{M}{|n| - L} \right) - \log \left( 1 - \frac{M}{|n|} \right) - \int_{|n| - L}^{L} \log \left( 1 - \frac{M}{t} \right) dt
\]

\[
= - \log \left( 1 - \frac{M}{|n| - L} \right) - \log \left( 1 - \frac{M}{|n|} \right) + M \log \left| \frac{|n| + L}{|n| - L} \right|
\]

\[
+ M \log \left( 1 - \frac{M}{|n| + L} \right) - M \log \left( 1 - \frac{M}{|n| - L} \right)
\]

\[
+ (|n| - L) \log \left( 1 - \frac{M}{|n| - L} \right) - (|n| + L) \log \left( 1 - \frac{M}{|n| + L} \right).
\]
\[-(1 + M) \log(1 - M) - \log \left(1 - \frac{M}{L}\right) + M \log(2L + 1) - M - 2L \log \left(1 - \frac{M}{2L}\right)\]

Assume that \(L \geq 3\) and that \(M \leq 1/2\). When adding the two last terms of each of the above estimates, we get

\[2M + \frac{M^2}{L} - M - 2L \log \left(1 - \frac{M}{2L}\right) \leq 2M + \frac{4M^2}{5}\]

Combining these estimates, we arrive at (9). (To simplify calculations, we have estimated everything in terms of \(M\) and \(L\). We could have proceeded with the term \(M \log |(|n| + L)/(|n| - L)|^M\) would appear in the sum in (10) and an upper bound for this sum would be \(CL^{-2L-1}\). However, the \(C\) would be more complicated, and the final results are not significantly better.)

REFERENCES


