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Nonlinear Ordinary Boundary Value Problems under a Combined Effect of Periodic and Attractive Nonlinearities

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1. INTRODUCTION

Let us consider the linear boundary value problem

$$\begin{gathered} -u''(t) - u(t) = h(t),\ t \in [0, \pi], \\ u(0) = u(\pi) = 0, \end{gathered} \tag{1.1}$$

where $h \in C[0, \pi]$. The Fredholm Alternative Theorem ([9]) shows that (1.1) has solution if and only if

$$\int_0^\pi h(t)\sin(t)\, dt = 0,$$

so that, we have a precise description of the range R_0, of the operator M_0: $C_0^2[0, \pi] \to C[0, \pi]$, defined by

$$C_0^2[0, \pi] = \{u \in C^2[0, \pi]: u(0) = u(\pi) = 0\},$$
$$M_0(u) = -u'' - u, \qquad \forall\ u \in C_0^2[0, \pi].$$

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Let us suppose that we are interested in a nonlinear bounded perturbation of (1.1), i.e., in the problem

$$-u''(t) - u(t) + p(u(t)) = h(t),\ t \in [0, \pi],$$
$$u(0) = u(\pi) = 0,$$

where $p: \mathbb{R} \to \mathbb{R}$ is a continuous and bounded function. For some general kinds of nonlinearities p, the range R_p of the operator $M_p: C_0^2[0, \pi] \to C[0, \pi]$, defined by

$$M_p(u)(t) = -u''(t) - u(t) + p(u(t)), \qquad \forall\ u \in C_0^2[0, \pi],$$

contains, in a strict manner, to R_0. For instance, this is the case if p is, moreover, a nontrivial T-periodic function with zero mean value ([8]). This type of problem models, for example, the motion of a clock pendulum (see [14]) and many other situations where the nonlinear force terms are of periodic type ([17]).

Other classes of nonlinearities where $R_0 \subset R_p$ are those where $p(u)u \leq 0$, $\forall\ u \in \mathbb{R}$ ([13]). This type of problem models, for example, the motion of a particle restrained by a nonlinear spring, when the force exerted by the spring is a sum of a linear term and a nonlinear soft one, and moreover, this motion is affected by an external force h.

On the other hand, there are many mechanical models which may be considered as a combination of the two previously mentioned; i.e., the nonlinear term p is a sum of the form $p(u) = f(u) + g(u)$, where f is continuous and T-periodic with zero mean value and g is continuous, bounded, and satisfies a sign condition as above, i.e., $g(u)u \leq 0$, $\forall\ u \in \mathbb{R}$. For example, this may be the case of a pendulum attached to a rolling wheel which is restrained by a nonlinear spring (see [18]). It seems that this type of nonlinearity has not been previously considered in the literature (see [11, 13] for the case $f \equiv 0$ and [8] for the case $g \equiv 0$) and we dedicate the first part of this paper to the study of it. More precisely, we consider problems like

$$\begin{aligned} -u''(t) - u(t) + f(u(t)) + g(u(t)) = h(t),\ t \in [0, \pi], \\ u(0) = u(\pi) = 0, \end{aligned} \tag{1.2}$$

where f and g are as above. It is clear that the resulting nonlinearity, $f + g$, does not need to be of any of the two types, i.e., in general, neither is $f + g$ periodic nor does $f + g$ satisfy a sign type condition (see the remarks after Theorem 2.1).

We prove that if f is nontrivial, then $R_0 \subset R_p$, in a strict manner. Also, we show some qualitative analogies and differences with respect to the case where $g \equiv 0$.

The main difficulty to study (1.2) is caused by the resonance character of its linear part and the undetermined behaviour of the nonlinearity $f(u) + g(u)$ at $\pm\infty$. In fact, under our hypotheses, the nonlinear term in (1.2) has not, in general, limits either at $-\infty$ or at $+\infty$. Moreover, we cannot precise its sign for u sufficiently large. However, taking into account some ideas from [2, 8, 20], we are able to prove the change of sign of the bifurcation equation in the Alternative Method.

The second part of this work deals with similar problems for systems of equations, which arise from Mechanics (coupled oscillators) and also from coupled circuits theory ([15, 18]). In fact, this was the original motivation of the present paper, and if we previously treat the scalar case, it is because we think that this way is much more convenient, to understand properly the proofs presented in the case of systems of equations.

A general kind of mechanical model, which motivates our study, is given by the following description: let us consider two masses connected to each other and to two fixed points by an arrangement of three springs. Suppose that all forces other than the spring one are ignored, and that we consider one-dimensional motion of each mass along the line of the springs. Then, the linear behaviour of this system is governed by two equations of the type

$$-u''(t) - au(t) + (a-1)v(t) = 0,$$
$$-v''(t) - av(t) + (a-1)u(t) = 0,$$

where a is a real positive constant satisfying some additional restrictions (see [15]).

If we assume that, moreover, there are time-dependent external forces affecting this system and that a nonlinear force which depends on the position of v acts on the first mass u and that a nonlinear force which depends on the position of u acts on the second mass v the system becomes

$$\begin{aligned} -u''(t) - au(t) + (a-1)v(t) + f(v(t)) &= p(t), \\ -v''(t) - av(t) + (a-1)u(t) + g(u(t)) &= q(t). \end{aligned} \tag{1.3}$$

We study the Dirichlet homogeneous boundary value problem for the previous system (in the interval $[0, \pi]$), in the case where f is a continuous and T-periodic function with zero mean value, g satisfies a sign condition as in Section 2, and the kernel of the linear part of the system has dimension one. More precisely, we prove that the presence of the nonlinearities f (nontrivial) and g causes a strict enlargement of the set of external forces (p, q) for which the previous problem has solution, with respect to the case where f and g are both identically zero (this fact is very important in the applications). To the best of our knowledge, this kind of system has not been previously considered in the literature (see [12, 16] for the periodic boundary value problem for systems which are weakly coupled in the nonlinear part or they have a variational structure, and see [4, 5] for systems which may be strongly coupled with respect to the nonlinear part, but where both nonlinear terms are always as g, i.e., satisfying a sign condition).

In the proofs we use the alternative method (Liapunov–Schmidt reduction). However, after applying this technique, and for this type of nonlinearity, the main difficulty is, again, the study of the sign of the bifurcation equation (think that in the case of systems, this is usually more difficult than in the case of scalar equations). By using different ideas about connectivity, which may be seen in [1, 2, 10, 19, 22], we have been able to prove the change of sign of the bifurcation equation.

Lastly, it must be remarked that we restrict ourselves to problems of the type (1.2) or (1.3), just to avoid tedious calculations and to keep the idea clear, but, it is easily deduced from our proofs that the same techniques may be used in many other situations. For instance, some generalizations of our results to nonlinear Sturm–Liouville problems are possible. Also, one may consider, instead of system (1.3), other systems which are weakly coupled with respect to the nonlinear part (i.e., $f(u)$ and $g(v)$ instead of $f(v)$ and $g(u)$) and also, systems of n equations, where one nonlinearity is of type f and the other nonlinearities are of type g.

2. THE SCALAR EQUATION

Let us consider the bvp (1.2), where the functions f and g satisfy the following hypothesis:

[H] *$f\colon \mathbb{R} \to \mathbb{R}$ is continuous, T-periodic, and with zero mean value, i.e., $\int_0^T f(s)\,ds = 0$; $g\colon \mathbb{R} \to \mathbb{R}$ is continuous, bounded, and $g(u)u \le 0, \forall\, u \in \mathbb{R}$ (or $g(u)u \ge 0, \forall\, u \in \mathbb{R}$),*

and $h \in C[0, \pi]$.

Taking into account our purpose (we want to prove that $R_0 \subset R_{f+g}$), each $h \in C[0, \pi]$ may be written in the form $h(t) = s \sin t + \tilde{h}(t)$, $s \in \mathbb{R}$, $\tilde{h} \in C[0, \pi]$, $\int_0^\pi \tilde{h}(t) \sin t\, dt = 0$, and problem (1.2) becomes

$$-u''(t) - u(t) + f(u(t)) + g(u(t)) = s \sin t + \tilde{h}(t), \qquad t \in [0, \pi],$$
$$u(0) = u(\pi) = 0. \tag{2.1}$$

Our main result in this section is the following.

THEOREM 2.1. *Consider bvp* (2.1) *where f and g satisfy* [H] *and f is not identically zero. Then, for any given $\tilde{h}$, there exists a bounded real number interval $I_{\tilde{h}}$, which contains negative and positive values such that* (2.1) *has solution if, and only if, $s \in I_{\tilde{h}}$. Moreover, the interval $I_{\tilde{h}}$ may be closed, open, or semiopen.*

Proof. Let V denote the Banach space $V = C([0, \pi], \mathbb{R})$, with the norm $\|v\|_0 = \max_{t \in [0, \pi]} |v(t)|$, for any $v \in V$. By U we denote the Banach space $U = \{u \in V : u(0) = u(\pi) = 0\}$ equipped with the same norm as V. If we define the operators

$$L\colon \operatorname{dom} L \to V, \qquad \operatorname{dom} L = U \cap C^2[0, \pi],$$
$$Lu = -u'' - u, \ \forall\ u \in \operatorname{dom} L,$$

and

$$N\colon U \to V,$$
$$(Nu)(t) = s \sin t + \tilde{h}(t) - f(u(t)) - g(u(t)), \ \forall\ u \in U, \ \forall\ t \in [0, \pi],$$

then problem (2.1) is equivalent to solving the operator equation

$$Lu = Nu. \tag{2.2}$$

It is well known that L is a linear Fredholm mapping of index zero, so that there exist continuous projections $P\colon U \to U$ and $Q\colon V \to V$, such that $\operatorname{Im} P = \ker L$, $\operatorname{Im} L = \ker Q$, and (2.2) is equivalent to the alternative system

$$\tilde{u} = K(I - Q)N(c \sin(\cdot) + \tilde{u}) \qquad \text{(auxiliary equation)}, \tag{2.3}$$
$$QN(c \sin(\cdot) + \tilde{u}) = 0 \text{ (bifurcation equation)}, \tag{2.4}$$

where K is the inverse of the mapping $L\colon \operatorname{dom} L \cap \ker P \to \operatorname{Im} L$ and any $u \in U$ is written in the form $u(t) = \bar{u}(t) + \tilde{u}(t) = c \sin t + \tilde{u}(t)$, $c \in \mathbb{R}$,

$$\int_0^{\pi} \tilde{u}(t) \sin t \, dt = 0.$$

Applying the Schauder fixed point theorem, we get that for any fixed $c \in \mathbb{R}$, there exists at least one solution $\tilde{u} \in \ker P$ of (2.3) ([13]).

Denote by Σ the "solution set" of Eq. (2.3), i.e.,

$$\Sigma = \{(c, \tilde{u}) \in \mathbb{R} \times \ker P\colon \tilde{u} = K(I - Q)N(c \sin(\cdot) + \tilde{u})\}.$$

Taking into account that

$$Qv(t) = \left(\frac{2}{\pi}\int_0^\pi v(t)\sin t\,dt\right)\sin t,\ \forall\ v \in V,$$

the bifurcation equation (2.4) becomes

$$s = \frac{2}{\pi}\left[\int_0^\pi f(c\sin t + \tilde{u}(t))\sin t\,dt + \int_0^\pi g(c\sin t + \tilde{u}(t))\sin t\,dt\right].$$

Hence, for a given $\tilde{h}$, bvp (2.1) has solution if and only if s belongs to the range of the function $\Gamma\colon \Sigma \to \mathbb{R}$, defined by $\Gamma = \Gamma_1 + \Gamma_2$, where

$$\Gamma_1(c,\tilde{u}) = \frac{2}{\pi}\int_0^\pi f(c\sin t + \tilde{u}(t))\sin t\,dt,$$

$$\Gamma_2(c,\tilde{u}) = \int_0^\pi g(c\sin t + \tilde{u}(t))\sin t\,dt.$$

Since f and g are bounded functions, we deduce that $\Gamma(\Sigma)$ is bounded. It is also known ([2, 7]) that $\Gamma(\Sigma)$ is a connected set (i.e., an interval). Let us write $\Gamma(\Sigma) = I_{\tilde{h}}$. Next, we prove that $I_{\tilde{h}}$ contains negative and positive values. To show this, we define

$$p_1\colon \Sigma \to \mathbb{R}, \qquad p_2\colon \Sigma \to \ker P, \text{ by } p_1(c,\tilde{u}) = c,$$
$$p_2(c,\tilde{u}) = \tilde{u},\ \forall\ (c,\tilde{u}) \in \Sigma.$$

Then, from (2.3), we deduce that there is a constant $M > 0$, independent of $c \in \mathbb{R}$, such that

$$\|\tilde{u}\|_0 \le M, \qquad \|(\tilde{u})'\|_0 \le M, \qquad \|(\tilde{u})''\|_0 \le M,\ \forall\ \tilde{u} \in p_2(\Sigma). \tag{2.5}$$

Now, we may use ideas similar to [8] (Lemma 2.2), to [6] (Lemmas 3 and 4), or to [20] (formula (24)), to prove the following lemma

LEMMA 2.2. *If F is the primitive of f with zero mean value, then there exists $c_0 > 0$ such that if $(c,\tilde{u}) \in \Sigma$ with $c \ge c_0$, we have*

$$\Gamma_1(c,\tilde{u}) = \int_0^\pi \left[-F(c\sin t + \tilde{u}(t)) + F(\|c\sin(\cdot) + \tilde{u}(\cdot)\|_0)\right] \times \frac{c + \cos t\cdot\tilde{u}'(t) - \sin t\cdot\tilde{u}''(t)}{(c\cos t + \tilde{u}'(t))^2}\,dt. \tag{2.6}$$

Analogously, if $(c, \tilde{u}) \in \Sigma$ *and* $c \leq -c_0$, *then*

$$\Gamma_1(c, \tilde{u}) = \int_0^{\pi} \left[-F(c \sin t + \tilde{u}(t)) + F(-\|c \sin(\cdot) + \tilde{u}(\cdot)\|_0) \right] \times \frac{c + \cos t \cdot \tilde{u}'(t) - \sin t \cdot \tilde{u}''(t)}{(c \cos t + \tilde{u}'(t))^2} \, dt. \tag{2.7}$$

Also, from the properties of the function $\sin(\cdot)$ and (2.5), it is easily proved that if c_0 is sufficiently large, then

$$c(c \sin t + \tilde{u}(t)) \geq 0, \ \forall \ t \in [0, \pi], \ \forall \ (c, \tilde{u}) \in \Sigma \colon |c| \geq c_0. \tag{2.8}$$

Now, let $b > 0$ be any real number which satisfies

$$b > 2 \max\{c_0, 2M + T\}.$$

Then, since (2.1) is a resonance problem at the principal eigenvalue and the nonlinearity $f + g$ is bounded, it is possible to prove the existence of a connected subset Σ_1 of Σ, such that $p_1(\Sigma_1) = [b/2, b]$ (see [2, 10, 19]). Also, the set $J = \{\|c \sin(\cdot) + \tilde{u}\|_0, (c, \tilde{u}) \in \Sigma_1\}$ is a real interval and since $\|\frac{b}{2}\sin(\cdot) + \tilde{u}\|_0 \leq \frac{b}{2} + M$ and $\|b \sin(\cdot) + \tilde{v}\|_0 \geq b - M$, for each $(\frac{b}{2}, \tilde{u})$, $(b, \tilde{v}) \in \Sigma_1$, the length of J is at least $\frac{b}{2} - 2M$ which is greater than T. Therefore, there exist $(c_1, \tilde{u}_1) \in \Sigma_1$ such that $F(\|c_1 \sin(\cdot) + \tilde{u}_1\|_0) = \min_{\mathbb{R}} F$ and consequently, from (2.6), we obtain

$$\Gamma_1(c_1, \tilde{u}_1) < 0. \tag{2.9}$$

Moreover, from (2.8), we prove

$$\Gamma_2(c_1, \tilde{u}_1) \leq 0. \tag{2.10}$$

Finally, from (2.9) and (2.10), we obtain

$$\Gamma(c_1, \tilde{u}_1) < 0. \tag{2.11}$$

By a similar reasoning, we deduce the existence of a connected subset Σ_2 of Σ, such that $p_1(\Sigma_2) = [-b, -b/2]$ and the existence of an element $(c_2, \tilde{u}_2) \in \Sigma_2$, verifying

$$F(-\|c_2 \sin(\cdot) + \tilde{u}_2\|_0) = \min_{\mathbb{R}} F, \qquad \Gamma_1(c_2, \tilde{u}_2) > 0. \tag{2.12}$$

Moreover, from (2.8), we have

$$\Gamma_2(c_2, \tilde{u}_2) \geq 0. \tag{2.13}$$

Finally, from (2.12) and (2.13), we obtain

$$\Gamma(c_2, \tilde{u}_2) > 0.$$

Just to finish the prove of the theorem, we present different situations which show that the interval $I_{\tilde{h}}$ may be closed, open, or semiopen. In fact if, moreover, of the hypotheses of Theorem 2.1, the nonlinearity g fulfills

$$\lim_{|u| \to +\infty} g(u) = 0, \tag{2.14}$$

then $I_{\tilde{h}}$ is closed. To prove this, let $\{s_n\} \subset I_{\tilde{h}}$ be such that $s_n \to \inf I_{\tilde{h}}$. Then, there exists a sequence $(c_n, \tilde{u}_n) \in \Sigma$ such that $s_n = \Gamma(c_n, \tilde{u}_n)$, $\forall\, n \in \mathbb{N}$. Since $\inf I_{\tilde{h}} < 0$, by using (2.5), a generalization of the Riemann–Lebesgue lemma ([21]) and (2.14), we deduce that the sequence $\{c_n\}$ must be bounded. From (2.5) and applying a standard compactness argument ([13]), we may prove that $\inf I_{\tilde{h}} \in I_{\tilde{h}}$. Analogously, $\sup I_{\tilde{h}} \in I_{\tilde{h}}$.

However, the interval $I_{\tilde{h}}$, may, in other situations, be an open (or semiopen) interval. For instance, take $f(u) = \sin(u)$, $g(u) = -\sin(u) + h(u)$, where the function h is defined by

$$h(u) = \begin{cases} -\arctan u + \arctan(-\pi) + 1, & u \le -\pi, \\ \dfrac{-u}{\pi}, & -\pi \le u \le \pi, \\ -\arctan u + \arctan \pi - 1, & u \ge \pi. \end{cases}$$

It is easily proved that $I_{\tilde{h}} = (-\frac{\pi}{2} + \arctan \pi - 1, \frac{\pi}{2} + \arctan(-\pi) + 1)$.

Lastly, from the previous two situations, it is easy to define functions f and g such that $I_{\tilde{h}}$ is semiopen.

Remark.

(1) First, it must be pointed out that the nonlinearity $f + g$ does not need to be periodic, as f, nor satisfy a sign condition as g. This is, for example the case if $f(u) = \sin(u)$, $g(u) = -u/(1 + u^2)$. In this case, $f + g$ is not periodic and there are sequences $\{a_n\} \to +\infty$, $\{b_n\} \to +\infty$, $\{c_n\} \to -\infty$, $\{d_n\} \to -\infty$, such that $(f+g)(a_n) > 0$, $(f+g)(b_n) < 0$, $(f+g)(c_n) > 0$, $(f+g)(d_n) < 0$.

(2) An important qualitative difference between (2.1), and the case studied in [8] has been shown in Theorem 2.1, since, in [8], the interval $I_{\tilde{h}}$ is always closed. Another qualitative distinction between these two kinds of problems is related to multiplicity results.

In fact, it was first proved in [20] (and after, by a different procedure, in [8]) that if $g \equiv 0$, then the bvp (2.1) has, for $s = 0$, infinitely many solutions. Next, we show that, under the hypotheses of Theorem 2.1, this is not necessarily true. To see this, let us suppose that, in addition to the hypotheses of such theorem, we assume:

[C]: f and g are analytic functions, $f'(u) + g'(u) \geq k > -3$, $\forall\ u \in \mathbb{R}$ and

$$\lim_{|u| \to +\infty} g(u) \neq 0. \tag{2.15}$$

Then, the bvp (2.1) has, for $s = 0$, a finite number of distinct solutions since if $s = 0$, all the solutions of (2.1) are of the form $c \sin(\cdot) + \tilde{u}(c)(\cdot)$, where $c \in \mathbb{R}$, $\tilde{u}(c)$ is the unique element in $\ker P$ satisfying (2.3) (see [3]), and

$$0 = \Gamma(c, \tilde{u}(c)) \equiv H(c).$$

Now, by using the asymptotic behaviour of g and, again, the mentioned generalization of the Riemann–Lebesgue lemma ([21]), we obtain that the set $H^{-1}\{0\}$ is bounded. Also, we have proved in the theorem that H is a nonconstant function and, moreover, it is known that it is analytic ([3]). Therefore, the identity principle provides that the set $H^{-1}\{0\}$ is finite.

3. SYSTEMS OF EQUATIONS

In this section we will study the bvp

$$\begin{aligned} -u''(t) - au(t) + (a-1)v(t) + f(v(t)) &= p(t), \qquad t \in [0, \pi], \\ -v''(t) - av(t) + (a-1)u(t) + g(u(t)) &= q(t), \qquad t \in [0, \pi], \\ u(0) = u(\pi) = v(0) = v(\pi) &= 0, \end{aligned} \tag{3.1}$$

where a is a real constant and the nonlinearities f and g are as in the previous section (i.e., they satisfy hypothesis [H]). First, under some additional assumptions on the constant a, we describe the kernel and the range of the linear part of (3.1).

PROPOSITION 3.1. *Let $a \in \mathbb{R}$ be such that $(2a - 1) \notin \{n^2,\ n \in \mathbb{N}\}$ and define the Banach spaces $X = U \times U$, $Z = V \times V$, where U and V are as in the previous section. If we define the operator*

$$L_s\colon \operatorname{dom} L_s \to Z, \operatorname{dom} L_s = (U \cap C^2[0, \pi]) \times (U \cap C^2[0, \pi]),$$

$$L_s\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -u'' - au + (a-1)v \\ -v'' - av + (a-1)u \end{pmatrix}, \ \forall \begin{pmatrix} u \\ v \end{pmatrix} \in \operatorname{dom} L_s$$

then, L_s is a linear Fredholm mapping of index zero. In fact,

$$\operatorname{Ker} L_s = \left\{ c \begin{pmatrix} \sin(\cdot) \\ \sin(\cdot) \end{pmatrix} : c \in \mathbb{R} \right\}$$

and

$$\operatorname{Im} L_s = \left\{ \begin{pmatrix} p \\ q \end{pmatrix} \in Z : \int_0^{\pi} (p(t) + q(t)) \sin(t)\, dt = 0 \right\}$$

Proof. If $\binom{u}{v} \in \operatorname{dom} L_s$ is such that $L_s \binom{u}{v} = 0$, then

$$-(u''(t) - v''(t)) - (2a - 1)(u(t) - v(t)) = 0, \ \forall\ t \in [0, \pi],$$
$$(u - v)(0) = (u - v)(\pi) = 0.$$

Since $(2a - 1) \notin \{n^2 : n \in \mathbb{N}\}$, we deduce $u \equiv v$. Now, from $L_s \binom{u}{u} = 0$, we obtain

$$-u''(t) - u(t) = 0, \qquad t \in [0, \pi], \qquad u(0) = u(\pi) = 0.$$

Therefore, if $\binom{u}{v} \in \ker L_s$ then $u \equiv v$ and there exists some constant $c \in \mathbb{R}$ such that $u(t) = v(t) = c \sin(t)$, $\forall\ t \in [0, \pi]$. The reciprocal result is trivially true. Moreover, if $\binom{p}{q} \in \operatorname{Im} L_s$, then there is some element $\binom{u}{v} \in \operatorname{dom} L_s$ satisfying $L_s \binom{u}{v} = \binom{p}{q}$. Then

$$-(u''(t) + v''(t)) - (u(t) + v(t)) = p(t) + q(t), \qquad t \in [0, \pi],$$
$$(u + v)(0) = (u + v)(\pi) = 0.$$

This implies, by the Fredholm Alternative Theorem ([9]), that

$$\int_0^{\pi} (p(t) + q(t)) \sin(t)\, dt = 0. \tag{3.2}$$

Reciprocally, if $\binom{p}{q} \in Z$ satisfies (3.2), let w be the unique solution of the scalar problem

$$-w''(t) - w(t) = p(t) + q(t), \qquad t \in [0, \pi], \qquad w(0) = w(\pi) = 0$$

which fulfills $\int_0^{\pi} w(t) \sin(t)\, dt = 0$, and u the unique solution of the problem

$$\begin{aligned} -u''(t) &- (2a - 1)u(t) \\ &= p(t) + (1 - a)w(t), \qquad t \in [0, \pi], \qquad u(0) = u(\pi) = 0. \end{aligned}$$

Now, by taking $v = w - u$, we have $L_s \binom{u}{v} = \binom{p}{q}$.

Finally, any $\binom{p}{q} \in Z$ may be written as

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix} + r \begin{pmatrix} \sin(\cdot) \\ \sin(\cdot) \end{pmatrix},$$

where

$$\begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix} = \begin{pmatrix} p(\cdot) - \dfrac{1}{\pi}\left(\displaystyle\int_0^{\pi} (p(t) + q(t))\sin(t)\, dt\right)\sin(\cdot) \\ q(\cdot) - \dfrac{1}{\pi}\left(\displaystyle\int_0^{\pi} (p(t) + q(t))\sin(t)\, dt\right)\sin(\cdot) \end{pmatrix}$$

and

$$r = \frac{1}{\pi}\left(\int_0^{\pi} (p(t) + q(t))\sin(t)\, dt\right)$$

Consequently, $\operatorname{Im} L_s$ is closed in Z and $\dim\ker L_s = \operatorname{codim}\operatorname{Im} L_s = 1$.

Next, we state and prove the main result of this section

THEOREM 3.2. *Let us consider the bvp*

$$\begin{aligned} &-u''(t) - au(t) + (a-1)v(t) + f(v(t)) \\ &\qquad = r\sin t + \tilde{p}(t), \qquad t \in [0, \pi], \\ &-v''(t) - av(t) + (a-1)u(t) + g(u(t)) \\ &\qquad = r\sin t + \tilde{q}(t), \qquad t \in [0, \pi], \\ &\qquad\qquad u(0) = u(\pi) = v(0) = v(\pi) = 0, \end{aligned} \tag{3.3}$$

where $a \in \mathbb{R}$ *is such that* $(2a-1) \notin \{n^2, n \in \mathbb{N}\}$, f, g *satisfy* [H] *and* f *is not identically zero. Then for any given function* $\binom{\tilde{p}}{\tilde{q}} \in \operatorname{Im} L_s$, *there exists a bounded real number interval* $I_{(\tilde{p}, \tilde{q})}$, *which contains negative and positive values, such that if* $r \in I_{(\tilde{p}, \tilde{q})}$, *the bvp* (3.3) *has solution.*

Proof. Since L_s is a linear Fredholm mapping of index zero, there exist continuous projections $P_s\colon X \to X$ and $Q_s\colon Z \to Z$, such that $\operatorname{Im} P_s = \ker L_s$, $\operatorname{Im} L_s = \ker Q_s$. In fact, P_s is defined by

$$P_s\begin{pmatrix} u \\ v \end{pmatrix}(t) = \left(\frac{1}{\pi}\int_0^{\pi} (u(t) + v(t))\sin t\, dt\right)\begin{pmatrix} \sin t \\ \sin t \end{pmatrix}$$

and an identical formula is valid for Q_s.

Now, if we define the nonlinear operator $N_s \colon X \to Z$ by

$$N_s \begin{pmatrix} u \\ v \end{pmatrix}(t) = \begin{pmatrix} r \sin t + \tilde{p}(t) - f(v(t)) \\ r \sin t + \tilde{q}(t) - g(u(t)) \end{pmatrix}$$

then, problem (3.3) is equivalent to solving the operator equation

$$L_s U_s = N_s U_s \tag{3.4}$$

where $U_s = \binom{u}{v}$.

Writing any $U_s \in X$ in the form $U_s(t) = \overline{U_s}(t) + \tilde{U}_s(t) = d\binom{\sin t}{\sin t} + \tilde{U}_s(t)$, $d \in \mathbb{R}$, and $\tilde{U}_s \in \ker P_s$, (3.4) is equivalent to the alternative system

$$\tilde{U}_s = K_s(I - Q_s) N_s \left(d \begin{pmatrix} \sin(\cdot) \\ \sin(\cdot) \end{pmatrix} + \tilde{U}_s \right) \text{ (auxiliary equation)}, \tag{3.5}$$

$$Q_s N_s \left(d \begin{pmatrix} \sin(\cdot) \\ \sin(\cdot) \end{pmatrix} + \tilde{U}_s \right) = 0 \text{ (bifurcation equation)}, \tag{3.6}$$

where the completely continuous (see the proof of Proposition 3.1) operator K_s is the inverse of the mapping

$$L_s \colon \operatorname{dom} L_s \cap \ker P_s \to \operatorname{Im} L_s.$$

Now, we get that for any fixed $d \in \mathbb{R}$, there exists at least one solution $\tilde{U}_s \in \ker P_s$ of (3.5). In fact, if d is a given real number, the operator

$$K_s(I - Q_s) N_s \left(d \begin{pmatrix} \sin(\cdot) \\ \sin(\cdot) \end{pmatrix} + \tilde{U}_s \right)$$

is bounded from $\ker P_s$ into itself and completely continuous on bounded subsets of $\ker P_s$. Then, applying the Schauder fixed point theorem we get that for any fixed $d \in \mathbb{R}$, there exists at least one solution $\tilde{U}_s \in \ker P_s$ of (3.5).

Denote by Σ_s the "solution set" of Eq. (3.5), i.e.,

$$\Sigma_s = \left\{ \left(d, \tilde{U}_s\right) \in \mathbb{R} \times \ker P_s \colon \tilde{U}_s = K_s(I - Q_s) N_s \left(d \begin{pmatrix} \sin(\cdot) \\ \sin(\cdot) \end{pmatrix} + \tilde{U}_s \right) \right\}.$$

Taking into account the explicit expression for Q_s, the bifurcation equation (3.6) becomes

$$r = \frac{1}{\pi} \left[\int_0^\pi f\left(d \sin t + \tilde{U}_{s_2}(t)\right) \sin t \, dt + \int_0^\pi g\left(d \sin t + \tilde{U}_{s_1}(t)\right) \sin t \, dt \right],$$

where

$$\tilde{U}_s = \begin{pmatrix} \tilde{U}_{s_1} \\ \tilde{U}_{s_2} \end{pmatrix}.$$

Hence bvp (3.3) has solution if and only if r belongs to the range of the function $\Gamma_s\colon \Sigma_s \to \mathbb{R}$, defined by $\Gamma_s = \Gamma_{s_1} + \Gamma_{s_2}$, where

$$\Gamma_{s_1}\left(d, \tilde{U}_s\right) = \frac{1}{\pi}\int_0^\pi f\left(d \sin t + \tilde{U}_{s_2}(t)\right)\sin t\, dt$$

and

$$\Gamma_{s_2}\left(d, \tilde{U}_s\right) = \frac{1}{\pi}\int_0^\pi g\left(d \sin t + \tilde{U}_{s_1}(t)\right)\sin t\, dt.$$

At this point, we must remark that, contrary to the case of the scalar equation, we do not know if $\Gamma_s(\Sigma_s)$ is an interval of real numbers. In fact, for the case of systems of equations like (3.3), we do not have a result similar to Theorem 3.1 in [2] (see also [7]). However, we will prove that $\Gamma_s(\Sigma_s)$ contains an interval with positive and negative values. To see this, define

$$p_{s_1}\colon \Sigma_s \to \mathbb{R}, \qquad p_{s_2}\colon \Sigma_s \to \ker P_s,$$

by

$$p_{s_1}\left(d, \tilde{U}_s\right) = d, \qquad p_{s_2}\left(d, \tilde{U}_s\right) = \tilde{U}_s, \ \forall \ \left(d, \tilde{U}_s\right) \in \Sigma_s.$$

Then, taking into account that f and g are bounded and (3.5), we deduce that there is a constant $M_s > 0$, independent of $d \in \mathbb{R}$, such that

$$\|\tilde{U}_s\|_0 \le M_s, \qquad \|\left(\tilde{U}_s\right)'\|_0 \le M_s, \qquad \|\left(\tilde{U}_s\right)''\|_0 \le M_s, \ \forall \ \tilde{U}_s \in p_{s_2}(\Sigma_s). \quad (3.7)$$

In fact, going into detail, think that $\tilde{U}_s$ satisfies

$$\begin{aligned} -\tilde{U}''_{s_1} - a\tilde{U}_{s_1} + (a-1)\tilde{U}_{s_2} &= \tilde{p} - f\left(d \sin t + \tilde{U}_{s_2}\right) \\ &\quad - \frac{1}{\pi}\left\{\int_0^\pi \left(f\left(d \sin t + \tilde{U}_{s_2}(t)\right) + g\left(d \sin t + \tilde{U}_{s_1}(t)\right)\sin t\, dt\right\}\sin t \\ -\tilde{U}''_{s_2} - a\tilde{U}_{s_2} + (a-1)\tilde{U}_{s_1} &= \tilde{q} - g\left(d \sin t + \tilde{U}_{s_1}\right) \\ &\quad - \frac{1}{\pi}\left\{\int_0^\pi \left(f\left(d \sin t + \tilde{U}_{s_2}(t)\right) + g\left(d \sin t + \tilde{U}_{s_1}(t)\right)\sin t\, dt\right\}\sin t \\ \tilde{U}_{s_1}(0) = \tilde{U}_{s_1}(\pi) &= \tilde{U}_{s_2}(0) = \tilde{U}_{s_2}(\pi) = 0 \end{aligned}$$

i.e., a system of the form

$$-\tilde{U}''_{s_1} - a\tilde{U}_{s_1} + (a-1)\tilde{U}_{s_2} = S^1\left(t, d, \tilde{U}_{s_1}, \tilde{U}_{s_2}\right)$$
$$-\tilde{U}''_{s_2} - a\tilde{U}_{s_2} + (a-1)\tilde{U}_{s_1} = S^2\left(t, d, \tilde{U}_{s_1}, \tilde{U}_{s_2}\right)$$
$$\tilde{U}_{s_1}(0) = \tilde{U}_{s_1}(\pi) = \tilde{U}_{s_2}(0) = \tilde{U}_{s_2}(\pi) = 0,$$

where the functions S^1 and S^2 are continuous and bounded. Then, taking into account the reasonings done in Proposition 3.1, we deduce (3.7).

By using Lemma 1.2 in [2], we obtain that for every $b > 0$, there exists a closed and connected subset Σ_{s_b} of $\Sigma_s \cap ([-b, b] \times \ker P_s)$ such that $p_{s_1}(\Sigma_{s_b}) = [-b, b]$. Let us write $\Gamma_s(\Sigma_{s_b}) = I_{(\tilde{p}, \tilde{q})}$. It remains to prove that if b is chosen sufficiently large, then $I_{(\tilde{p}, \tilde{q})}$ contains negative and positive values. The main difficulty here is that if we want to use a formula like (2.6) (or (2.7)), we need that the constant c be sufficiently large, but this is not necessarily true in Σ_{s_b}. This will be overcome, with the help of the following lemma:

LEMMA 3.3. *There are connected subsets $\Sigma^1_{s_b}, \Sigma^2_{s_b}$ of Σ_{s_b} such that $p_{s_1}(\Sigma^1_{s_b}) = [\frac{b}{2}, b]$ and $p_{s_1}(\Sigma^2_{s_b}) = [-b, -\frac{b}{2}]$.*

Proof. Since $\Sigma_s \cap ([-b, b] \times \ker P_s)$ is a compact metric space, Σ_{s_b} is compact. Let us consider now the compact metric space $X^b = \Sigma_{s_b} \cap ([\frac{b}{2}, b] \times \ker P_s)$. Then, the sets $X^b \cap (\{\frac{b}{2}\} \times \ker P_s)$ and $X^b \cap (\{b\} \times \ker P_s)$ are two closed subsets which are not separated in X^b (if these two subsets are separated in X^b we would have the conclusion that Σ_{s_b} is not connected). Therefore, by [1, Corollary 4] or [22, 9.3, p. 12], there exists a connected subset $\Sigma^1_{s_b}$ of X^b such that $\Sigma^1_{s_b} \cap X^b \cap (\{\frac{b}{2}\} \times \ker P_s)$ and $\Sigma^1_{s_b} \cap X^b \cap (\{b\} \times \ker P_s)$ are both different from the empty set. Consequently, we obtain that $\Sigma^1_{s_b}$ is a connected subset of Σ_{s_b} such that $p_{s_1}(\Sigma^1_{s_b}) = [\frac{b}{2}, b]$. Analogously, one may prove the existence of $\Sigma^2_{s_b}$, a connected subset of Σ_{s_b}, such that $p_{s_1}(\Sigma^2_{s_b}) = [-b, -\frac{b}{2}]$.

Now, by doing a similar reasoning to the previous section, we obtain that if b is sufficiently large, then there exists some element $(d_1, \tilde{U}_{s_1}) \in \Sigma^1_{s_b} \subset \Sigma_{s_b}$ such that $\Gamma_s(d, \tilde{U}_{s_1}) < 0$. Also, there exists some element $(d_2, \tilde{U}_{s_2}) \in \Sigma^2_{s_b} \subset \Sigma_{s_b}$ such that $\Gamma_s(d_2, \tilde{U}_{s_2}) > 0$. Since $\Gamma_s(\Sigma_{s_b})$ must be an interval, we obtain the conclusion of the theorem.

Remarks.

(1) Let us point out that the conclusion of the previous theorem is that if $r \in I_{(\tilde{p}, \tilde{q})}$, then bvp (3.3) has at least one solution, but, as we have previously mentioned, under the hypotheses of this theorem, we are not able to prove that the set of values of the parameter r, for which (3.3) has

solution, is a real number interval. Of course, this will be the case if we add some restrictions on the nonlinearities f and g. For example, if moreover of the conditions of Theorem 3.2, f and g are lipschitzian functions with a sufficiently small Lipschitz constant, then for each $d \in \mathbb{R}$, system (3.5) will have a unique solution $\tilde{U}_s(d) \in \ker P_s$ which depends continuously on d. Therefore, in this case, $\Gamma_s(\Sigma_s)$ is an interval.

(2) It is clear that the method of the proof allows to deal with other types of bvp different from (3.3). For instance, we may study the case where the system is only linearly coupled, i.e., f is a function of u and g is a function of v (in this case, it would be possible to use a variational method as in [16], but in the case of system (3.3) this is not possible). Moreover, we may consider bvps where the kernel of the linear part is one dimensional and it is spanned by a function of the form $\binom{a_1 \sin(\cdot)}{a_2 \sin(\cdot)}$, where a_1 and a_2 are real numbers different from zero. Also, we may treat systems of n equations, where one nonlinearity is of the type of f and the others are of the type of g.

(3) There are other different problems which seem difficult to treat with our methods. For instance problems like (3.3), where both nonlinearities are of the type f. In the case of periodic boundary conditions and linearly coupled systems, these kinds of problems have been studied in [12, 16]. Notice, however, that our theorem provides the existence of solutions, not only if $r = 0$ (which is the result of [12, 16]) but also if r is sufficiently small.

(4) Some interesting questions may arise if we assume different hypotheses on the constant a in Theorem 3.2. For instance, if $a = 1$, the kernel of the linear part is of dimension two. In this case, and under the same hypotheses of Theorem 3.2 on the nonlinearities f and g, the description of the range of the corresponding nonlinear operator has not been given yet.

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