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# Marcinkiewicz–Zygmund inequalities and interpolation by spherical harmonics

Jordi Marzo [1]

*Departament de Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, Gran via 585, 08071 Barcelona, Spain*

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## Abstract

We find necessary density conditions for Marcinkiewicz–Zygmund inequalities and interpolation for spaces of spherical harmonics in $\mathbb{S}^d$ with respect to the $L^p$ norm. Moreover, we prove that there are no complete interpolation families for $p \neq 2$.



*Keywords:* Spherical harmonics; Marcinkiewicz–Zygmund inequalities; Interpolation; Landau densities; Ball multiplier; Paley–Wiener spaces

## 1. Introduction

Let $\mathbb{S}^d$ be the unit sphere in $\mathbb{R}^{d+1}$. We consider the Banach spaces $L^p(\mathbb{S}^d)$ of measurable functions defined in $\mathbb{S}^d$ such that

$$\|f\|_p^p = \int_{\mathbb{S}^d} |f(z)|^p \, d\sigma(z) < \infty,$$

if $1 \leqslant p < \infty$, and

$$\|f\|_\infty = \sup_{z \in \mathbb{S}^d} |f(z)| < \infty,$$

when $p = \infty$. Here $d\sigma$ stands for the Lebesgue surface measure in $\mathbb{S}^d$.

*E-mail address:* jmarzo@mat.ub.es.

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Now we recall some facts about spherical harmonics, see [25]. For any integer $\ell \geqslant 0$, let $\mathcal{H}_\ell$ be the space of spherical harmonics of degree $\ell$ in $\mathbb{S}^d$. Then $\mathcal{H}_\ell$ is the restriction to $\mathbb{S}^d$ of the homogeneous harmonic polynomials of degree $\ell$ in $\mathbb{R}^{d+1}$. For any integer $L \geqslant 0$ we denote the space of spherical harmonics of degree not exceeding $L$ by $\Pi_L = \operatorname{span} \bigcup_{\ell=0}^{L} \mathcal{H}_\ell$. One can see by using Stirling's formula[2] that $\dim \Pi_L = \pi_L \sim L^d$, when $L \to \infty$.

For any degree $L$ we take $m_L$ points in $\mathbb{S}^d$

$$\mathcal{Z}(L) = \{ z_{Lj} \in \mathbb{S}^d \colon 1 \leqslant j \leqslant m_L \}, \quad L \geqslant 0,$$

and assume that $m_L \to \infty$ as $L \to \infty$. This yields a triangular family of points $\mathcal{Z} = \{\mathcal{Z}(L)\}_{L \geqslant 0}$ in $\mathbb{S}^d$.

**Definition 1.1.** Let $\mathcal{Z} = \{\mathcal{Z}(L)\}_{L \geqslant 0}$ be a triangular family with $m_L \geqslant \pi_L$ for all $L$. We call $\mathcal{Z}$ an $L^p$-*Marcinkiewicz–Zygmund* family, denoted by $L^p$-MZ, if there exists a constant $C_p > 0$ such that for all $L \geqslant 0$ and $Q \in \Pi_L$,

$$\frac{C_p^{-1}}{\pi_L} \sum_{j=1}^{m_L} \left| Q(z_{Lj}) \right|^p \leqslant \int_{\mathbb{S}^d} \left| Q(\omega) \right|^p d\sigma(\omega) \leqslant \frac{C_p}{\pi_L} \sum_{j=1}^{m_L} \left| Q(z_{Lj}) \right|^p, \tag{1}$$

if $1 \leqslant p < \infty$, and

$$\sup_{\omega \in \mathbb{S}^d} \left| Q(\omega) \right| \leqslant C \sup_{j=1,\dots,m_L} \left| Q(z_{Lj}) \right|,$$

when $p = \infty$.

Then the $L^p$-norm in $\mathbb{S}^d$ of a polynomial of degree $L$ is comparable to the discrete version given by the weighted $\ell^p$-norm of its restriction to $\mathcal{Z}(L)$. In fact we observe that $\mathcal{Z}$ is $L^2$-MZ if and only if, for all $L \geqslant 0$, the normalized reproducing kernels of $\Pi_L$ centered at the points $\mathcal{Z}(L)$ form a frame in $\Pi_L$, with frame bounds independent of $L$.

A concept that can be seen as dual of MZ is that of interpolation.

**Definition 1.2.** Let $\mathcal{Z} = \{\mathcal{Z}(L)\}_{L \geqslant 0}$ be a triangular family with $m_L \leqslant \pi_L$ for all $L$. We say that $\mathcal{Z}$ is $L^p$-*interpolating*, if for all family $\{c_{Lj}\}_{L \geqslant 0,\ 1 \leqslant j \leqslant m_L}$ of values such that

$$\sup_{L \geqslant 0} \frac{1}{\pi_L} \sum_{j=0}^{m_L} |c_{Lj}|^p < \infty,$$

there exists a polynomial $Q \in \Pi_L$ such that $Q(z_{Lj}) = c_{Lj}$, $1 \leqslant j \leqslant m_L$.

Roughly speaking in order to recover the $L^p$-norm of a polynomial of degree $L$ from the evaluation at the points in $\mathcal{Z}(L)$ we need a sufficiently big number of points in $\mathcal{Z}(L)$. On the other hand, it is possible to have a spherical harmonic of degree at most $L$ attaining some prescribed

[2] Here and in what follows $\sim$ means that the ratio of the two sides is bounded from above and from below by two positive constants.

values on $\mathcal{Z}(L)$ only when $\mathcal{Z}(L)$ is sparse. When we have both MZ and interpolation the points of the family can be thought as placed in some sort of equilibrium.

**Definition 1.3.** Let $\mathcal{Z} = \{\mathcal{Z}(L)\}_{L\geqslant 0}$ be a triangular family. We say that $\mathcal{Z}$ is an $L^p$-complete interpolating family if it is both $L^p$-MZ and $L^p$-interpolating.

We denote by $d(u,v) = \arccos\langle u,v\rangle$ the geodesic distance between $u, v \in \mathbb{S}^d$, where $\langle u,v\rangle$ is the scalar product in $\mathbb{R}^{d+1}$. The ball $B(u,\theta) \subset \mathbb{S}^d$ is, therefore, the spherical cap of radius $0 < \theta < \pi$ and center $u \in \mathbb{S}^d$.

A first measure of sparsity is the uniform separation between points of the same generation. This leads to the following definition.

**Definition 1.4.** A triangular family $\mathcal{Z}$ is *uniformly separated* if there is a positive number $\epsilon > 0$ such that

$$d(z_{Lj}, z_{Lk}) \geqslant \frac{\epsilon}{L+1}, \quad \text{if } j \neq k,$$

for all $L \geqslant 0$.

The precise formulation of the sparsity requirement is expressed in terms of the following Beurling type densities [18].

**Definition 1.5.** For $\mathcal{Z}$ a triangular family in $\mathbb{S}^d$ we define the upper and lower density respectively as

$$D^-(\mathcal{Z}) = \liminf_{\alpha\to\infty}\ \liminf_{L\to\infty} \frac{\min_{z\in\mathbb{S}^d} \#(\mathcal{Z}(L) \cap B(z, \frac{\alpha}{L+1}))}{\alpha^d},$$

$$D^+(\mathcal{Z}) = \limsup_{\alpha\to\infty}\ \limsup_{L\to\infty} \frac{\max_{z\in\mathbb{S}^d} \#(\mathcal{Z}(L) \cap B(z, \frac{\alpha}{L+1}))}{\alpha^d}.$$

Now we can formulate our main result which we will prove in Section 6.

**Theorem 1.6.** *Let $1 \leqslant p \leqslant \infty$. If $\mathcal{Z}$ is an $L^p$-Marcinkiewicz–Zygmund family there exists a uniformly separated $L^p$-MZ family $\tilde{\mathcal{Z}} \subset \mathcal{Z}$ such that*

$$D^-(\tilde{\mathcal{Z}}) \geqslant \frac{2}{d!d\sqrt{\pi}} \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})}.$$

*If $\mathcal{Z}$ is an $L^p$-interpolating family then it is uniformly separated and*

$$D^+(\mathcal{Z}) \leqslant \frac{2}{d!d\sqrt{\pi}} \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})}.$$

This result together with Theorem 4.10, that shows that an interpolating family has to be uniformly separated, proves that $L^p$-complete interpolation families must have

$$D^-(\mathcal{Z}) = D^+(\mathcal{Z}) = \frac{2}{d!d\sqrt{\pi}} \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})}.$$

In order to stress the relationship between our problem and the problems of sampling and interpolation in the Paley–Wiener space, $PW^p$, of $L^p$-functions bandlimited to the unit ball, we recall some results. A reference for material on sampling and interpolation is [23].

As in the Paley–Wiener case, in the study of $L^p$-MZ and interpolation families much more is known in $d = 1$ than in $d > 1$. The main reason for such gap is that for $d = 1$ the family given by the roots of the unity is both MZ and interpolating. We recall the classical result due to A. Zygmund and J. Marcinkiewicz: there exists a constant $C_p > 0$ such that for any $q$ polynomial of degree smaller or equal than $n$

$$\frac{C_p^{-1}}{n} \sum_{j=0}^{n} \left|q(\omega_{n,j})\right|^p \leqslant \int_0^{2\pi} \left|q\left(e^{i\theta}\right)\right|^p d\theta \leqslant \frac{C_p}{n} \sum_{j=0}^{n} \left|q(\omega_{n,j})\right|^p,$$

where $\omega_{n,j}$ are the $(n+1)$th roots of the unity, see [13] or [27, Theorem 7.5, Chapter X].

In the case $d > 1$ that we deal with in this paper we do not have an even distribution of points analogous to the roots of unity, although a lot of schemes have been proposed. We refer to N.J.A. Sloane [24] for further information. In fact, in contrast with the situation for $d = 1$ we will prove the following result about complete interpolating families.

**Theorem 1.7.** *For $d > 1$, there are no $L^p$-complete interpolating families if $p \neq 2$.*

The one-dimensional case was treated by A. Zygmund and J. Marcinkiewicz and can be seen as the $\mathbb{S}^1$ analogue to the Whittaker–Kotelnikov–Shannon theorem. Moreover, there is a complete characterization for $L^p$-complete interpolating families in terms of Muckenhoupt's condition, due to C.K. Chui, X.-C. Shen and L. Zhong [3,4] analogous to that of B.S. Pavlov, Y.I. Lyubarskii and K. Seip [12,19], in the case of the Paley–Wiener space.

Also the classical results, for $d = 1$, about sampling and interpolation for Bernstein's space given by A. Beurling [1] using densities and weak limits have their counterparts for $L^p$-MZ and interpolation families in the recent results given in [18]. Indeed, it is shown in [18] that if a triangular family is $L^p$-MZ then its lower density has to be greater or equal to $1/2\pi$, and that the converse holds for families with densities greater to $1/2\pi$. The corresponding result for interpolation can be proved without a lot of effort.

Finally, for $d = 1$, one can consider the space of holomorphic polynomials in one variable as a model space and obtain a full characterization of $L^2$-MZ families in terms of the invertibility of certain Toeplitz operators, see [23, Theorem 8, p. 88].

In the Paley–Wiener case and for greater dimensions there are classical necessary conditions for sampling and interpolation in terms of densities due to H. Landau [10]. It can be easily seen that these densities cannot characterize sampling and interpolation sequences. In previous work [14] we have shown how to obtain sampling and interpolation sequences with densities arbitrarily close to the critical one (Nyquist density) for functions bandlimited in the Euclidean space. In particular this applies to functions in $PW^p$.

Concerning the question of sufficient conditions in $\mathbb{S}^d$, in 2000 H.N. Mhaskar, F.J. Narcowich and J.D. Ward [16] using the doubling weights construction due to G. Mastroianni and V. Totik [15] obtained a sufficient condition for being $L^p$-MZ in terms of a mesh norm condition that is far from being optimal.

Our main result, Theorem 1.6, can be seen as the analogue of the Paley–Wiener space result due to H. Landau [10]. Instead of using the approach provided by J. Ramanathan T. Steger [20], that was adapted in [18] to the $\mathbb{S}^1$ case, we are going to adapt the classical operator theoretic proof given by H. Landau. We deal with the case $d > 1$ but the result for $d = 1$ follows also with minor changes.

We prove also that for $p \neq 2$ there are no triangular families that are both $L^p$-MZ and interpolating. Indeed, if such a family exists one can construct a bounded multiplier that turns out to be the multiplier for the ball. Finally the well-known result of C. Fefferman [7] brings us the contradiction.

Up to here we have seen that the knowledge is similar in both spaces. Therefore the Paley–Wiener case provides us the inspiration but technically the situation is completely different. In further work we will focus on this relation.

The main technical difficulties in the case $d > 1$ is that we cannot use the techniques for holomorphic polynomials used in [18].

The outline of this paper is as follows. In the next section we summarize some well-known facts about spherical harmonics and Jacobi polynomials.

In Section 3 we calculate the traces of the concentration operator and its square over a spherical cap, which are the main tools in proving the density conditions. Controlling these quantities we can estimate the number of "big" eigenvalues of the concentration operator, and this quantity can be thought of as the local dimension of the space of spherical harmonics. Now, to get a MZ or interpolating family we will need locally to have respectively more or less points than this local dimension.

In Section 4 we prove several general results concerning MZ and interpolating families. Our main tool, Lemma 4.2, says that the $L^p$-norm of a spherical harmonic is equivalent, with constants that do not depend on the degree, to the $L^p$-norm computed in any other sphere with radius close to 1. A perturbative argument allows us to restrict ourselves to the case $p = 2$ with uniformly separated family in the proof of our main result, Theorem 1.6. We characterize also in this section the Carleson families of measures in $\mathbb{S}^d$.

In Section 5 we prove the result about nonexistence of complete interpolating families, Theorem 1.7, using the approach outlined above.

Finally, in Section 6 we prove two technical lemmas that we use to prove the main result.

## 2. Spherical harmonics

In this section we recall some facts about spherical harmonics and Jacobi polynomials, see [25,26].

Let $Z_\eta^\ell \in \mathcal{H}_\ell$ be such that for $Q \in \mathcal{H}_\ell$

$$Q(\eta) = \int_{\mathbb{S}^d} Q(\xi)\overline{Z_\eta^\ell(\xi)}\, d\sigma(\xi), \quad \xi \in \mathbb{S}^d.$$

We call it the *zonal harmonic* of degree $\ell$ with pole $\eta \in \mathbb{S}^d$. Let $\mathcal{P}(\mathbb{S}^d)$ be the linear span of $\bigcup_{\ell=0}^{\infty} \mathcal{H}_\ell$.

**Definition 2.1.** A *zonal multiplier* is a linear map from $\mathcal{P}(\mathbb{S}^d)$ into $\mathcal{C}(\mathbb{S}^d)$ which commutes with rotations.

The following explains why the term multiplier is used in this last definition.

**Theorem 2.2.** *(See [5, Chapter 3].) Let $T$ be a zonal multiplier in $\mathbb{S}^d$. For any $\ell \geqslant 0$, $Y_\ell \in \mathcal{H}_\ell$ are eigenvectors of $T$ corresponding to the same eigenvalue.*

Then for $T$ as above there exists a sequence $\{m_\ell\}_{\ell=0}^{\infty} \subset \mathbb{C}$ such that for $\sum_{\ell=0}^{N} Y_\ell \in \mathcal{P}(\mathbb{S}^d)$

$$T\left(\sum_{\ell=0}^{N} Y_\ell\right) = \sum_{\ell=0}^{N} m_\ell Y_\ell.$$

**Definition 2.3.** We say that $T$ is a *bounded zonal multiplier* if for some $1 \leqslant p < \infty$ we have $A_p > 0$ such that for any $Y \in \mathcal{P}(\mathbb{S}^d)$

$$\|TY\|_p \leqslant A_p \|Y\|_p.$$

**Definition 2.4.** We call a function in $\mathbb{S}^d$ *zonal* if it is invariant by the action of $SO(d)$, i.e. if

$$f \circ \rho(\omega) = f(\omega), \quad \omega \in \mathbb{S}^d,$$

for $\rho \in SO(d+1)$ such that $\rho N = N$.

Observe that this is equivalent to saying that $f$ is constant on

$$L_\theta = \left\{\omega \in \mathbb{S}^d \colon d(\omega, N) = \theta\right\}, \quad 0 \leqslant \theta \leqslant \pi,$$

so the value of a zonal function in one point depends only on its geodesic distance to the north pole.

For functions $f, g \in L^1(\mathbb{S}^d)$ with $g$ zonal we define the convolution product

$$(g * f)(\omega) = \int_{\mathbb{S}^d} g^\flat\big(\langle \omega, x\rangle\big) f(x)\, d\sigma(x),$$

where $g^\flat$ is the function in $[-1, 1]$ defined by

$$g^\flat\big(\langle \omega, N\rangle\big) = g(\omega).$$

From now on we denote $\dim \mathcal{H}_\ell = h_\ell$. In the Hilbert space $L^2(\mathbb{S}^d)$ we can take an orthonormal basis of $\mathcal{H}_\ell$, that we denote by $Y_\ell^1, \dots, Y_\ell^{h_\ell}$, which can be chosen in such a way that $Y_\ell^1$ is the only vector non-vanishing at the north pole. The spaces $\mathcal{H}_\ell$ are orthogonal so taking all these basis together (for $\ell = 0, \dots, L$) we get an orthonormal basis for $\Pi_L$. Given $f \in L^2(\mathbb{S}^d)$ we define its Fourier coefficients as the triangular family

$$\hat{f}(\ell, j) = \int_{\mathbb{S}^d} f(z)\overline{Y_\ell^j(z)}\, d\sigma(z),$$

for $\ell \geqslant 0$ and $1 \leqslant j \leqslant h_\ell$.

It is well known that the reproducing kernel for $\Pi_L$ is

$$K_L(u, v) = \sum_{\ell=0}^{L} \sum_{j=1}^{h_\ell} Y_\ell^j(u)\overline{Y_\ell^j(v)}, \quad u, v \in \mathbb{S}^d,$$

and that this expression does not depend on the basis.

Now we will compute the kernel $K_L$. The zonal harmonic of degree $\ell \geqslant 0$ is the reproducing kernel in $\mathcal{H}_\ell$, so

$$Z_u^\ell(v) = \sum_{j=1}^{h_\ell} Y_\ell^j(u)\overline{Y_\ell^j(v)} = \frac{h_\ell}{\sigma(\mathbb{S}^d)} P_\ell\big(d+1; \langle u, v\rangle\big),$$

where $P_\ell(d+1; x)$ is the $\ell$th Legendre polynomial in $d+1$ dimensions [17]. Using the Christoffel–Darboux formula we get

$$\sum_{\ell=0}^{L} \frac{h_\ell}{\sigma(\mathbb{S}^d)} P_\ell\big(d+1; \langle u, v\rangle\big) = \binom{d+L-1}{L} \frac{P_L(d+1; \langle u, v\rangle) - P_{L+1}(d+1; \langle u, v\rangle)}{\sigma(\mathbb{S}^d)(1 - \langle u, v\rangle)}.$$

Finally,

$$P_L(d+1; x) - P_{L+1}(d+1; x) = (1-x)\binom{L + (d-2)/2}{L}^{-1} P_L^{(d/2,(d-2)/2)}(x),$$

where $P_L^{(\alpha,\beta)}$ stands for the Jacobi polynomial of degree $L$ and index $(\alpha, \beta)$.

From now on we denote $\lambda = (d-2)/2$. So the reproducing kernel is given by

$$K_L(u, v) = \frac{C_{d,L}}{\sigma(\mathbb{S}^d)} P_L^{(1+\lambda,\lambda)}\big(\langle u, v\rangle\big),$$

where $C_{d,L} = \binom{d+L-1}{L}/\binom{L+\frac{d-2}{2}}{L}$, and using Stirling's formula one can see that $C_{d,L} \sim L^{d/2}$, if $L \to \infty$.

To estimate the $L^p$ norm of this kernel, all we need is to estimate the $L^p$-norm of the Jacobi polynomial. For the case $p = \infty$ it is well known that

$$\sup_{t\in[-1,1]} \big|P_L^{(1+\lambda,\lambda)}(t)\big| = \binom{L+\lambda+1}{L} \sim L^{d/2}.$$

For $1 \leqslant p < \infty$ we can use the estimate in [26, p. 391] and the fact that $P_L^{(1+\lambda,\lambda)}(t) = (-1)^L P_L^{(1+\lambda,\lambda)}(-t)$ to obtain, for any $v \in \mathbb{S}^d$

$$\int_{\mathbb{S}^d} \left| P_L^{(1+\lambda,\lambda)}\big(\langle u, v\rangle\big)\right|^p d\sigma(u) \sim \begin{cases} L^{d(\frac{p}{2}-1)}, & p > \frac{2d}{d+1}, \\ L^{-p/2}\log L, & p = \frac{2d}{d+1}, \\ L^{-p/2}, & p < \frac{2d}{d+1}. \end{cases} \tag{2}$$

Finally we recall an estimate that will be used later on [26, p. 198]:

$$P_L^{(1+\lambda,\lambda)}(\cos\theta) = \frac{k(\theta)}{\sqrt{L}}\left\{\cos\big((L+\lambda+1)\theta+\gamma\big) + \frac{O(1)}{L\sin\theta}\right\}, \tag{3}$$

if $c/L \leqslant \theta \leqslant \pi - (c/L)$, where

$$k(\theta) = \pi^{-1/2}\left(\sin\frac{\theta}{2}\right)^{-\lambda-3/2}\left(\cos\frac{\theta}{2}\right)^{-\lambda-1/2}, \qquad \gamma = -\left(\lambda + \frac{3}{2}\right)\frac{\pi}{2}.$$

## 3. Concentration operator

In this section we estimate the trace of the concentration operator and its square in order to obtain an estimate for the eigenvalues of this operator, Proposition 3.1. In the next section we will show how the cardinality of the set of "big" eigenvalues can be related with the density of the triangular family when it is MZ or interpolating.

Let $\mathcal{K}_A$ be the concentration operator over $A \subset \mathbb{S}^d$ defined for $Q \in \Pi_L$ and given by

$$\mathcal{K}_A Q(u) = \int_A K_L(u, v) Q(v)\, dv. \tag{4}$$

This operator results from the composition of the restriction operator

$$\begin{aligned} \Pi_L &\longrightarrow L^2(\mathbb{S}^d) \\ Q &\longmapsto \chi_A Q, \end{aligned}$$

with the orthogonal projection

$$\begin{aligned} L^2(\mathbb{S}^d) &\longrightarrow \Pi_L \\ f &\longmapsto \sum_{\ell=0}^{L}\sum_{j=1}^{h_\ell} \langle f, Y_\ell^j\rangle Y_\ell^j. \end{aligned}$$

The operator $\mathcal{K}_A$ is self-adjoint and by the spectral theorem its eigenvalues are all real and $\Pi_L$ has an orthonormal basis of eigenvectors of $\mathcal{K}_A$. We can compute the trace of this operator using $Z_u^\ell(u) = h_\ell/\sigma(\mathbb{S}^d)$ and the expression of $K_L$ as sum of zonal harmonics

$$\operatorname{tr}(\mathcal{K}_A) = \int_A K_L(u, u)\, d\sigma(u) = \pi_L \frac{\sigma(A)}{\sigma(\mathbb{S}^d)}.$$

Now we take $A$ a spherical cap with radius $\alpha/(L+1)$ and we want to obtain an estimate for $\mathrm{tr}(\mathcal{K}_A^2)$.

**Proposition 3.1.** *Let $A \subset \mathbb{S}^d$ be a spherical cap with radius $\alpha/(L+1)$ and let $\mathcal{K}_A$ be the concentration operator defined in* (4)*. Then*

$$\mathrm{tr}(\mathcal{K}_A) - \mathrm{tr}\big(\mathcal{K}_A^2\big) = O\big(\alpha^{d-1}\log\alpha\big),$$

*when $L \to \infty$, with constants depending only on $d$.*

**Remark.** The invariance of the zonal harmonic, $Z_{\rho u}^{\ell}(\rho v) = Z_u^{\ell}(v)$, for $\rho \in SO(d+1)$, gives $\mathrm{tr}(\mathcal{K}_A^2) = \mathrm{tr}(\mathcal{K}_{\rho A}^2)$.

**Proof.** Using the reproducing property we have

$$\begin{aligned}
\mathrm{tr}\big(\mathcal{K}_A^2\big) &= \int_A \int_A \big|K_L(u,v)\big|^2\, d\sigma(u)\, d\sigma(v) \\
&= \int_A \int_{S^d} \big|K_L(u,v)\big|^2\, d\sigma(u)\, d\sigma(v) - \int_A \int_{S^d\setminus A} \big|K_L(u,v)\big|^2\, d\sigma(u)\, d\sigma(v) \\
&= \int_A K_L(u,u)\, d\sigma(u) - \int_A \int_{S^d\setminus A} \big|K_L(u,v)\big|^2\, d\sigma(u)\, d\sigma(v) \\
&= \mathrm{tr}(\mathcal{K}_A) - \frac{C_{d,L}^2}{\sigma(\mathbb{S}^d)^2} \int_A \int_{\mathbb{S}^d\setminus A} \big|P_L^{(1+\lambda,\lambda)}\big(\langle u,v\rangle\big)\big|^2\, d\sigma(u)\, d\sigma(v).
\end{aligned}$$

In $\mathbb{S}^d$ we take the spherical coordinates

$$\begin{cases}
x_1 = \sin\theta_d \ldots \sin\theta_2 \sin\theta_1, \\
x_2 = \sin\theta_d \ldots \sin\theta_2 \cos\theta_1, \\
\vdots \\
x_d = \sin\theta_d \cos\theta_{d-1}, \\
x_{d+1} = \cos\theta_d,
\end{cases}$$

where $0 \leqslant \theta_k < \pi$ if $k \neq 1$ and $0 \leqslant \theta_1 < 2\pi$. Using the rotation invariance we get

$$\begin{aligned}
\int_{\mathbb{S}^d\setminus A} \big|P_L^{(1+\lambda,\lambda)}\big(\langle u,v\rangle\big)\big|^2\, d\sigma(u) &\leqslant \int_{\mathbb{S}^d\setminus B(N,d(v,\partial A))} \big|P_L^{(1+\lambda,\lambda)}\big(\langle u,N\rangle\big)\big|^2\, d\sigma(u) \\
&= \sigma\big(\mathbb{S}^{d-1}\big) \int_{d(v,\partial A)}^{\pi} \big|P_L^{(1+\lambda,\lambda)}(\cos\theta)\big|^2 \sin^{d-1}\theta\, d\theta.
\end{aligned}$$

Let $\theta_\alpha = \alpha/(L+1)$ be the radius of the spherical cap $A$ and let $v \in A$ be fixed. Since we want an asymptotic result we will take an $\alpha \gg 1$ and an even bigger $L$, in such a way that $\theta_\alpha \ll 1$. Integrating over $A$ we get

$$\int_A \int_{\mathbb{S}^d \setminus A} \left| P_L^{(1+\lambda,\lambda)}\bigl(\langle u, v\rangle\bigr)\right|^2 d\sigma(u)\, d\sigma(v)$$

$$\leqslant \sigma\bigl(\mathbb{S}^{d-1}\bigr)^2 \int_0^{\theta_\alpha} \sin^{d-1}\eta \int_{\theta_\alpha-\eta}^{\pi} \left| P_L^{(1+\lambda,\lambda)}(\cos\theta)\right|^2 \sin^{d-1}\theta\, d\theta\, d\eta.$$

Split the innermost integral depending on whether $\theta > L^{-1}$ or $\theta < L^{-1}$. In the first case (recall that $\theta_\alpha > L^{-1}$)

$$L^d \int_0^{\theta_\alpha} \sin^{d-1}\eta \int_{\theta_\alpha-\eta,\, \theta > L^{-1}}^{\pi} \left| P_L^{(1+\lambda,\lambda)}(\cos\theta)\right|^2 \sin^{d-1}\theta\, d\theta\, d\eta$$

$$\lesssim \int_0^{\alpha} \eta^{d-1} \int_{\pi - m(\alpha,\eta,L)}^{\pi} \left| P_L^{(1+\lambda,\lambda)}(\cos\theta)\right|^2 \sin^{d-1}\theta\, d\theta\, d\eta$$

$$+ \int_0^{\alpha} \eta^{d-1} \int_{m(\alpha,\eta,L)}^{\pi - m(\alpha,\eta,L)} \left| P_L^{(1+\lambda,\lambda)}(\cos\theta)\right|^2 \sin^{d-1}\theta\, d\theta\, d\eta = A1 + A2,$$

where $m(\alpha,\eta,L) = \max((\alpha-\eta)/L, 1/L)$.

For part $A1$ we use that $|P_L^{(1+\lambda,\lambda)}(x)| = O(L^\lambda)$, for $-1 \leqslant x \leqslant 0$, [26, p. 168]. Then, for a fixed $\alpha$

$$A1 \lesssim L^{2\lambda} \int_0^{\alpha} \eta^{d-1} m(\alpha,\eta,L)^d\, d\eta = L^{-2} \int_0^{\alpha} \eta^{d-1} \max(\alpha-\eta, 1)^d\, d\eta,$$

which goes to zero as $L \to \infty$.

Using the Szegő estimate (3) we get

$$A2 \lesssim \int_0^{\alpha} \eta^{d-1} \int_{m(\alpha,\eta,L)}^{\pi - m(\alpha,\eta,L)} \frac{k^2(\theta)}{L} \sin^{d-1}\theta\, d\theta\, d\eta = \int_0^{\alpha} \eta^{d-1} \int_{m(\alpha,\eta,L)}^{\pi - m(\alpha,\eta,L)} \frac{2^{d-1}}{L \sin^2 \frac{\theta}{2}}\, d\theta\, d\eta$$

$$\sim \frac{1}{L} \int_0^{\alpha} \eta^{d-1} \cot\frac{m(\alpha,\eta,L)}{2}\, d\eta \lesssim \frac{1}{L} \cot\frac{1}{L} \int_{\alpha-1}^{\alpha} \eta^{d-1}\, d\eta + \int_1^{\alpha} (\alpha-\eta)^{d-1} \frac{1}{L} \cot\frac{\eta}{L}\, d\eta$$

$$\lesssim \frac{\alpha^{d-1}}{L} \cot\frac{1}{L} + \int_1^{\alpha} \frac{(\alpha-\eta)^{d-1}}{\eta}\, d\eta = O\bigl(\alpha^{d-1}\log\alpha\bigr).$$

For the second part ($\theta < L^{-1}$) we obtain

$$\int_0^{\theta_\alpha} \sin^{d-1}\eta \int_{\theta_\alpha-\eta,\,\theta<L^{-1}}^{\pi} \left|P_L^{(1+\lambda,\lambda)}(\cos\theta)\right|^2 \sin^{d-1}\theta\, d\theta\, d\eta$$

$$= \int_{\theta_\alpha-L^{-1}}^{\theta_\alpha} \sin^{d-1}\eta \int_{\theta_\alpha-\eta}^{L^{-1}} \left|P_L^{(1+\lambda,\lambda)}(\cos\theta)\right|^2 \sin^{d-1}\theta\, d\theta\, d\eta.$$

Observe that $\eta < \theta_\alpha - L^{-1}$ would imply $\theta > L^{-1}$. Then

$$L^d \int_{\theta_\alpha-L^{-1}}^{\theta_\alpha} \sin^{d-1}\eta \int_{\theta_\alpha-\eta}^{L^{-1}} \left|P_L^{(1+\lambda,\lambda)}(\cos\theta)\right|^2 \sin^{d-1}\theta\, d\theta\, d\eta$$

$$\leqslant L^{2d} \int_{\theta_\alpha-L^{-1}}^{\theta_\alpha} \sin^{d-1}\eta \int_{\theta_\alpha-\eta}^{L^{-1}} \sin^{d-1}\theta\, d\theta\, d\eta$$

$$\sim \int_{\alpha-1}^{\alpha} \left(1-(\alpha-t)^d\right)t^{d-1}\, dt = O\left(\alpha^{d-1}\right).$$

Taking all the estimates together we get the result. □

## 4. General results about MZ and interpolating families

In this section we prove some results about MZ and interpolation triangular families. Also we characterize the families of Carleson measures for the spherical harmonics $\Pi_L$ on $\mathbb{S}^d$.

The first thing we need to show is that in calculating densities we can restrict ourselves to uniformly separated families. Following [18] we will compare the norm of a polynomial in $\mathbb{S}^d$ with the norm in a shell sufficiently small containing $\mathbb{S}^d$. This comparison result is harder than in dimension one [18, Lemma 2] because Hadamard's three circle principle is no longer available.

For $r > 0$ we denote $S_r^d = r\mathbb{S}^d$ and for a measurable function $f$ defined in $S_r^d$ we have

$$\frac{1}{r^d}\int_{S_r^d} f(\omega)\, d\sigma(\omega) = \int_{S^d} f(r\omega)\, d\sigma(\omega).$$

First we prove a result which we will use later on.

**Proposition 4.1.** *There exists a bounded zonal multiplier $T : L^p(\mathbb{S}^d) \to L^p(\mathbb{S}^d)$ for $1 \leqslant p \leqslant \infty$, such that $\|T\|_p \leqslant C < \infty$, with $C$ independent of $p$ and $L$, and such that range $T \subset \Pi_{3L}$, $T|\Pi_L = Id$.*

**Proof.** Let $g \in L^1(\mathbb{S}^d)$ be a zonal function. For any $1 \leqslant p \leqslant \infty$ we have

$$\|g * f\|_p \leqslant \|g\|_1 \|f\|_p,$$

so the operator $T_g : \mathcal{P}(\mathbb{S}^d) \to \mathcal{C}(\mathbb{S}^d)$ defined as $T_g(f) = g * f$ is bounded in $L^p(\mathbb{S}^d)$, commutes with rotations and has norm $\|g\|_1$.

Using Hölder's inequality it is easy to see that the function

$$g = \frac{\binom{2L+\lambda+1}{2L}}{\binom{L+\lambda+1}{L}} P_L^{(1+\lambda,\lambda)}\big(\langle N, \cdot\rangle\big) P_{2L}^{(1+\lambda,\lambda)}\big(\langle N, \cdot\rangle\big),$$

has $L^1$-norm independent of $L$. Also, for $f \in \mathcal{P}(\mathbb{S}^d)$

$$\begin{aligned} g * f(\omega) &= \int_{\mathbb{S}^d} g^\flat\big(\langle \omega, x\rangle\big) f(x)\, d\sigma(x) \\ &= \frac{\binom{2L+\lambda+1}{2L}}{\binom{L+\lambda+1}{L}} \int_{\mathbb{S}^d} P_L^{(1+\lambda,\lambda)}\big(\langle \omega, x\rangle\big) P_{2L}^{(1+\lambda,\lambda)}\big(\langle \omega, x\rangle\big) f(x)\, d\sigma(x), \end{aligned}$$

is a polynomial of degree $\leqslant 3L$ in $\omega$, hence range $T_g \subset \Pi_{3L}$. Finally taking the polynomial $f \in \Pi_L$ and applying the reproducing property we obtain $g * f(\omega) = f(\omega)$, so $T_g|\Pi_L = Id$. □

The next lemma shows that the $L^p$-norm of a spherical harmonic in the unit sphere is equivalent to the $L^p$-norm in any other sphere with radius close to 1.

**Lemma 4.2.** *Let $p \in [1, \infty]$ and $Q \in \Pi_L$. For any $|r - 1| \leqslant \rho/L$ there exists a constant $C$ depending only on $\rho$ and $d$ such that*

$$C\|Q\|_{L^p(\mathbb{S}^d)} \leqslant \|Q\|_{L^p(S_r^d)} \leqslant C^{-1}\|Q\|_{L^p(\mathbb{S}^d)}. \tag{5}$$

**Proof.** First we consider the right-hand side inequality. For $Q \in \Pi_L$, $|Q|^p$ is subharmonic, thus, for $0 < r < 1$ and $1 \leqslant p < \infty$, $\|Q\|_{L^p(S_r^d)} \leqslant \|Q\|_{L^p(\mathbb{S}^d)}$ [8, Theorem 2.12]. For $p = \infty$ the same inequality follows using the maximum principle.

Using the orthogonal decomposition in spherical harmonics of a harmonic functions in $\mathbb{S}^d$ it can be proved that Hadamard's three circle principle for harmonic functions holds in $L^2$-norm [9, Lemma 2.1]. Then, for $Q \in \Pi_L$, $1 < r < 1 + \rho/L$ and $R \gg 1$, we have

$$\log\|Q\|_{L^2(S_r^d)} \leqslant \left(1 - \frac{\log r}{\log R}\right)\log\|Q\|_{L^2(\mathbb{S}^d)} + \frac{\log r}{\log R}\log\|Q\|_{L^2(S_R^d)},$$

and using that $\|Q\|^2_{L^2(S_R^d)} = O(R^L)$ we obtain $\|Q\|_{L^2(S_r^d)} \leqslant e^\rho \|Q\|_{L^2(S^d)}$.

Let $Q_L \in \Pi_L$ be such that $\|Q_L\|_\infty = 1 = Q_L(N)$ and let $1 - \rho/L < r < 1$. Restricting $Q_L$ to a great circle of $\mathbb{S}^d$ through $N$ we get a trigonometric polynomial of degree at most $L$. So using Bernstein's inequality we get $Q_L(z) \geqslant 1 - \epsilon$ for all $z \in B(N, \epsilon/L)$.

We want to estimate the integral

$$Q_L(rN) = \frac{1}{\sigma(\mathbb{S}^d)} \int_{\mathbb{S}^d} \frac{1-r^2}{|rN-u|^{d+1}} Q_L(u)\,d\sigma(u).$$

For any $0<\theta<1-r$ we have

$$\frac{1-r^2}{1+r^2-2r\cos\theta} \geqslant \frac{1}{1-r},$$

then the integral over $B(N,\epsilon/L)$ is bounded below by a constant independent of $r$

$$\begin{aligned}
&\frac{1}{\sigma(\mathbb{S}^d)} \int_{B(N,\epsilon/L)} \frac{1-r^2}{|rN-u|^{d+1}} Q_L(u)\,d\sigma(u) \\
&= (1-\epsilon)\frac{\sigma(\mathbb{S}^{d-1})}{\sigma(\mathbb{S}^d)} \int_0^{\epsilon/L} \left[\frac{(1-r^2)^2}{1+r^2-2r\cos\theta}\right]^{(d+1)/2} \frac{\sin^{d-1}\theta}{(1-r^2)^d}\,d\theta \\
&\gtrsim \frac{1-\epsilon}{(1-r)^d} \int_0^{\epsilon/L} \chi_{(0,1-r)}(\theta)\sin^{d-1}\theta\,d\theta \gtrsim (1-\epsilon)\left(\frac{\epsilon}{\rho}\right)^d.
\end{aligned}$$

Since

$$\frac{(1-r^2)^2}{1+r^2-2r\cos\theta} = \frac{(1-r^2)^2}{2r(1-\cos\theta)+(1-r)^2} \leqslant \frac{2(1-r)^2}{1-\cos\theta},$$

then

$$\int_{B(N,\epsilon/L)^c} \frac{1-r^2}{|rN-u|^{d+1}} Q_L(u)\,d\sigma(u) \leqslant C(1-r)\frac{L}{\epsilon} \leqslant C\frac{\rho}{\epsilon}.$$

We have seen that there exists a constant $\delta_d>0$, depending only on $d$, such that for $0<\rho<\delta_d$, $0<1-\rho/L<r$ and $Q\in\Pi_L$ $\|Q\|_{L^\infty(\mathbb{S}^d_r)} \geqslant C_\rho \|Q\|_{L^\infty(\mathbb{S}^d)}$. Now, iterating the process, and therefore changing the constant, we can obtain the same result for arbitrary $\rho>0$ getting for any $0<1-\rho/L<r$ and $Q\in\Pi_L$

$$\|Q\|_{L^\infty(\mathbb{S}^d_r)} \geqslant C_\rho \|Q\|_{L^\infty(\mathbb{S}^d)}.$$

So the dilation operator $T_r$ in $\Pi_L$ given by $Q\mapsto Q(r\cdot)$ is such that, if we denote by $|T_r|_p$ the norm of $T_r$ defined in $(\Pi_L,\|\cdot\|_p)$, we get $|T_r|_2\leqslant e^\rho$ and $|T_r|_\infty\leqslant C_\rho$. Being $\Pi_L$ finite-dimensional spaces we always have $|T_r|_p<\infty$. By [6, Part 1, Theorem VI.10.10, p. 524] we know that $\log|T_r|_p$ is a convex function of $1/p$, then for all $2\leqslant p\leqslant\infty$ we have $|T_r|_p\leqslant\max\{C_\rho,e^\rho\}$.

For $1<p<2$ we consider the multiplier $M=M_L$ given by Proposition 4.1. Then for $Q\in\Pi_L$ and $1<p<2$,

$$\begin{aligned}\big\|T_r(Q)\big\|_{L^p(\mathbb{S}^d)} &= \sup_{\|R\|_q\leqslant 1}\big|\langle T_r(Q),R\rangle\big| = \sup_{\|R\|_q\leqslant 1}\left|\sum_{k=0}^{L} r^k\langle M_L Q_k,R\rangle\right|\\ &= \sup_{\|R\|_q\leqslant 1}\left|\sum_{k=0}^{L} r^k\langle Q_k,M_L R\rangle\right| \lesssim \sup_{\|R\|_q=1}\|Q\|_p\big\|T_r(M_L R)\big\|_q\\ &\lesssim |T_r|_q\|Q\|_p \leqslant C_\rho\|Q\|_p.\end{aligned}$$

We observe that we cannot use the projection onto $\Pi_L$ instead of $M_L$ in the calculation above because for $p \neq 2$ it is not bounded by a constant independent of $L$, see Section 5.

So far we have seen that for $1 < p \leqslant \infty$, $1 < r < 1 + \rho/L$ and $Q \in \Pi_L$

$$\|Q\|_{L^p(\mathbb{S}^d)} \leqslant C_\rho\|Q\|_{L^p(\mathbb{S}^d)}.$$

For $p = 1$ we can just take the limit.

For the left-hand side inequality in (5) with $r > 1$ we define, given $Q \in \Pi_L$, the polynomial $\tilde{Q}(\omega) = Q(r\omega)$ and apply the former result. $\square$

Integrating with respect to the radius we get the following analog of [18, Corollary 1].

**Corollary 4.3.** *Let*

$$C_{\rho,L} = \big\{\omega \in \mathbb{R}^{d+1}\colon \big||\omega| - 1\big| < \rho/L\big\}.$$

*For $Q \in \Pi_L$ and $1 \leqslant p \leqslant \infty$ we have*

$$\|Q\|^p_{L^p(\mathbb{S}^d)} \asymp L\|Q\|^p_{L^p(C_{\rho,L},dm)},$$

*where the constants depend on $\rho$ and $p$, but not on the polynomial.*

Now we want to prove that a triangular family $\mathcal{Z}$ is a finite union of uniformly separated families if and only if the left-hand inequality in (1) holds. This is the generalization to $d \geqslant 1$ of [18, Theorem 3] and will be used to show that a MZ family contains a separated family which is also MZ. The problem in proving this result comes from the fact that there is no analogue of the Bernstein inequality for spherical harmonics if $p \neq \infty$. Instead of proving our result directly, we will derive it from the next characterization for Carleson measures on $\mathbb{S}^d$ that can be of interest on their own.

**Definition 4.4.** Let $M = \{\mu_L\}_{L\geqslant 0}$ a family of measures on $\mathbb{S}^d$ and $1 \leqslant p < \infty$. We say that $M$ is an $L^p$-*Carleson* family for $\Pi_L$ if there exists a positive constant $C$ such that for any $Q \in \Pi_L$

$$\int_{\mathbb{S}^d}\big|Q(z)\big|^p\,d\mu_L(z) \leqslant C\int_{\mathbb{S}^d}\big|Q(z)\big|^p\,d\sigma(z).$$

**Theorem 4.5.** *Let $1 \leqslant p < \infty$. The family of measures $M = \{\mu_L\}_{L \geqslant 0}$ on $\mathbb{S}^d$ is $L^p$-Carleson for $\Pi_L$ if and only if there exists a $C > 0$ such that*

$$\sup_{z \in \mathbb{S}^d} \mu_L\big(B\big(z, L^{-1}\big)\big) \leqslant \frac{C}{\pi_L}. \tag{6}$$

**Remark.** We want to point out that condition (6) is independent of $p$ and that we could take balls of any other radius $\alpha/L$ for $\alpha > 0$.

**Proof.** Let $0 < m_d$ be the first extremum of the Bessel function $J_{d/2}$ and let $\eta_L$ be such that $\eta_L L \to m_d$ when $L \to \infty$. Now, using Mehler–Heine formula [26, Theorem 8.1.1] we see that there exist $\delta_d > 0$ and $L_0$ such that for $L \geqslant L_0$ and $0 \leqslant \eta \leqslant \eta_L$

$$1 \geqslant L^{-d/2} P_L^{(1+\lambda,\lambda)}(\cos\eta) \geqslant L^{-d/2} P_L^{(1+\lambda,\lambda)}(\cos\eta_L) \geqslant \delta_d > 0.$$

We argue by contradiction. Suppose that for all $n \in \mathbb{N}$ there exist $L_n$ and a geodesic ball $B_n$ with radius $m_d/L_n$ such that $\pi_{L_n}\mu_{L_n}(B_n) > n$. Let $b_n \in \mathbb{S}^d$ be the center of $B_n$ and define for $\omega \in \mathbb{S}^d$

$$K_n(\omega) = P_{L_n}^{(1+\lambda,\lambda)}\big(\langle b_n, \omega\rangle\big) \in \Pi_{L_n}.$$

For any Carleson family of measures $M$ we get

$$\begin{aligned}\big\|L_n^{-d/2} K_n\big\|_p^p &\gtrsim \int_{\mathbb{S}^d} \big|L_n^{-d/2} K_n(z)\big|^p \, d\mu_{L_n}(z) \geqslant \int_{B_n} \big|L_n^{-d/2} K_n(z)\big|^p \, d\mu_{L_n}(z) \\ &\geqslant \delta_d^p \mu_{L_n}(B_n).\end{aligned}$$

Then $L_n^{-d(p/2-1)} \| P_{L_n}^{(1+\lambda,\lambda)}(\langle b_n, \cdot\rangle)\|_p^p \geqslant Cn$ with $C$ depending on $p$ and $d$, so if we take $p \geqslant 2d/(d+1)$ this contradicts (2).

For other $p \geqslant 1$ we consider $\ell$ such that $q = \ell p > 2d/(d+1)$. Then for

$$K_n(\omega) = P_{[L_n/\ell]}^{(1+\lambda,\lambda)}\big(\langle b_n, \omega\rangle\big)^\ell \in \Pi_{L_n},$$

and spherical balls $B_n$ with radius $\ell m_d/L_n$ we have

$$\begin{aligned}L_n^{-dq/2} \big\| P_{[L_n/\ell]}^{(1+\lambda,\lambda)}\big(\langle b_n, \cdot\rangle\big)\big\|_q^q = L_n^{-dq/2} \|K_n\|_p^p &\gtrsim \int_{\mathbb{S}^d} \big|L_n^{-d\ell/2} K_n(z)\big|^p \, d\mu_{L_n}(z) \\ &\geqslant \int_{B_n} \big|L_n^{-d\ell/2} K_n(z)\big|^p \, d\mu_{L_n}(z) \geqslant \delta_d^p \mu_{L_n}(B_n)\end{aligned}$$

and this together with (2) brings us the contradiction.

Conversely, for any $z \in \mathbb{S}^d$ and $Q \in \Pi_L$ we have

$$\big|Q(z)\big|^p \leqslant C_{d,\delta} L^{d+1} \int_{\mathbb{B}(z,1/L)} \big|Q(u)\big|^p \, dm(u),$$

where $\mathbb{B}(z, 1/L)$ stands for the Euclidean ball in $\mathbb{R}^{d+1}$. Using Corollary 4.3 we have

$$\begin{aligned}\int_{\mathbb{S}^d} \left|Q(z)\right|^p d\mu_L(z) &\lesssim L^{d+1} \int_{\mathbb{S}^d} \int_{\mathbb{B}(z,1/L)} \left|Q(u)\right|^p dm(u)\, d\mu_L(z) \\ &\leqslant L^{d+1} \int_{C_{1,L}} \left|Q(u)\right|^p \int_{\mathbb{S}^d} \chi_{\mathbb{B}(z,1/L)}(u)\, d\mu_L(z)\, dm(u) \\ &\leqslant L^{d+1} \int_{C_{1,L}} \left|Q(u)\right|^p \int_{\mathbb{S}^d} \chi_{B(u/|u|,1/L)}(z)\, d\mu_L(z)\, dm(u) \\ &\leqslant \frac{C}{\pi_L} L^{d+1} \int_{C_{1,L}} \left|Q(u)\right|^p dm(u) \sim \int_{\mathbb{S}^d} \left|Q(u)\right|^p d\sigma(u). \qquad \square\end{aligned}$$

**Corollary 4.6.** *Let $1 \leqslant p < \infty$. The family $\mathcal{Z} \subset \mathbb{S}^d$ is a finite union of uniformly separated families if and only if there exists $C_p > 0$ such that for all $L \geqslant 1$ and $Q \in \Pi_L$*

$$\frac{1}{\pi_L} \sum_{j=1}^{m_L} \left|Q(z_{Lj})\right|^p \leqslant C_p \int_{\mathbb{S}^d} \left|Q(\omega)\right|^p d\sigma(\omega). \tag{7}$$

**Proof.** It is enough to take the family of measures

$$\mu_L = \frac{1}{\pi_L} \sum_{j=1}^{m_L} \delta_{z_{Lj}}, \quad L \geqslant 0,$$

and apply the previous result. $\square$

**Theorem 4.7.** *Any $L^p$-MZ family $\mathcal{Z}$ contains a uniformly separated family $\tilde{\mathcal{Z}} \subset \mathcal{Z}$ which is also an $L^p$-MZ family.*

**Proof.** First consider $1 \leqslant p < \infty$. Using Corollary 4.6 we can assume that $\mathcal{Z}$ is a finite union of $N$ uniformly $\epsilon$-separated families, that we call $\mathcal{Z}^{(j)}$, $j = 1, \ldots, N$. Now, following [22, p. 141] we can construct for $0 < \delta < \epsilon/4$ a uniformly separated family $\tilde{\mathcal{Z}} \subset \mathcal{Z}$ such that for all $L \geqslant 0$ and $j = 1, \ldots, m_L$

$$d\big(z_{Lj}, \tilde{\mathcal{Z}}(L)\big) < \delta/L.$$

Let $\tilde{z}$ be the closest point in $\tilde{\mathcal{Z}}(L)$ to $z \in \mathcal{Z}(L)$. Given $Q \in \Pi_L$ there exists $z' \in \mathbb{R}^{d+1}$ in the segment joining $z$ and $\tilde{z}$ such that

$$\left|Q(z) - Q(\tilde{z})\right| \leqslant \left|\nabla Q(z')\right| |z - \tilde{z}| \leqslant \frac{\delta}{L} \left|\nabla Q(z')\right|.$$

Differentiating Poisson's formula

$$Q(v)=\frac{1}{\sigma(\mathbb{S}^d)}\int\limits_{\partial B(z',r)}\frac{r^2-|v-z'|^2}{r|u-v|^{d+1}}Q(u)\,d\sigma_r(u),$$

and evaluating in $z'$ we obtain

$$\left|\nabla Q(z')\right|^p r^{d+p}\leqslant C\|Q\|^p_{L^p(\partial B(z',r))}$$

where $C$ only depends on $p$ and $d$. Integrating with respect to $r$ in $[0,\epsilon/2L]$ we get

$$\left|\nabla Q(z')\right|^p\leqslant C_\epsilon L^{d+p+1}\int\limits_{\mathbb{B}(z',\epsilon/2L)}\left|Q(v)\right|^p dm(v).$$

Observe that the balls $\mathbb{B}(z',\epsilon/2L)$ are mutually disjoint therefore

$$\begin{aligned}\|Q\|^p_{L^p(\mathbb{S}^d)}&\sim\frac{1}{\pi_L}\sum_{z\in\mathcal{Z}(L)}\left|Q(z)\right|^p\lesssim\frac{1}{\pi_L}\sum_{j=1}^N\sum_{z\in\mathcal{Z}^{(j)}(L)}\left(\left|Q(z)-Q(\tilde z)\right|^p+\left|Q(\tilde z)\right|^p\right)\\&\leqslant\frac{1}{\pi_L}\sum_{j=1}^N\delta^pL^{d+1}\int\limits_{C_{\epsilon/2,L}}\left|Q(v)\right|^p dm(v)+\frac{CN}{\pi_L}\sum_{z\in\tilde{\mathcal{Z}}(L)}\left|Q(z)\right|^p\\&\lesssim C_{\epsilon,p,d,N}\delta^p\|Q\|^p_{L^p(\mathbb{S}^d)}+\frac{CN}{\pi_L}\sum_{z\in\tilde{\mathcal{Z}}(L)}\left|Q(z)\right|^p.\end{aligned}$$

We finish by taking $\delta$ small enough. The reverse inequality follows from Corollary 4.6.

For $p=\infty$ take $\epsilon>0$ such that $2C\epsilon<1$, where $C$ is the constant in the MZ inequality. Let $u,v\in\mathbb{S}^d$ be such that $d(u,v)<\epsilon/L$. Bernstein's inequality for trigonometric polynomials applied to the restriction of $Q$ to a great circle gives us

$$\left|Q(u)-Q(v)\right|\leqslant\varepsilon\|Q\|_\infty,$$

for $Q\in\Pi_L$. Now it is easy to construct a $\tilde{\mathcal{Z}}(L)\subset\mathcal{Z}(L)$ such that $d(u,v)>\epsilon/L$ for $u,v\in\tilde{\mathcal{Z}}(L)$ and any $z\in\mathcal{Z}(L)$ belongs to a ball of center one point in $\tilde{\mathcal{Z}}(L)$ and radius $\epsilon/L$. We denote $\tilde{\mathcal{Z}}(L)=\{z_{Lk_j}\}_{j=1,\dots,N}$ and for $Q\in\Pi_L$

$$\begin{aligned}\|Q\|_\infty&\leqslant C\sup_{z\in\mathcal{Z}(L)}\left|Q(z)\right|=C\max_{j=1,\dots,N}\sup_{z\in\mathcal{Z}(L),\,d(z,z_{Lk_j})<\epsilon/L}\left|Q(z)\right|\\&\leqslant C\epsilon\|Q\|_\infty+C\max_{z\in\tilde{\mathcal{Z}}(L)}\left|Q(z)\right|.\end{aligned}$$

So we obtain a $\epsilon$-uniformly separated family $\tilde{\mathcal{Z}}$ such that for $Q\in\Pi_L$

$$\|Q\|_\infty\leqslant 2C\max_{z\in\tilde{\mathcal{Z}}(L)}\left|Q(z)\right|.\qquad\square$$

Proposition 3.1 works only when $p=2$. For other $p\in[1,\infty]$ we use a perturbative result.

**Definition 4.8.** Given a family $\mathcal{Z}$ and $\delta>0$, we denote by $\mathcal{Z}_\delta$ the family $\mathcal{Z}_\delta(L)=\mathcal{Z}(L_{1+\delta})$, where $L_{1+\delta}=[(1+\delta)L]$.

**Lemma 4.9.** *Let $p\in[1,\infty]$ and $\mathcal{Z}$ be a uniformly separated $L^p$-MZ family, then for $\delta>0$ and $q\in[1,\infty]$ the family $\mathcal{Z}_\delta$ is $L^q$-MZ.*

**Proof.** Using Riesz–Thorin theorem on interpolation of operators, see [6, Part 1, p. 524], it is enough to show that $\mathcal{Z}_\delta$ is an $L^q$-MZ family for $q=1,\infty$. Fixed $z\in\mathbb{S}^d$ the evaluation operator $e_z(Q_L)=Q_L(z)$ defined in $(\Pi_L,\|\cdot\|_p)$ can be written as

$$e_z(Q_L)=\frac{1}{\pi_L}\sum_{j=1}^{m_L}Q_L(z_{Lj})a_{Lj}(z),$$

where $a_{Lj}(z)\in\mathbb{C}$ are such that $\sum_{j=1}^{m_L}|a_{Lj}(z)|^{p'}<C\pi_L$, where $1/p+1/p'=1$. Let $p_{L_\delta}(t)$ be a polynomial in one variable of degree $L_\delta$ such that $p_{L_\delta}(1)=1$ and

$$\int_{-1}^{1}\left|p_{L_\delta}(t)\right|^p\left(1-t^2\right)^\lambda\,dt=1.$$

We have

$$Q_L(z)=\frac{1}{\pi_{L_{1+\delta}}}\sum_{j=1}^{m_{L_{1+\delta}}}Q_L(z_{L_{1+\delta}j})p_{L_\delta}(z\cdot z_{L_{1+\delta}j})a_{L_{1+\delta}j}(z),$$

so

$$\begin{aligned}\left|Q_L(z)\right|&\leqslant C\sup_j\left|Q_L(z_{L_{1+\delta}j})\right|\left(\frac{1}{\pi_{L_{1+\delta}}}\sum_{j=1}^{m_{L_{1+\delta}}}\left|p_{L_\delta}(z\cdot z_{L_{1+\delta}j})\right|^p\right)^{1/p}\\&\leqslant C\left(\int_{-1}^{1}\left|p_{L_\delta}(t)\right|^p\left(1-t^2\right)^\lambda\,dt\right)^p\sup_j\left|Q_L(z_{L_{1+\delta}j})\right|.\end{aligned}$$

For $q=1$ we take $p_{L_\delta}(t)$ polynomial of degree $L_\delta$ in one variable such that $p_{L_\delta}(1)=1$ and

$$\int_{-1}^{1}\left|p_{L_\delta}(t)\right|\left(1-t^2\right)^\lambda\,dt=\pi_{L_{1+\delta}}^{-1}$$

and we get the result

$$\int_{\mathbb{S}^d} \left|Q_L(z)\right| d\sigma(z) \leqslant C \sum_{j=1}^{m_{L_{1+\delta}}} \left|Q_L(z_{L_{1+\delta}j})\right| \int_{\mathbb{S}^d} \left|p_{L_\delta}(z \cdot z_{L_{1+\delta}j})\right| d\sigma(z). \qquad \square$$

Finally we prove the corresponding result for interpolation. But first we want to estimate the norm of the evaluation operator. As in the proof of Theorem 4.6 we have for $Q \in \Pi_L$ and $u \in \mathbb{S}^d$

$$\begin{aligned}\left|Q(u)\right|^p &\lesssim L^{d+1} \int_{B(u,1/L)} \left|Q(v)\right|^p d\sigma(v) \leqslant L^{d+1} \int_{C_{1,L}} \left|Q(v)\right|^p dm(v)\\ &\sim L^d \int_{\mathbb{S}^d} \left|Q(v)\right|^p d\sigma(v),\end{aligned}$$

so

$$\pi_L^{-1} \|Q\|_\infty^p \lesssim \|Q\|_p^p.$$

**Theorem 4.10.** *If $\mathcal{Z}$ is an interpolation family for $L^p$, then it is uniformly separated.*

**Proof.** Standard arguments based on the open mapping theorem for Banach spaces, see [22], show that the interpolation can be done with polynomials $P_L$ such that

$$\|P_L\|^p \lesssim \frac{1}{\pi_L} \sum_{j=0}^{m_L} \left|P_L(z_{Lj})\right|^p.$$

Then, for a given $L_0 \geqslant 0$ and $1 \leqslant j_0 \leqslant \pi_{L_0}$, we can take polynomials $P_{L_0 j_0} \in \Pi_{L_0}$ such that $P_{L_0 j_0}(z_{Lj}) = \delta_{LL_0}\delta_{jj_0}$ and $\|P_{L_0 j_0}\|_p^p \lesssim \pi_L^{-1}$. Then for $j \neq j_0$ restricting the polynomial to a great circle and using Bernstein's inequality for trigonometric polynomials

$$\begin{aligned}1 = \left|P_{L_0 j_0}(z_{L_0 j_0}) - P_{L_0 j_0}(z_{L_0 j})\right| &\leqslant \sup_\gamma |D_T P_{L_0 j_0}| d(z_{L_0 j_0}, z_{L_0 j})\\ &\leqslant L_0 \|P_{L_0 j_0}\|_\infty d(z_{L_0 j_0}, z_{L_0 j}) \lesssim L_0 \pi_L^{1/p} \|P_{L_0 j_0}\|_p d(z_{L_0 j_0}, z_{L_0 j})\\ &\lesssim L_0 d(z_{L_0 j_0}, z_{L_0 j}),\end{aligned}$$

where $D_T$ stands for any unitary tangential derivative. $\square$

**Lemma 4.11.** *Let $p \in [1,\infty]$ and let $\mathcal{Z}$ be an $L^p$-interpolation family. For $\delta > 0$ and $q \in [1,\infty]$ $\mathcal{Z}_{-\delta}$ (as in Definition 4.8) is an $L^q$-interpolation family.*

**Proof.** As in the previous lemma we will show that $\mathcal{Z}_{-\delta}$ is an $L^q$-interpolation family for $q = 1, \infty$. The hypothesis implies that there exist polynomials $Q_{L_{1-\delta},j} \in \Pi_{L_{1-\delta}}$ such that

$$Q_{L_{1-\delta},j}(z_{L_{1-\delta},k}) = \delta_{jk}, \quad 1 \leqslant j,k \leqslant m_{L_{1-\delta}},$$

with

$$\|Q_{L_{1-\delta},j}\|_p^p \lesssim \pi_{L_{1-\delta}}^{-1}.$$

Now take polynomials $p_{L_\delta}$ in one variable of degree $L_\delta$, such that $p_{L_\delta}(1) = 1$,

$$\int_{-1}^{1} \left|p_{L_\delta}(t)\right|^{p'} \left(1 - t^2\right)^\lambda \, dt = \pi_{L_{1-\delta}}^{-1} \quad \text{for } \frac{1}{p} + \frac{1}{p'} = 1.$$

Given a triangular family $\{c_{L_{1-\delta}j}\}_{L,j}$ such that

$$\frac{1}{\pi_{L_{1-\delta}}} \sum_{j=1}^{m_{L_{1-\delta}}} |c_{L_{1-\delta}j}| < C,$$

construct the polynomial

$$Q_L(z) = \sum_{j=1}^{m_{L_{1-\delta}}} c_{L_{1-\delta}j} Q_{L_{1-\delta},j}(z) p_{L_\delta}(z \cdot z_{L_{1-\delta}j}) \in \Pi_L,$$

which satisfies $Q_L(z_{L_{1-\delta}j}) = c_{L_{1-\delta}j}$ and

$$\begin{aligned}
\int_{\mathbb{S}^d} \left|Q_L(z)\right| d\sigma(z) &\leqslant \sum_{j=1}^{m_{L_{1-\delta}}} |c_{L_{1-\delta}j}| \|Q_{L_{1-\delta},j}\|_p \left\| p_{L_\delta}\left(\langle \cdot, z_{L_{1-\delta}j}\rangle\right)\right\|_{p'} \\
&\lesssim \frac{1}{\pi_{L_{1-\delta}}} \sum_{j=1}^{m_{L_{1-\delta}}} |c_{L_{1-\delta}j}|.
\end{aligned}$$

For $q = \infty$ we take polynomials $p_{L_\delta}$ as before, but with

$$\int_{-1}^{1} \left|p_{L_\delta}(t)\right| \left(1 - t^2\right)^\lambda \, dt = \pi_{L_{1-\delta}}^{-1}.$$

And defining $Q_L$ as before we obtain the interpolation property and

$$\left|Q_L(z)\right| \leqslant C \sup_j |c_{L_{1-\delta},j}| \sum_{j=1}^{m_{L_{1-\delta}}} \left|p_{L_\delta}\left(\langle z, z_{L_\delta}\rangle\right)\right| \leqslant C \sup_j |c_{L_{1-\delta},j}|. \qquad \square$$

## 5. There are no complete interpolation families in $L^p$ for $p \neq 2$

In this section we show that there are no $L^p$-complete interpolation families for $p \neq 2$. We construct, using transference methods (see [2, Theorem 1.1]), a projection in $L^p(\mathbb{S}^d)$ that yields a bounded ball multiplier in $L^p(\mathbb{R}^d)$. Finally the celebrated result of C. Fefferman [7] says that this can happen only for $p = 2$.

**Proof of Theorem 1.7.** We argue by contradiction. Let $\mathcal{Z}$ be an $L^p$-complete interpolation family. By Theorem 4.10 we know that it is uniformly separated. Let $\epsilon > 0$ be the separation constant. Let $\ell^p_L$ be the vector space of $\{c_j\} \in \mathbb{C}^{L^d}$ with norm given by $\|\{c_j\}\|^p_{\ell^p_L} = \frac{1}{L^p}\sum_{j=1}^{L^p} |c_j|^p$. For $L \geqslant 0$ we consider the map $R_L : L^p(\mathbb{S}^d) \to \ell^p_L$ defined as

$$L^p(\mathbb{S}^d) \ni f \longmapsto \big\{\big\langle f, L^{-d} K_L(\cdot, z_{Lj}) K_{2L}(\cdot, z_{Lj})\big\rangle\big\}_{j=1,\dots,L^d}.$$

We want to show that $R_L$ is bounded for $p = 1, \infty$, with constant independent of $L$. So let $f \in L^1(\mathbb{S}^d)$,

$$\begin{aligned}
&\frac{1}{L^d}\sum_{j=1}^{L^d}\big|\big\langle f, L^{-d} K_L(\cdot, z_{Lj}) K_{2L}(\cdot, z_{Lj})\big\rangle\big| \\
&\quad\leqslant \frac{1}{L^d}\sum_{j=1}^{L^d}\int_{\mathbb{S}^d}\big|f(\omega)\big|\left|\frac{K_L(\omega, z_{Lj})}{L^d} K_{2L}(\omega, z_{Lj})\right| d\omega \\
&\quad\leqslant \|f\|_1 \frac{1}{L^d}\sup_{\omega\in\mathbb{S}^d}\sum_{j=1}^{L^d}\left|\frac{K_L(\omega, z_{Lj})}{L^d} K_{2L}(\omega, z_{Lj})\right|.
\end{aligned}$$

Let $\omega \in \mathbb{S}^d$ be fixed. Then

$$\begin{aligned}
\sum_{j=1}^{L^d}\left|\frac{K_L(\omega, z_{Lj})}{L^d} K_{2L}(\omega, z_{Lj})\right| &\sim \sum_{j=1}^{L^d}\big|P_L^{(1+\lambda,\lambda)}\big(\langle z_{Lj}, \omega\rangle\big) P_{2L}^{(1+\lambda,\lambda)}\big(\langle z_{Lj}, \omega\rangle\big)\big| \\
&\leqslant L^d + \sum_{j\in\mathcal{I}}\big|P_L^{(1+\lambda,\lambda)}\big(\langle z_{Lj}, \omega\rangle\big) P_{2L}^{(1+\lambda,\lambda)}\big(\langle z_{Lj}, \omega\rangle\big)\big| + L^{d-2},
\end{aligned}$$

where $\mathcal{I}$ are the indices $j$ such that $\frac{\epsilon}{2(L+1)} \leqslant d(\omega, z_{Lj}) \leqslant \pi - \frac{\epsilon}{2(L+1)}$. Observe that there are only two points $z_{Lj}$ such that $j \notin \mathcal{I}$ (one on each cap), and the value of the polynomial is bounded by the local maximum. In between we use Szegő's estimate (3) to get

$$\sum_{j\in\mathcal{I}}\big|P_L^{(1+\lambda,\lambda)}\big(\langle z_{Lj}, \omega\rangle\big) P_{2L}^{(1+\lambda,\lambda)}\big(\langle z_{Lj}, \omega\rangle\big)\big| \lesssim \frac{1}{L}\sum_{j\in\mathcal{I}} k^2\big(d(z_{Lj}, \omega)\big).$$

Using rotation invariance we can suppose that $\omega = N$. The function $k$ is decreasing in $(0, \pi/2)$ and a lot bigger around 0 than around $\pi$. Then to increase the sum we place the points $z_{Lj}$, the closer the better, in "bands" around the north pole. Coarse estimates using the uniform separation yields $\#\mathcal{I} = O(L^d)$, a maximum of $O(L^{d-1})$ "bands" and $O(\frac{L}{\epsilon}\sin\frac{\ell\epsilon}{L})$ points in the $\ell$th "band," if we start counting from $N$. So

$$\frac{1}{L}\sum_{j\in\mathcal{I}} k^2\bigl(d(\omega, z_{Lj})\bigr) \lesssim \frac{1}{L}\sum_{\ell=1}^{L^{d-1}} \left(\frac{L}{\epsilon}\right)^{d-1} \sin^{d-1}\frac{\ell\epsilon}{L} k^2\left(\frac{\ell\epsilon}{L}\right) \lesssim L^d \sum_{\ell=1}^{L^{d-1}} \frac{1}{\ell^2},$$

and we get

$$\|R_L f\|_{\ell^1_L} = \frac{1}{L^d}\sum_{j=1}^{L^d} \bigl|\bigl\langle f, L^{-d} K_L(\cdot, z_{Lj}) K_{2L}(\cdot, z_{Lj})\bigr\rangle\bigr| \lesssim \|f\|_1,$$

where the constant depends on $\epsilon$ but is independent of $L$. To prove the $L^\infty$ case is a lot easier:

$$\begin{aligned} L^{-d}\bigl\|K_L(\cdot, z_{Lj})K_{2L}(\cdot, z_{Lj})\bigr\|_1 &\sim \bigl\|P_L^{(1+\lambda,\lambda)}\bigl(\langle\cdot, z_{Lj}\rangle\bigr)P_{2L}^{(1+\lambda,\lambda)}\bigl(\langle\cdot, z_{Lj}\rangle\bigr)\bigr\|_1 \\ &\leqslant \bigl\|P_L^{(1+\lambda,\lambda)}\bigl(\langle\cdot, z_{Lj}\rangle\bigr)\bigr\|_2\bigl\|P_L^{(1+\lambda,\lambda)}\bigl(\langle\cdot, z_{Lj}\rangle\bigr)\bigr\|_2 = \sigma\bigl(\mathbb{S}^d\bigr). \end{aligned}$$

Now let $E_L$ be the map from $\ell^p_L$ to $\Pi_L$, sending $v = \{v_j\} \in \ell^p_L$ to $P_L \in \Pi_L$ such that $P_L(z_{Lj}) = v_j$. By hypothesis $\|E_L(v)\|_p \asymp \|v\|_{\ell^p_L}$, so $E_L \circ R_L$ is bounded from $L^p(\mathbb{S}^d)$ to $L^p(\mathbb{S}^d)$ for $p = 1, \infty$ and by Riesz–Thorin theorem on interpolation of operators, see [6, Part 1, p. 524], we get that it is bounded for all $1 \leqslant p \leqslant \infty$. Denoting $\mathcal{P}_L = E_L \circ R_L$ we get $\mathcal{P}_{L|\Pi_L} = I_{\Pi_L}$.

Following [21, Theorem 1] we define

$$\mathfrak{P}_L f = \int_{SO(d+1)} \nu^{-1}\mathcal{P}_L \nu f\, d\nu,$$

that turns out to be a projection from $L^p(\mathbb{S}^d)$ to $\Pi_L$, commuting with rotations and such that $\|\mathfrak{P}_L\| \leqslant \|\mathcal{P}_L\|$.

According to Theorem 2.2 we have $\mathfrak{P}_L Y = m_\ell Y$, for $Y \in \mathcal{H}_\ell$ and for $m_\ell \in \mathbb{C}$. The properties of $\mathfrak{P}_L$ impose that $m_\ell = 1$ for $\ell \leqslant L$ and zero otherwise. So $\mathfrak{P}_L f$ is just the sum of the orthogonal projections of $f$ over $\mathcal{H}_\ell$ (denoted by $P_{\mathcal{H}_\ell} f$) for $\ell = 0, \ldots, L$.

Now we can put

$$\mathfrak{P}_L f = \sum_{j=0}^{\infty} m_L(\ell) P_{\mathcal{H}_\ell} f,$$

with $m_L(\ell) = m(\frac{\ell}{L})$ and $m(|x|) = \chi_{\mathbb{B}}(x)$. The sequence $\{m_L(\ell)\}_{\ell\geqslant 0}$ defining a multiplier in $L^p(\mathbb{S}^d)$ with

$$\sup_{L\geqslant 0} \|\mathfrak{P}_L\|_p < \infty,$$

where $m_0(\ell) = \delta_{0\ell}$.

Now using the transference result in [2, Theorem 1.1] we see that the multiplier in $L^p(\mathbb{R}^d)$ given by

$$f \longmapsto \mathcal{F}^{-1}(\chi_{\mathbb{B}} \mathcal{F} f),$$

is bounded. Finally C. Fefferman's result [7] says that this is only possible for $p = 2$. $\square$

## 6. Proofs

We need some notation and two technical lemmas before proving Theorem 1.6.

Given $L \geqslant 0$ and $\alpha > 0$, let $A_L$, $A_L^+$ and $A_L^-$ be the geodesic balls centered at the north pole with respective radius $\alpha/(L+1)$, $(\alpha+\epsilon)/(L+1)$ and $(\alpha-\epsilon)/(L+1)$, where $\epsilon$ will denote the separation constant.

Denote the eigenvalues of the concentration operator $\mathcal{K}_{A_L}$ as

$$1 > \lambda_1^L \geqslant \cdots \geqslant \lambda_{\pi_L}^L > 0.$$

**Lemma 6.1.** *Let $\mathcal{Z}$ be a $\epsilon$-uniformly separated $L^2$-MZ family and let*

$$N_L = \#\big(\mathcal{Z}(L) \cap A_L^+\big).$$

*There exists a constant $0 < \gamma < 1$ independent of $\alpha$ and $L$ such that*

$$\lambda_{N_L+1}^L \leqslant \gamma.$$

**Remark.** In the conditions of Lemma 6.1

$$\#\big\{\lambda_j^L > \gamma\big\} \leqslant N_L = \#\big(\mathcal{Z}(L) \cap A_L^+\big) \leqslant \#\big(\mathcal{Z}(L) \cap A_L\big) + C\big(1 + o\big(\alpha^d\big)\big), \quad \alpha \to \infty,$$

where the constant $C$ depends on $d$ and $\epsilon$. This follows from the estimates $L^d \sigma(A_L^+ \setminus A_L) = 1 + o(\alpha^d)$ if $\alpha \to \infty$ and

$$\#\big(\mathcal{Z}(L) \cap \big(A_L^+ \setminus A_L\big)\big) \frac{\epsilon^d}{L^d} \lesssim \sigma\big(A_L^+ \setminus A_L\big).$$

**Lemma 6.2.** *Let $\mathcal{Z}$ be an $L^2$-interpolation family and let*

$$n_L = \#\big(\mathcal{Z}(L) \cap A_L^-\big).$$

*There exists a constant $0 < \delta < 1$ independent of $\alpha$ and $L$ such that*

$$\lambda_{n_L-1}^L \geqslant \delta.$$

**Remark.** In the conditions of Lemma 6.2 we have, as before,

$$\#\big(\mathcal{Z}(L) \cap A_L\big) - C\big(1 + o\big(\alpha^d\big)\big) \leqslant n_L = \#\mathcal{Z}(L) \cap A_L^- \leqslant \#\big\{\lambda_j^L \geqslant \delta\big\} + 1.$$

**Proof of Theorem 1.6.** Using Theorems 4.7 and 4.10 we can suppose that $\mathcal{Z}$ is a uniformly separated family. Now given $\eta > 0$ and taking either $\mathcal{Z}_\eta$ or $\mathcal{Z}_{-\eta}$ we have by Lemmas 4.11 and 4.9 that our family is respectively $L^2$-MZ or interpolating. Now we relabel the family as before and defining the measures $d\mu_L = \sum_{j=1}^{\pi_L} \delta_{\lambda_j^L}$ we have

$$\operatorname{tr}(\mathcal{K}_{A_L}) = \int_0^1 x\, d\mu_L(x) \quad \text{and} \quad \operatorname{tr}(\mathcal{K}_{A_L}^2) = \int_0^1 x^2\, d\mu_L(x).$$

Let $\mathcal{Z}$ be an $L^2$-MZ and let $\gamma$ be given by Lemma 6.1. We get

$$\begin{aligned}\#\{\lambda_j^L > \gamma\} &= \int_\gamma^1 d\mu_L(x) \geqslant \int_0^1 x\, d\mu_L(x) - \frac{1}{1-\gamma}\int_0^1 x(1-x)\, d\mu_L(x)\\ &= \operatorname{tr}(\mathcal{K}_{A_L}) - \frac{1}{1-\gamma}\big(\operatorname{tr}(\mathcal{K}_{A_L}) - \operatorname{tr}(\mathcal{K}_{A_L}^2)\big).\end{aligned}$$

The remark following Lemma 6.1 and Proposition 3.1 yields

$$\frac{\#(\mathcal{Z}(L)\cap A_L) + C(1+o(\alpha^d))}{\alpha^d} \geqslant \frac{\pi_L \sigma(A_L)}{\alpha^d \sigma(\mathbb{S}^d)} - \frac{O(\alpha^{d-1}\log\alpha)}{\alpha^d(1-\gamma)},$$

and taking limits we get, for any $\eta > 0$,

$$D^-(\mathcal{Z}_\eta) \geqslant \frac{2}{d!d\sqrt{\pi}} \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})},$$

what implies the result.

Assume now that $\mathcal{Z}$ is an $L^2$-interpolation family and let $\delta > 0$ be the value provided by Lemma 6.2. Using the estimate of Proposition 3.1 we get

$$\begin{aligned}\#\{\lambda_j^L \geqslant \delta\} &\leqslant \frac{-1}{\delta}\operatorname{tr}(\mathcal{K}_{A_L}^2) + \frac{1+\delta}{\delta}\operatorname{tr}(\mathcal{K}_{A_L})\\ &= \operatorname{tr}(\mathcal{K}_{A_L}) + \frac{1}{\delta}\big(\operatorname{tr}(\mathcal{K}_{A_L}) - \operatorname{tr}(\mathcal{K}_{A_L}^2)\big) = \frac{\pi_L \sigma(A_L)}{\sigma(\mathbb{S}^d)} + \frac{1}{\delta} O\big(\alpha^{d-1}\log\alpha\big).\end{aligned}$$

Using as before the remark following Lemma 6.2 and taking limits we get for any $\eta > 0$

$$D^+(\mathcal{Z}_\eta) \leqslant \frac{2}{d!d\sqrt{\pi}} \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})},$$

what finishes the proof. □

In the proof of Lemmas 6.1 and 6.2 we follow [11]. For the definition of the Gegenbauer polynomials and related notions see [17].

Given $\delta > 0$ consider the functions

$$h(\omega) = \big(\frac{L}{\delta}\big)^d \chi_{B(N, \frac{\delta}{2(L+1)})}(\omega), \quad \omega \in \mathbb{S}^d. \tag{8}$$

The polynomial $Y_\ell^1$ (a multiple of the Legendre harmonic) is just the Gegenbauer polynomial $C_\ell^{\frac{d-1}{2}}$ normalized in the $L^2$-norm. Applying Funk–Hecke theorem to $h$ we get

$$\begin{aligned}\hat{h}(\ell, 1) &= \left(\frac{L}{\delta}\right)^d \int\limits_{\mathbb{S}^d} \chi_{(\cos \frac{\delta}{2(L+1)}, 1)}\big(\langle \omega, N\rangle\big) Y_\ell^1(\omega)\, d\sigma(\omega) \\ &= \frac{L^d \sigma(\mathbb{S}^{d-1})}{\delta^d C_\ell^{\frac{d-1}{2}}(1) \|C_\ell^{\frac{d-1}{2}}(\langle N, \cdot\rangle)\|_2} \int\limits_0^{\frac{\delta}{2(L+1)}} C_\ell^{\frac{d-1}{2}}(\cos\theta) \sin^{d-1}\theta\, d\theta.\end{aligned}$$

Given $f \in L^2(\mathbb{S}^d)$, $0 \leqslant \ell$, $1 \leqslant m \leqslant h_\ell$, and applying Funk–Hecke as before, we deduce that

$$\begin{aligned}(f * h)\hat{}(\ell, m) &= \int\limits_{\mathbb{S}^d} (f * h)(\omega) \overline{Y_\ell^m(\omega)}\, d\sigma(\omega) \\ &= \frac{L^d}{\delta^d} \int\limits_{\mathbb{S}^d} f(u) \left( \int\limits_{\mathbb{S}^d} \chi_{(\cos \frac{\delta}{2(L+1)}, 1)}\big(\langle u, \omega\rangle\big) \overline{Y_\ell^m(\omega)}\, d\sigma(\omega) \right) d\sigma(u) \\ &= \big\| C_\ell^{\frac{d-1}{2}}\big(\langle N, \cdot\rangle\big) \big\|_2 \hat{h}(\ell, 1) \hat{f}(\ell, m)\end{aligned}$$

thus

$$\big|(f * h)\hat{}(\ell, m)\big| = \frac{C_{L,\delta} \sigma(\mathbb{S}^{d-1})}{C_\ell^{\frac{d-1}{2}}(1) \sigma(\mathbb{S}^d)} \big|\hat{f}(\ell, m)\big| \left| \int\limits_0^{\frac{\delta}{2(L+1)}} C_\ell^{\frac{d-1}{2}}(\cos\theta) \sin^{d-1}\theta\, d\theta \right|.$$

Now we want to show that for $0 \leqslant \ell \leqslant L$ and $\delta$ sufficiently small

$$\left| \int\limits_0^{\frac{\delta}{2(L+1)}} C_\ell^{\frac{d-1}{2}}(\cos\theta) \sin^{d-1}\theta\, d\theta \right| \gtrsim C_\ell^{\frac{d-1}{2}}(1) \left(\frac{\delta}{L}\right)^d, \tag{9}$$

and in particular for all $Q \in \Pi_L$

$$\big|(Q * h)\hat{}(\ell, m)\big| \gtrsim \big|\hat{Q}(\ell, m)\big|, \quad 0 \leqslant \ell \leqslant L,\ 1 \leqslant m \leqslant h_\ell.$$

To prove (9) let $x_\ell$ be the largest zero in $[-1, 1]$ of $C_\ell^{\frac{d-1}{2}}$. It is known that $x_\ell \sim \cos C/L$, for some constant $C > 0$, so for $\delta$ sufficiently small independent of $L$, the polynomial $C_\ell^{\frac{d-1}{2}}$ has no zeros in the spherical cap centered in $N$ with radius $\delta/2(L+1)$ [26]. The integral in (9) can be written as

$$\int\limits_{B(N,\frac{\delta}{2(L+1)})} C_\ell^{\frac{d-1}{2}}\big(\langle\omega, N\rangle\big)\,d\sigma(\omega),$$

and for $d(\omega, N) < \delta/2(L+1)$

$$C_\ell^{\frac{d-1}{2}}\big(\langle\omega, N\rangle\big) \geqslant C_\ell^{\frac{d-1}{2}}(1)\left(1-\frac{2(L+1)d(\omega,N)}{\delta}\right),$$

or equivalently

$$C_\ell^{\frac{d-1}{2}}(x) \geqslant C_\ell^{\frac{d-1}{2}}(1)\left(1-\frac{2(L+1)\arccos x}{\delta}\right),$$

if $\cos\frac{\delta}{2(L+1)} \leqslant x \leqslant 1$. This can be deduced using the concavity of the polynomial and the convexity of the function in the right-hand side of the last expression. So

$$\begin{aligned}
&\int\limits_{B(N,\frac{\delta}{2(L+1)})} C_\ell^{\frac{d-1}{2}}(\omega\cdot N)\,d\sigma(\omega)\\
&\quad\geqslant \int\limits_{B(N,\frac{\delta}{2(L+1)})} C_\ell^{\frac{d-1}{2}}(1)\left(1-\frac{2(L+1)d(\omega,N)}{\delta}\right)d\sigma(\omega)\\
&\quad\sim \int_0^{\frac{\delta}{2(L+1)}} C_\ell^{\frac{d-1}{2}}(1)\sin^{d-1}\eta\left(1-\frac{2(L+1)\eta}{\delta}\right)d\eta\\
&\quad= C_\ell^{\frac{d-1}{2}}(1)\int_0^1 \frac{\delta}{2(L+1)}\sin^{d-1}\left(\frac{\delta}{2(L+1)}(1-\eta)\right)d\eta\\
&\quad\gtrsim C_\ell^{\frac{d-1}{2}}(1)\left(\frac{\delta}{2(L+1)}\right)^d,
\end{aligned}$$

and (9) follows.

**Proof of Lemma (6.1).** Let $Q \in \Pi_L$, let $0 < \delta < \epsilon$, where $\epsilon > 0$ is the separation constant of $\mathcal{Z}$ and let $h$ be as in (8). Defining $g = Q * h \in \Pi_L$ we have

$$\|Q\|^2 = \sum_{\ell=0}^{L}\sum_{k=1}^{h_\ell}\big|\hat{Q}(\ell,k)\big|^2 \lesssim \sum_{\ell=0}^{L}\sum_{k=1}^{h_\ell}\big|(Q*h)\hat{\ }(\ell,k)\big|^2 = \|g\|^2 \lesssim \frac{1}{\pi_L}\sum_{k=1}^{m_L}\big|g(z_{Lj})\big|^2.$$

Applying Schwarz's inequality, we get

$$\left|g(z_{Lj})\right|^2 = \left| \int\limits_{\nu \in SO(d+1)} Q(\nu N) h\left(\nu^{-1} z_{Lj}\right) d\nu \right|^2 \leqslant \frac{\|h\|^2_{L^2(\mathbb{S}^d)}}{\sigma(\mathbb{S}^d)} \int\limits_{d(\nu N, z_{Lj}) < \frac{\epsilon}{2(L+1)}} \left|Q(\nu N)\right|^2 d\nu.$$

Now suppose that

$$g(z_{Lj}) = 0, \quad \text{for every } z_{Lj} \in A_L^+,$$

and denote by $\mathcal{I}$ the set of indices of those points $z_{Lj}$ where $g$ vanishes. Then

$$\begin{aligned} \|Q\|^2 &\lesssim \frac{1}{\pi_L} \sum_{j \notin \mathcal{I}} \left|g(z_{Lj})\right|^2 \leqslant \frac{\|h\|^2_{L^2(\mathbb{S}^d)}}{\pi_L} \sum_{j \notin \mathcal{I}} \int\limits_{d(\nu N, z_{Lj}) < \frac{\epsilon}{2(L+1)}} \left|Q(\nu N)\right|^2 d\nu \\ &\leqslant C_\delta \int\limits_{\mathbb{S}^d \setminus A_L} \left|Q(\omega)\right|^2 d\sigma(\omega), \end{aligned}$$

where we have used the separation in the last inequality.

Now we consider an orthonormal basis of eigenvectors $G_j^L$, corresponding to the eigenvalues $\lambda_j^L$ and let $c_j^L$ in

$$Q(z) = \sum_{j=1}^{N_n+1} c_j^L G_j^L \in \Pi_L,$$

be such that $g(z_{Lj}) = (Q * h)(z_{Lj}) = 0$ for $z_{Lj} \in A_L^+$. Then

$$\begin{aligned} \lambda_{N_L+1}^L \sum_{j=0}^{N_L+1} \left|c_j^L\right|^2 &\leqslant \sum_{j=0}^{N_L+1} \lambda_j^L \left|c_j^L\right|^2 = \|\chi_{A_L} Q\|^2 = \|Q\|^2 - \|\chi_{\mathbb{S}^d \setminus A_L} Q\|^2 \\ &\leqslant \left(1 - \frac{1}{C_\delta}\right) \sum_{j=0}^{N_L+1} \left|c_j^L\right|^2, \end{aligned}$$

and we get the result. □

**Proof of Lemma 6.2.** Let $\widetilde{\Pi}_L$ be the subspace of those polynomials in $\Pi_L$ vanishing in $\mathcal{Z}(L)$. Let $Q_j \in \Pi_L \ominus \widetilde{\Pi}_L$ be such that

$$Q_j(z_{Lj'}) = \delta_{jj'},$$

and let $h$ be as in (8) with $0 < \delta < \epsilon$ where $\epsilon > 0$ is the separation constant of $\mathcal{Z}$.

Let $\tilde{Q}_j \in \Pi_L$ be such that $Q_j(\omega) = (\tilde{Q}_j * h)(\omega)$, and for

$$Q \in \operatorname{span}\left\{\tilde{Q}_j \colon z_{Lj} \in A_L^-\right\}$$

we take $g = Q * h$.

It is clear that $g \in \Pi_L \ominus \widetilde{\Pi}_L$ and vanishes in those points such that $z_{Lj} \notin A_L^-$. Now following the same steps of Lemma 6.2 and using that $g \in \Pi_L \ominus \widetilde{\Pi}_L$ we get

$$\|Q\|^2 \leqslant C \int_{A_L} \left|Q(\omega)\right|^2 d\sigma(\omega).$$

Applying Weyl–Courant's lemma [6, Part 2, p. 908],

$$\lambda_{k-1}^L \geqslant \inf_{Q \in \Pi_L, Q \in E} \frac{\|\chi_{A_L} Q\|^2}{\|Q\|^2}, \quad \text{if } \dim E = k.$$

Taking $E = \operatorname{span}\{\tilde{Q}_j \colon z_{Lj} \in A_L^-\}$, that has dimension $n_L$, we get the result. □


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