A Pedagogical Discussion on Neutrino Wave-Packet Evolution

Cheng-Hsien Li*, Yong-Zhong Qian

School of Physics and Astronomy, University of Minnesota, Minneapolis, Minnesota 55455, USA

Abstract

We present a pedagogical discussion on the time evolution of a Gaussian neutrino wave packet in free space. A common treatment is to keep momentum terms up to the quadratic order in the expansion of the energy-momentum relation so that the Fourier transform can be evaluated analytically via Gaussian integrals. This leads to a solution representing a flat Gaussian distribution with a constant longitudinal width and a spreading transverse width, which suggests that special relativity would be violated if the neutrino wave packet were detected on its edge. However, we demonstrate that by including terms of higher order in momentum the correct geometry of the wave packet is restored. The corrected solution has a spherical wave front so that it complies with special relativity.

Keywords: neutrino wave packet, quantum mechanics, special relativity

1. Introduction

After the OPERA Collaboration reported their false superluminal neutrino results in 2011, the authors of [1] claimed that the off-axis neutrinos may be detected earlier than those traveling along the source-detector axis due to the spreading of a neutrino wave packet (WP) in the transverse direction. For a WP with a finite size, its momentum uncertainty allows it to expand transversely as it propagates in space. These authors claimed that, when the edge of an off-axis WP reaches the detector, the distance traveled by the center of the WP is shorter than the source-detector distance, and therefore, a superluminal effect is observed as depicted in Fig. 1.

The argument was soon questioned by the author of [2], who pointed out that a Gaussian neutrino WP will asymptotically develop a spherical wave front. Consequently, the arrival time of an off-axis neutrino WP will be the same as one traveling along the source-detector axis. However, the argument of [2] was based on the description of a Gaussian beam that is localized only in the transverse direction but not in the longitudinal direction. The spherical wave front was inferred from the phase term in the Gaussian beam solution, which is well known in laser optics.

The evolution of an initial WP can be found by two different but equivalent methods. The first method involves solving a wave equation suitable to the context. The massless Klein-Gordon equation is appropriate in the case of neutrinos or photons propagating in free space. The solution usually requires some

*Corresponding author. Tel.: +1-612-624-2327

Email addresses: cli@physics.umn.edu (Cheng-Hsien Li), qian@physics.umn.edu (Yong-Zhong Qian)
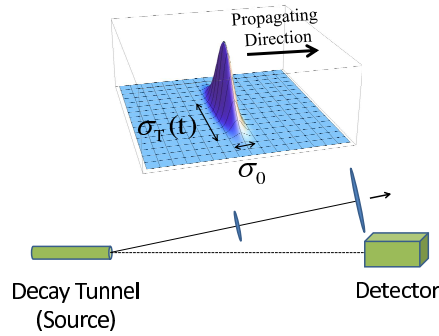
Fig. 1. An off-axis neutrino WP may be detected due to its spreading in the transverse direction.

approximation to this equation and employs an ansatz with desired boundary conditions (e.g., [3, 4]). The second method involves evolving each Fourier mode of the initial WP with a specific angular frequency and superposing each mode at a later time (e.g., [5, 6, 7, 8]). Tackling an oscillatory integral is the major difficulty of the second method.

For pedagogical purpose, we revisit the time evolution of a Gaussian WP with a sharply peaked momentum distribution. In particular, we review the necessary computational techniques of the second method and extend the solution commonly found in literature by including higher-order terms in the expansion of the energy-momentum relation. We demonstrate explicitly that these terms lead to the spherical shape of the WP. We end with a brief discussion of the corrected solution.

2. Time Evolution of a Gaussian Neutrino WP

In order to simplify the calculations, we assume an initial neutrino WP with a momentum distribution sharply peaked around the average momentum. Specifically, at time $t = 0$, the neutrino state is an isotropic Gaussian WP with a spatial width σ_0 corresponding to a momentum width $(2\sigma_0)^{-1}$. Its average momentum is $\vec{k}_0 = k_0\hat{z}$ with $k_0\sigma_0 \gg 1$. Under the above assumption, the spinor part of the neutrino wave function is essentially a constant factor and can be neglected, and only the evolution of the scalar part needs to be followed. In addition, we assume that the neutrino is massless and without flavor oscillations. The calculations for massive neutrinos are straightforward but introduce unnecessary complications to the discussion. Natural units are used throughout the presentation.

The position-space wave function for the initial neutrino WP is

$$\Psi(\vec{r}, 0) = \frac{1}{(2\pi\sigma_0^2)^{3/4}} \exp\left(-\frac{r^2}{4\sigma_0^2} + ik_0 z\right), \tag{1}$$

and its Fourier transform is

$$\tilde{\Psi}(\vec{k}) = (8\pi\sigma_0^2)^{3/4} \exp\left[-\sigma_0^2 k_\perp^2 - \sigma_0^2(k_z - k_0)^2\right], \tag{2}$$

where $k_\perp^2 \equiv k_x^2 + k_y^2$. At a later time, each plane-wave component of the WP evolves with a specific angular frequency and gains a phase factor $\exp(-i\omega t)$, where $\omega = |\vec{k}|$ is the dispersion relation (the energy-momentum relation for a massless neutrino). Superposing each plane-wave component at $t > 0$, we can express the resulting wave function as

$$\Psi(\vec{r}, t) = \int \frac{d^3\vec{k'}}{(2\pi)^3} (8\pi\sigma_0^2)^{3/4} \exp\left[-\sigma_0^2 k'^2 - i\omega_{\vec{k'}+\vec{k}_0} t + i(\vec{k'} + \vec{k}_0) \cdot \vec{r}\right], \tag{3}$$

where $\vec{k}' = \vec{k} - \vec{k}_0$. To simplify the notation, we will use $\vec{k}$ for $\vec{k}'$ below. Noting that the integrand in the above equation is significant only for $|\vec{k}| \lesssim O(\sigma_0^{-1}) \ll k_0$, we expand the dispersion relation around $\vec{k} = 0$:

$$\omega_{\vec{k}+\vec{k}_0} \equiv \omega_q + \delta\omega = k_0 \left(1 + \frac{k_z}{k_0} + \frac{k_\perp^2}{2k_0^2} - \frac{k_z k_\perp^2}{2k_0^3} + \frac{4k_z^2 k_\perp^2 - k_\perp^4}{8k_0^4} + \cdots \right), \tag{4}$$

where ω_q represents the terms up to the quadratic order in the parentheses and $\delta\omega$ those beyond.

If we approximate $\omega_{\vec{k}+\vec{k}_0}$ as ω_q, the exponent in Eq. (3) can be expressed as quadratic polynomials of k_x, k_y, and k_z. Upon completing the square for each momentum variable, we can evaluate Eq. (3) via Gaussian integrals to obtain the zeroth-order solution

$$\Psi^{(0)}(\vec{r}, t) = \frac{1}{(2\pi\sigma_0^2)^{3/4} \left[1 + it(2k_0\sigma_0^2)^{-1} \right]} \exp \left\{ -\frac{x^2 + y^2}{4\sigma_0^2 \left[1 + it(2k_0\sigma_0^2)^{-1} \right]} - \frac{(z-t)^2}{4\sigma_0^2} + ik_0(z-t) \right\}. \tag{5}$$

This solution is a plane wave with momentum $\vec{k}_0$ modulated by a complex envelope that is traveling along the z-direction at the speed of light. The modulus squared of Eq. (5), or the probability density, essentially corresponds to a "flat disc" with a time-dependent transverse width $\sigma_T(t) = \sigma_0 \sqrt{1 + t^2(4k_0^2\sigma_0^4)^{-1}}$ as shown in Fig. 1.

To go beyond the zeroth-order solution, we include $\delta\omega$ in $\omega_{\vec{k}+\vec{k}_0}$ and use $\exp(i\delta\omega t) = \sum_{n=0}^\infty (i\delta\omega t)^n/n!$ to rewrite Eq. (3) as

$$\Psi(\vec{r}, t) = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} (8\pi\sigma_0^2)^{3/4} \exp \left\{ -\sigma_0^2 k^2 - i\omega_q t + i(\vec{k} + \vec{k}_0) \cdot \vec{r} \right\}$$
$$\times \left[1 + \frac{it}{2k_0^2} k_z k_\perp^2 - \frac{it}{8k_0^3} \left(4k_z^2 k_\perp^2 - k_\perp^4 \right) + \text{higher orders} \right], \tag{6}$$

which again can be evaluated via Gaussian integrals. The correction to $\Psi^{(0)}(\vec{r}, t)$ can be efficiently computed by using the following identity:

$$\frac{\int_{-\infty}^\infty \mathrm{d}k \, k^n e^{-a(k-b)^2+c}}{\int_{-\infty}^\infty \mathrm{d}k \, e^{-a(k-b)^2+c}} = \sum_{l=0}^n \binom{n}{l} b^{n-l} \times \begin{cases} 0, & \text{for odd } l, \\ (l-1)!!/(2a)^{l/2}, & \text{for even } l. \end{cases} \tag{7}$$

The complex coefficients a, b, and c in the above equation are obtained from completing the square for each momentum variable in the exponent of Eq. (6). In principle, $\Psi(\vec{r}, t)$ can be calculated to any desired order: $\Psi(\vec{r}, t) = \Psi^{(0)}(\vec{r}, t) + \Psi^{(1)}(\vec{r}, t) + \cdots$. For illustration, we present here only the leading-order correction:

$$\Psi^{(1)}(\vec{r}, t) = -\frac{t}{4k_0^2\sigma_0^3} \times \frac{(z-t)}{\sigma_0 \left[1 + it(2k_0\sigma_0^2)^{-1} \right]} \left\{ 1 - \frac{x^2 + y^2}{4\sigma_0^2 \left[1 + it(2k_0\sigma_0^2)^{-1} \right]} \right\} \Psi^{(0)}(\vec{r}, t), \tag{8}$$

which is obtained from the second term in the square brackets of Eq. (6).

3. Discussion

A comparison of the contour of constant probability density for $\Psi^{(0)}(\vec{r}, t)$ and $\Psi(\vec{r}, t)$ is made in Fig. 2. We can observe that significant deviation of $\Psi(\vec{r}, t)$ from $\Psi^{(0)}(\vec{r}, t)$ appears when the phase correction $\delta\omega t$ in Eq. (6) is no longer negligible. The leading correction term in Eq. (8) is proportional to $(z-t)$, which gives rise to the asymmetry between the space-like and time-like regions, thereby "bending" the flat Gaussian distribution. Thus the initial Gaussian WP develops a spherical wave front after higher-order terms in the phase are taken into account.
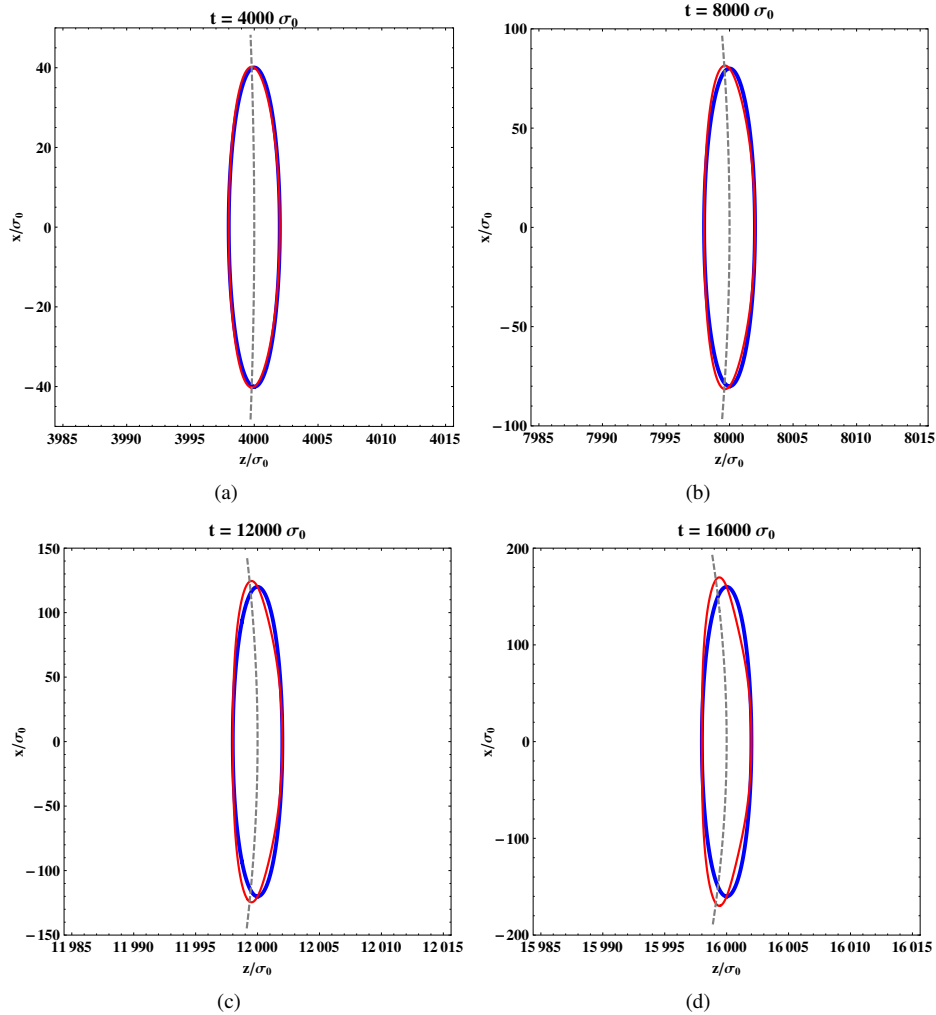
Fig. 2. The 2σ contours of the probability density calculated from $\Psi^{(0)}(\vec{r}, t)$ and $\Psi(\vec{r}, t)$ are shown in blue and red, respectively. The gray dashed curve indicates a circular arc of radius t. These plots are for $y = 0$ and assume $k_0 \sigma_0 = 100$. The horizontal scale is the same for all figures while the vertical scale is set in proportion to the increasing transverse size of the WP.

The result from Eq. (6) is ensured by the infinitely large convergence radius of the complex exponential function. It is just a matter of determining how many correction terms are required in order to produce a good approximation. To address this issue, we define $\delta\omega^{(i)}$ to correspond to the ith-order term ($i \geq 3$) in the parentheses of Eq. (4). The result in Eq. (8) obtained with the approximation $\exp(i\delta\omega t) \approx 1 + i\delta\omega^{(3)}t$ is valid for $|\delta\omega^{(3)}t| < 1$. This implies a limit on the propagation time $t \lesssim 2k_0^2\sigma_0^3$. It is not difficult to envision that, as time increases, a tiny contribution to $\delta\omega$ could totally change the value of $\exp(i\delta\omega t)$. Unfortunately, if the propagation time exceeds $2k_0^2\sigma_0^3$, the required computational effort to obtain $\Psi(\vec{r}, t)$ from Eq. (6) dramatically increases.

If one would like to find an acceptable solution at $t > 2k_0^2\sigma_0^3$ through the laborious computation using Eqs. (6) and (7), a reasonable procedure is as follows:

1. Find the term $\delta\omega^{(m)}$ in the expansion series of $\omega_{\vec{k}+\vec{k}_0}$ such that $\pi \sim |\delta\omega^{(m)}t| \gg |\delta\omega^{(m+1)}t|$ based on the estimate $|\delta\omega^{(m)}t| \sim O(k_0 t (k_0\sigma_0)^{-m})$. Make the approximation $\exp(i\delta\omega t) \approx \exp(i\delta\tilde{\omega}t)$, where $\delta\tilde{\omega} \equiv \sum_{i=3}^{m} \delta\omega^{(i)}$.

2. Make another approximation $\exp(i\delta\tilde{\omega}t) \approx \sum_{n=0}^{p}(i\delta\tilde{\omega}t)^n/n!$, where the integer p should be at least an order of magnitude larger than $|\delta\omega^{(3)}t|$ to ensure good convergence.

3. Compute the contribution to $\Psi(\vec{r}, t)$ in Eq. (6) from each term in $\sum_{n=0}^{p}(i\delta\tilde{\omega}t)^n/n!$ using Eq. (7).

We have not found an efficient way to simplify the cumbersome algebra in the above steps and conclude that the perturbative method discussed above is practical only for limited propagation time.

By evolving the different Fourier modes in an initial Gaussian neutrino WP, we have shown that the WP develops a spherical wave front, thereby obeying the limit on its propagation speed imposed by special relativity. We note that the apparent overlap of the WP with the space-like region shown in Fig. 2 must not be mistaken as a superluminal effect: this is purely due to the intrinsic position uncertainty of the initial WP. We also note that the transverse expansion of the WP indeed allows detection of the neutrino over an increasing spatial extent. This may have interesting consequences for long-baseline neutrino experiments, which we plan to study in the future.

Acknowledgments

References

[1] D. V. Naumov, V. A. Naumov, Neutrino Velocity Anomalies: A Resolution without a Revolution, arXiv:1110.0989.
[2] D. Bernard, Comment on: 'Neutrino Velocity Anomalies: A Resolution without a Revolution', arXiv:1110.2321.
[3] M. A. Porras, Ultrashort Pulsed Gaussian Light Beams, Phys. Rev. E 58 (1998) 1086–1093.
[4] Z. Wang, Z. Zhang, Z. Xu, Q. Lin, Space-time Profiles of an Ultrashort Pulsed Gaussian Beam, IEEE J. Quantum Electron. 33 (1997) 566–573.
[5] C. Almeida, A. Jabs, Spreading of a Relativistic Wave Packet, Am. J. Phys 52 (1984) 921–925.
[6] M. Andrews, The Evolution of Free Wave Packets, Am. J. Phys 76 (2008) 1102–1107.
[7] I. Christov, Propagation of Femtosecond Light Pulses, Opt. Commun. 53 (1985) 364 – 366.
[8] A. E. Bernardini, S. D. Leo, Analytic Approach to the Wave Packet Formalism in Oscillation Phenomena, Phys. Rev. D 70 (2004) 053010.