IDENTIFYING NON-INVERTIBLE KNOTS

RICHARD HARTLEY

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In this paper, the thirty-six non-invertible knots with ten crossings or fewer are identified. These are the knots listed by Conway[6] as non-invertible on the basis of empirical observation. The method used involves the calculation of covering invariants of certain irregular metabelian covering spaces. For most of the knots, this is the first proof of non-invertibility to appear, however I understand that Bonahon and Siebenmann (in work discussed below) have an alternative proof for a large number of them. However, details of this work are apparently not yet available.

Although most small knots are easily observed to be invertible (defined below) the existence of non-invertible knots has long been presumed[1]. However the existence of even a single non-invertible knot was not proven until Trotter[24] gave examples, the smallest of which had 15 crossings. Further examples were given by Whitten[26]. However until recently, no method has been available for determining whether any "naturally occurring" knot is invertible. The knot 8_27, for instance, has long been thought to be non-invertible, but this was only recently proven by Kawauchi[16] though prior proofs are claimed by Riley (using methods similar to those of [23]) and Bonahon and Siebenmann[3].

Kawauchi[16] gave a method which applied to amphicheiral hyperbolic knots, and he was able to identify the amphicheiral non-invertible knots with up to ten crossings (seven in all). The assumption that the knot be hyperbolic was removed by the present author[9]. This provided a useful criterion for an amphicheiral knot to be invertible.

Bonahon and Siebenmann[3] and Boileau[4, 5] have studied algebraic knots using their 2-fold branched covering space, and are able to classify them and determine their complete symmetry group (including invertibility). Algebraic knots are those which have a projection with a 1* type diagram—their Conway notation contains no dots. As Conway observes ([6], p. 335), those knots which he lists as 6** types—their notation begins with a dot—may be given a 1* type projection at the cost of adding two extra crossings. Hence, they are also algebraic, and are amenable to the methods of Bonahon/Siebenmann and Boileau. Of the 36 non-invertible knots, 27 are algebraic (see table). Of the remaining ones, three are covered by Kawauchi[16]. For the invertibility of the knots 10_{120}, 10_{106}, 10_{107}, 10_{110}, 10_{117}, 10_{119}, the methods of this paper give the only known proof.

The method used involves a calculation of covering invariants of the knot, that is, the homology groups of the branched and unbranched covering spaces and linking numbers between components of the covering link in some suitable covering. This has been dubbed "the universal method" by Riley[22] and has had many applications over a long period. The 2- and 3-fold cyclic coverings were used by Alexander and Briggs[1] to distinguish most knots to nine crossings, but the importance of homology groups of cyclic covers diminished with the invention of the Alexander polynomial. Linking numbers in dihedral covering spaces were first used by Reidemeister[21] and Bankwitz and Schumann[2] to complete the classification of knots to nine crossings. The method has been applied to the classification of 10 and 11 crossing knots by Perko[19] (dihedral and S, coverings) and Riley[22] (PSL coverings), and also as a powerful tool for showing non-amphicheirality[7, 19]. The present article is the first
Table 1. Table of covering invariants for non-invertible knots

<table>
<thead>
<tr>
<th>Knot</th>
<th>Conway</th>
<th>$r$</th>
<th>$p^N$</th>
<th>$p_n^0(t)$</th>
<th>$I'_1(t)$</th>
<th>$I'_n(t)$</th>
<th>Linking</th>
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<tr>
<td>$a_1$</td>
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<td>t-9</td>
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<td>07, (s)</td>
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<tr>
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<td>5</td>
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<td>t-3</td>
<td>105</td>
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</tr>
<tr>
<td>1099</td>
<td>(3,2)(3,2)</td>
<td>3</td>
<td>13</td>
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</tr>
<tr>
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<td>(3,2)(21,2)</td>
<td>7</td>
<td>6</td>
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<td>t-3</td>
<td>6, 6, 6, (s)</td>
<td>(s)</td>
</tr>
<tr>
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<td>11</td>
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<td>t-9</td>
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<td>.210.2.2</td>
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<td>t-2</td>
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<td>(s)</td>
</tr>
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<td>4</td>
<td>5</td>
<td>t-2</td>
<td>t-3</td>
<td>3, 3, 15</td>
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<tr>
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<td>.3.21.20</td>
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<td>5</td>
<td>t-2</td>
<td>t-3</td>
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<td>3</td>
<td>19</td>
<td>t-2</td>
<td>t-11</td>
<td>639, (7)</td>
<td>(s)</td>
</tr>
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</table>
to adapt it to proving that individual knots are non-invertible. The covering spaces used are certain irregular metacyclic and metabelian covers.

Other aspects of symmetry and invertibility of knots have been treated by the present author (and Kawauchi) in [12-14].

§1. TWIN COVERINGS SPACES OF INVERTIBLE KNOTS

Let $K$ be an oriented knot in $S^3$, and $G = \pi_1(S^3 - K, b)$ its knot group. $K$ is said to be invertible if there is an orientation-preserving homeomorphism $H:S^3 \to S^3$ taking $K$ to $K$ with reversed orientation. Let $N(K)$ be a regular neighbourhood of $K$ and $\mu$ a simple closed curve on $\partial N(K)$ which is contractible in $N(K)$ and loops once around $K$ in such a way that it has linking number $+1$ with $K$. (The reader is free to choose between right- and left-handed screw rules to define positive linking number.) Let $a$ be a path in $S^3 - N(K)'$ joining the base point $b$ to $b$. The element $c = a^{-1}$ of $G$ is called a meridian and will be denoted by $m$. Since it is defined only up to conjugacy, we will call any element conjugate to $m$ a meridian. If $K$ is invertible, then one can easily arrange that the inverting homeomorphism, $h$, takes $b$ to $b$ and a given meridian, $m$, to $m^{-1}$.

We will say that an element, $x$, in a group, $H$, is distinguishable from its inverse if there exists no automorphism of $H$ taking $x$ to $x^{-1}$. If a meridian of $G$ is distinguishable from its inverse, then $K$ is non-invertible. In this paper, it is shown that if $H$ is a finite group in which there are elements distinguishable from their inverse, then one may be able to use homomorphisms of $G$ onto $H$ to prove that $K$ is non-invertible.

Denote by $(\hat{M}, \hat{K}, \nu)$ a branched covering of $S^3$ branched over $K$. Here $\nu$ is called the covering projection, $(\hat{M}, \hat{K})$ is called the covering space pair and $\hat{K} = \nu^{-1}(K)$.

Convention. I write maps between topological spaces on the left and other maps, particularly algebraic homomorphisms, on the right and I observe the corresponding rule for composition. To avoid confusion, I write $f \circ g$ for the composition of topological maps.

Let $S(J_n) = S_n$ be the group of all permutations of a set, $J_n$, of $n$ elements, and let $\phi: G \to S(J_n)$ be a homomorphism onto a transitive subgroup of $S(J_n)$. (Thus, $\phi$ is a permutation representation of $G$.) By a standard construction, (see for instance [18] Chapter III, §5) there is, corresponding to $\phi$ an $n$-fold branched covering $(\hat{M}, \hat{K}, \nu)$ uniquely determined up to covering equivalence by the following consideration: the points lying above $b$ may be numbered $\hat{b}_i; i \in J_n$ in such a way that if $x \in G =
\( \pi_i(S^3 - K, b) \) and \( \nu_{i}^{-1}(x) \) denote the path in \( \tilde{M} \) starting at \( \tilde{b}_i \) obtained by lifting \( x \), then the end point of \( \nu_{i}^{-1}(x) \) is \( \tilde{b}_{i,\phi} \). Two permutation representations \( \phi \) and \( \phi' \) are called equivalent if there is an inner automorphism, \( \eta \), of \( S(J_\phi) \) such that \( \phi' = \phi \eta \). Two permutation representations are equivalent if and only if the corresponding coverings are covering-equivalent.

(1.1) Theorem. Let \( \phi: G \to H \) be a permutation representation of \( G \) such that \( m \phi \) is distinguishable from its inverse in \( H \). If \( K \) is invertible then there exists a homomorphism \( \phi': G \to H \) such that \( m \phi' = (m \phi)^{-1} \) and the covering space pairs corresponding to permutation representations \( \phi \) and \( \phi' \) are homeomorphic. But \( \phi \) and \( \phi' \) are not equivalent representations.

Proof. Let \( h: (S^3, K) \to (S^3, K) \) be the inverting homeomorphism and let \( \phi' = h_{\ast}^{-1} \phi \). Then \( m \phi' = m^{-1} \phi = (m \phi)^{-1} \). Since \( m \phi \) is distinguishable from its inverse, there is no inner automorphism of \( S(J_\phi) \) fixing \( H \) and taking \( m \phi \) to \( m \phi' \), so the permutation representations are inequivalent.

Let \( (\tilde{M}, \tilde{K}, \nu) \) be the branched covering corresponding to \( \phi \). Now, \( h \circ \nu \) maps \((\tilde{M}, \tilde{K})\) to \((S^3, K)\), and it is clearly a branched covering projection. To complete the proof, it remains to show that \((\tilde{M}, \tilde{K}, h \circ \nu)\) is the covering corresponding to \( h_{\ast}^{-1} \phi \). Let \( x \) be in \( \pi_i(S^3 - K, b) \). Then \( (h \circ \nu)_{i}^{-1}(x) = \nu_{i}^{-1}(h^{-1}(x)) = \nu_{i}^{-1}(xh_{\ast}^{-1}) \), and the end point of this path is \( \tilde{b}_{i,\phi} \), which completes the proof.

Note. If the knot \( K \) is positive amphicheiral, then there is an orientation reversing homeomorphism of \( S^3 \) taking a meridian of \( K \) to its inverse. So, in this case also each representation will have its inequivalent "twin", and there will be an orientation reversing homeomorphism between the two covering spaces. The homology groups of the two covering spaces will therefore be isomorphic. However unless the knot is invertible, the linking numbers in the two covering spaces will not in general coincide (they will be negatives of each other). Furthermore, the invariants \( H(G, \phi, ml) \) discussed in ([11], §9) will in general be different, which may be useful if linking numbers are undefined.

The particular groups, \( H \), to which this theorem is applied here are certain extensions of elementary abelian \( p \)-groups by a cyclic group, \( Z_n \), where \( p \) is a prime which does not divide \( r \). These groups were also studied in [8] to which paper the reader may refer for additional information.

Let \( X \) be a faithful, irreducible \( Z_r \)-module over \( Z_p \). Thus, \( X \) is a \( Z_p \)-vector space with a \( Z \)-action. Module multiplication by the generator of \( Z \), is a \( Z_p \)-linear transformation, \( T_\sigma \), of \( X \) of order \( r \). Since \( p \) does not divide \( r \), the polynomial \( t^r - 1 \) (over \( Z_p \)) has no repeated irreducible factors. So, neither does \( f(t) \), the characteristic polynomial of \( T_\sigma \) which must divide \( t^r - 1 \). Since \( X \) is irreducible, \( f(t) \) must be irreducible, otherwise the rational canonical form of \( T_\sigma \) splits into blocks. In fact, \( f(t) \) must be an irreducible factor over \( Z_p \) of the \( r \)-th cyclotomic polynomial, \( \sigma_r \), otherwise \( T_\sigma \) would have order less than \( r \). Therefore, ([25], Theorem III.12.E) \( f(t) \) has degree equal to the order of \( p \) modulo \( r \), and the \( Z \)-action is uniquely determined, up to a change of \( Z_r \)-basis by \( f(t) \).

Let \( H \) be the corresponding split extension of \( X \) by \( Z_r \). (By the Schur–Zassenhaus lemma, any such extension splits.) In [8] it was shown that up to isomorphism, the group \( H \) does not depend on the particular irreducible factor \( f(t) \) occurring. Thus, \( H \) is unambiguously determined by \( p \) and \( r \). It fits into an exact sequence
1 \to X \to H \to Z \to 1$, and $Z$ acts on $X$ by conjugation. In this paper, the notation $Z \bigoplus (Z_p \oplus \cdots \oplus Z_p)$ will be used to denote just this group, $H$. The number of $p$'s occurring is equal to the order of $p$ modulo $r$. An important special case is that of the metacyclic group $Z \bigoplus Z_p$, where $r$ divides $p - 1$.

Now, $X$ is a characteristic subgroup of $H$, in fact, the commutator subgroup. For since $Z$ is abelian, $X$ contains $H'$. However, $H'$ must then be a $Z$-submodule of $X$, nonzero, since $H$ is non-abelian, and so equal to $X$, since $X$ is irreducible. If $x \in H$, then conjugation by $x$ gives a linear transformation $T_x$ of $X$, the characteristic polynomial of which may be denoted by $f_x$. Since $X$ is a characteristic subgroup, $f_x$ is unambiguously determined.

Let $Z_p[t, t^{-1}]$ denote the ring of Laurent polynomials over $Z_p$. Given $f(t)$ and $g(t)$ in $Z_p[t, t^{-1}]$ we write $f(t) \sim g(t)$ if $f(t) = \alpha g(t)$ for some integer $j$ and non-zero $\alpha$ in $Z_p$. That is, $f(t)$ and $g(t)$ differ by a unit. Using this notation we observe:

(1.2) If $x \in H$ and $y = x^{-1}$, then $f_x(t) \sim f_y(t^{-1})$.

Proof. $T_x = T_y^{-1}$ and so the eigenvalues of $T_y$ are the inverses of those of $T_x$.

(1.3) Let $H = Z \bigoplus (Z \oplus \cdots \oplus Z)$ and let $x$ be an element of $H$, the normal closure of which is the whole of $H$. If $r$ is greater than $2$ and $p$ is of odd order modulo $r$, (that is, there are an odd number of $Z_p$'s) then $x$ is distinguishable from its inverse.

Proof. In view of (1.2) it suffices to show that $f_x(t) \neq f_y(t^{-1})$. Suppose $f_x(t) = \alpha f_y(t^{-1})$. Using this relation twice gives $f_x(t) = \alpha^2 f_y(t)$ whence $\alpha^2 = 1$. By hypothesis, degree (f) = $2k + 1$ is odd, say $f(t) = \alpha_0 t + \alpha_1 t^2 + \cdots + \alpha_{2k} t^{2k} + \alpha_0 t^{2k+1}$. Then $f(-\alpha) = 0$. Now the mapping of $H$ onto $Z$ must take $x$ to a generator of $Z$ say $t$, where $j$ is coprime with $r$. Then $T_x = T_y$ and the eigenvalues of $T_x$ are primitive $r$th roots of unity. So, then, are the eigenvalues of $T_y$. However, $-\alpha$ is not a primitive $r$th root of unity, since $r > 2$ and $\alpha^2 = 1$.

Note that the normal closure of a meridian, $m$, in $G$ is the whole of $G$. (1.2) and (1.1) Theorem then give:

(1.4) Theorem. If $\phi$ is a homomorphism of the knot group, $G$, onto a permutation group $H \cong Z \bigoplus (Z \oplus \cdots \oplus Z)$ and if $K$ is invertible, then there exists a homomorphism, $\phi'$ of $G$ onto $H$ such that $f_{m\phi}(t) = f_{m\phi'}(t^{-1})$, and the covering space pairs corresponding to $\phi$ and $\phi'$ are homeomorphic.

Note. As a consequence of the symmetry of the Alexander polynomial and (1.5) below, there will always exist a homomorphism $\phi'$ such that $f_{m\phi}(t) = f_{m\phi'}(t^{-1})$ whenever $\phi$ exists. The important point of (1.4) is not the existence of $\phi'$, but rather that the corresponding covering space pairs are homeomorphic.

Finding which groups, $Z \bigoplus (Z \oplus \cdots \oplus Z)$ a given knot group maps onto is a simple task because of the following results, proven in [8].

Let $S$ be a matrix with characteristic polynomial equal to the $r$th cyclotomic polynomial, $\sigma_r$ (that is, $S$ is a companion matrix for $\sigma_r$). Let $\Delta(t)$ be the Alexander polynomial of the knot, $K$. 

\[ \Delta(t) = \sum_{i=0}^{r-1} \sigma_r(t^{r-1-i}) t^i \]
(1.5) $G$ maps onto $\mathbb{Z} \otimes (\mathbb{Z}/p \oplus \cdots \oplus \mathbb{Z}/p)$ if and only if $p$ divides $\prod_{i=1}^{k} \Delta(\xi_i) = \det(\Delta(S))$, where $\xi_1, \ldots, \xi_k$ are the (complex) roots of $\sigma_n$ that is, the primitive $r$th roots of unity.

This comes from [8] Theorem 1.11 and Lemma 1.10. Theorem 1.7 of [8] allows one to find $f_{m_b}(t)$.

(1.6) Let $g(t)$ be an irreducible factor of $\sigma$, over $\mathbb{Z}/p$. There is a homomorphism $\phi$ of $G$ onto $\mathbb{Z} \otimes (\mathbb{Z}/p \oplus \cdots \oplus \mathbb{Z}/p)$ with $f_{m_b}(t) = g(t)$ if and only if $g(t)$ divides $\Delta_p(t)$, the Alexander polynomial with coefficients reduced modulo $p$.

For the case of metacyclic representations, (1.6) may be written simply as:

(1.7) Let $r$ be the order of $\alpha$ modulo $p$. There is a homomorphism $\phi$ of $G$ onto $\mathbb{Z} \otimes \mathbb{Z}/r$ with $f_{m_b}(t) = t - \alpha$ if and only if $p$ divides $\Delta(\alpha)$.

This was first proven by Fox [7].

§2. CALCULATION OF COVERING INVARIANTS

Calculation of representations and their corresponding covering invariants was carried out on a VAX 11/780 computer. Programs were written in Fortran. Most of them I translated from old programs I had written in PL/1, a language not available to me at present. The programs have received a considerable amount of use, and in some cases, the invariants were calculated using two different methods as explained below. The correctness of the calculations is further confirmed by internal consistency. The results are therefore almost surely correct. I will try to make copies of the programs used available on request.

First, a presentation for the knot group was calculated with meridian generators, $x_b$, as few in number as could conveniently be achieved by Tietze transformations on the Wirtinger presentation. Representations of the knot group of the type $\mathbb{Z}_r \otimes \mathbb{Z}_p$ were sought. Using the Alexander polynomial, it was easily determined using (1.7) which groups of this type (up to some moderate value of $p$) were homomorphic images of $G$. The so-called irregular permutation representation of degree $p$ was used for $\mathbb{Z}_r \otimes \mathbb{Z}_p$. This may be described in the following way. For $\beta \neq 0$, let $\tau_{\beta} \in \mathbb{Z}_p$ denote the permutation of the set $J_p = \{0, \ldots, p-1\}$ given by $j\tau_{\beta} = j\beta + i$ (modulo $p$) for $j \in J_p$. The set $H = \{\tau_{\beta} \mid \beta \equiv 1 \pmod{p}\}$ forms a permutation group isomorphic to $\mathbb{Z}_r \otimes \mathbb{Z}_p$, whereby $\mathbb{Z}_p$ corresponds to the subgroup generated by $\tau_{1,1}$. The set of elements, $x$, such that $f_i(t) = t - \alpha$ are just those permutations of the form $\tau_{n,1}$.

Representations of $G$ onto $H$ may be normalised according to the following scheme.

(2.1) If $G = \langle x_1, \ldots, x_n \mid R_1, \ldots, R_{n-1} \rangle$ and $\phi$ is some representation of $G$ onto $H$ for which $f_{m_b}(t) = t - \alpha$, then $\phi$ is equivalent to exactly one representation, $\phi'$ such that for some $k$ less than $n$, $x_1 \phi = x_2 \phi = \cdots = x_k \phi = x_{k+1} \phi = \tau_{n,1}$.

Proof. Let $x_i \phi = \tau_{n,1}$. Let $c$ be chosen so that $c(\alpha - 1) = b$ (modulo $p$), which is possible, since $\alpha \neq 1$ (modulo $p$). Conjugation by the permutation $\tau_{1,c}$ is an inner automorphism of $H$ taking $x_1 \phi$ to $\tau_{n,0}$. Thus, we may assume $x_1 \phi = \tau_{n,0}$. Let $x_k$ be the first generator such that $x_k \phi \neq x_1 \phi$. Let $x_k \phi = \tau_{n,p}$. Conjugation by the permutation $\tau_{n,0}$, where $h$ is the inverse of $j$ modulo $p$ takes $H$ to itself, $\tau_{n,0}$ to $\tau_{n,0}$ and $\tau_{n,j}$ to $\tau_{n,1}$. So we
may replace $\phi$ by an equivalent representation of the desired form. Since $\tau_{a,b}$ and $\tau_{a,1}$
together generate $H$, the images of the other generators are determined, and so $\phi'$ is
unique.

Using this result, it is an easy matter to find a complete set of non-equivalent
permutation representations of $G$ onto $H$ simply by testing which such normalised
assignments of permutation to generator satisfy the relators. For a presentation with $n$
generators, this involves testing only $1 + p + \cdots + p^n - 2$ candidates. This took no more
than two seconds CPU time for the worst case encountered. For larger knots than
those considered here, a more sophisticated approach may be preferable, such as
solving the congruences (*) on page 194 of [7].

Having obtained a permutation representation of $G$, one proceeds to calculate the
covering invariants. The method used here is discussed in some detail in [11] where an
example of calculation of linking numbers is given. Perko [17] also sets forth a method
of calculation. One constructs a matrix,

$$F = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

(see [11], (10.1)) the columns of which correspond to generators of $H_{\cdot}(M - \tilde{K})$, and in
particular the columns of

$$\begin{pmatrix} B \\ D \end{pmatrix}$$
correspond to meridians of the components of the covering link. The rows of $(A|B)$ are relators for $H_{\cdot}(M - \tilde{K})$ obtained via a Reidemeister-Schreier
process by rewriting the relators $R_i$. The rows of $(C|D)$ correspond to "longitudes" of
the components of the covering link, and are obtained by rewriting the longitude of $K$.
The matrix $(A|B)$ is a relation matrix for $H_{\cdot}(M - \tilde{K})$. Matrix $A$ is a relation matrix for
$H_{\cdot}(\tilde{M})$, and if there exists a matrix $X$ such that $XA = C$, then $D - XB$ is the matrix of
linking numbers of the components of the covering link. If there is no such $X$, this
means that some (and hence, for metacyclic covers, all) $\tilde{K}_i$ are of infinite order in
$H_{\cdot}(\tilde{M})$. In that case, the linking numbers do not exist. Construction of the matrix $F$ is
rapid and easy, requiring only a fraction of a second in the worst case.

From the relation matrix for an abelian group, one can in principle calculate the
rank (Betti number) and torsion numbers for the group by diagonalisation using a
well-known method ([17], §3.3). A solution of the equation $XA = C$ is also easily
found, particularly if the matrix $A$ has already been diagonalised. In fact these
straightforward techniques work efficiently if the matrix $F$ is no bigger than about
$30 \times 30$, that is, for primes up to about $p = 11$ for 3-bridged knots. Beyond this one
encounters the phenomenon of "entry explosion", in which the entries of the matrix
become extremely large during diagonalisation (even if the entries in the diagonalised
matrix are relatively small) and cause integer overflow. Heuristic methods used to keep
down the size of the entries prove expensive in terms of time. After using 40 minutes
CPU time in an unsuccessful effort to calculate the $Z_7 \otimes Z_{31}$ invariants for the knot
$10_{69}$, which required diagonalisation of a matrix of size $73 \times 62$, I turned to a less naive
approach. For matrix diagonalisation, I used my modified version of a diagonalisation
program devised by G. Havas and L. Sterling, and I thank them for making their
program available. In essence, this program carries out the diagonalisation modulo
appropriate prime powers. One can then reconstruct the diagonal form for the matrix. Their paper [15] gives the very interesting details of this program as well as a discussion of the problems involved in diagonalising matrices.

A similar approach proved successful for calculating the linking numbers also. Although the linking numbers, \( \lambda_{ij} \), are not in general integers, they have a common denominator dividing the largest torsion number, \( T \), of \( H_i(\overline{M}) \), (which has already been calculated). The values of \( T \lambda_{ij} \) may be calculated modulo a number of large primes, and the Chinese remainder theorem then serves to retrieve the actual values of \( T \lambda_{ij} \). When these techniques were used on the previously mentioned case of 10_{58}, calculation of the invariants associated with a given covering required only about 75 seconds CPU time, 60 seconds for the diagonalisation of the two matrices, and 15 seconds to calculate the linking numbers.

The non-invertibility of all knots except 10_{42} could be established using the first (and simplest) type of representation considered. The group of the knot 10_{42} has a pair of representations on \( \mathbb{Z}_2 \otimes \mathbb{Z}_2 \), but the two covering spaces could not be distinguished. Neither could the two coverings associated with \( \mathbb{Z}_2 \otimes \mathbb{Z}_3 \), however the two \( \mathbb{Z}_{10} \otimes \mathbb{Z}_{17} \) coverings were different. In three cases (10_{50}, 10_{148}, and 10_{158}) representations onto \( \mathbb{Z}_7 \otimes (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \) were used so as to avoid the larger calculation which would have been involved if metacyclic representations had been used instead.

In the Table I list each of the non-invertible knots with ten crossings or less, giving Rolfsen's numbering and also Conway's notation. Unfortunately, the diagrams of 10_{58} and 10_{66} given by Rolfsen do not correspond to their Conway notation or Alexander polynomial—they are interchanged. I take the view that the knot diagram is more basic than Conway's notation. Accordingly, I have called 10_{58} that knot drawn as 10_{66} by Rolfsen, and similarly 10_{66} has notation 3.12.0. Thus 10_{58} has notation 31.20 and polynomial \(-2 + 9t - 9t^2 + 25t^3 - 19t^4 + 9t^5 - 2t^6\), and 10_{66} has notation 31.2 and polynomial \(-2 + 9t - 19t^2 + 25t^3 - 19t^4 + 9t^5 - 2t^6\). With each knot is listed a group along with all non-equivalent representations onto that group. For each representation, the homology groups of \( \overline{\rho} \) and \( \overline{\rho} - \mathbb{Z} \) are then given in codified form. The number in brackets is the Betti number (the rank of the torsion-free summand) and the other numbers are torsion coefficients. Thus, for example 7126.1 represents the group \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z} \). The final column gives a linking invariant. The covering link consists of one component, \( \overline{K}_0 \), of branching index 1 and \( N = (p^n - 1)/r \) components of branching index \( r \). The number given is \( \sum_{i=1}^{N} \text{link} (\overline{K}_0, \overline{K}_i) \) multiplied by \( p^n/r \). I multiplied by \( p^n/r \) simply because it usually made the numbers smaller. A dash means that the linking numbers do not exist, since the \( \overline{K}_i \) are of infinite order in \( H_1(\overline{M}) \). If the knots were to be invertible, pairs of covering spaces would have to be homeomorphic. A glance reveals that they are not. This proves that the knots listed by Conway as non-invertible are indeed non-invertible. I did not undertake the task of verifying that the remaining knots are in fact invertible.

Using (1.7) it is not hard to show that if \( \Delta(t) \neq 1 \) for a knot group, \( G \), then \( G \) has infinitely many different quotients of type \( \mathbb{Z}_2 \otimes \mathbb{Z}_p \). For if \( \Delta(t) = a_0 + a_1 t + \cdots + a_n t^n \) and \( q \) is any sufficiently large integer, then \( \Delta(a_0^q t) = a_0(1 + a_0qC) \) for some \( C \), and this is divisible by a prime not contained in \( a_0q \). So there are an infinite number of different primes dividing \( \Delta(a) \) for some \( a \). The above method of proving knots non-invertible therefore seems to stand a good chance of eventual success. In the case of more difficult knots, it might be necessary to consider non-metabelian quotients. In [10] it is shown that a homomorphism of a knot group onto \( \mathbb{Z}_2 \otimes \mathbb{Z}_p \) lifts to any one of an infinite class of finite groups of soluble length three. Via the projection onto \( \mathbb{Z}_2 \otimes \mathbb{Z}_p \), elements in these groups are distinguishable from their inverse. In order to
deal with knots with trivial Alexander polynomial, it would be necessary to consider non-soluble quotient groups of \( G \) which contain elements distinguishable from their inverse. Presumably such groups exist, but I know of no example. In particular, the alternating groups and the projective special linear groups do not work, for elements in these groups are indistinguishable from their inverse.

REFERENCES