Perturbation Theory of Completely Mixed Matrix Games

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ABSTRACT

How do the value v and the solution x and y of a zero-sum two-person completely mixed game vary as the elements of the $n \times n$ payoff matrix $A = (a_{ij})$ are perturbed? Assuming $v > 0$, we show that, for $i, j, k = 1, \dots, n$, $0 < dv/da_{ij} = x_i y_j < 1$ (by a little-known theorem of Gross), that $dx_k/da_{ij} = x_i v \phi_{jk}$, $dy_k/da_{ij} = y_j v \phi_{ki}$, and (for $g, h = 1, \dots, n$) $d^2v/da_{gh}\, da_{ij} = v[x_i y_h \phi_{jg} + x_g y_j \phi_{hi}]$, where ϕ_{ij} is defined in terms of $B = (b_{ij}) = A^{-1}$ by $\phi_{ij} = (\Sigma_h b_{ih})(\Sigma_g b_{gj}) - b_{ij}\Sigma_{g,h} b_{gh}$. If A is a nonsingular M-matrix, then for $i, j = 1, \dots, n$ we have $d^2v/da_{ii}^2 < 0$, $dx_j/da_{ij} < 0$, and $dy_j/da_{ji} < 0$, but v is not concave as a function of the vector of diagonal elements $(a_{11}, \dots, a_{nn})$.

1. INTRODUCTION

The value $v(A)$ of an $m \times n$ real matrix A, $1 < m, n < \infty$, is defined [13] by

$$v(A) = \max_{x \in P_m} \min_{y \in P_n} x^T A y,$$

where ^{T} denotes transpose and $P_n = \{x \in R^n : x_i \geqslant 0,\ i = 1, 2, \dots, n, \text{ and } \Sigma_{i=1}^n x_i = 1\}$. A pair (x, y), where $x \in P_m$ and $y \in P_n$, is a *solution* of A if

$$(Ay)_i \leqslant v(A) \leqslant (x^T A)_j$$

for $i = 1, \dots, m$ and $j = 1, \dots, n$. These concepts arise in the theory of two-person zero-sum games (e.g. [4]).

How do the value $v(A)$ and solutions depend on the elements of A? It is known that if J is the $m \times n$ matrix with all elements equal to 1, then $v(A + aJ) = v(A) + a$ for $a \in R$. The solutions of $A + aJ$ are identical to those of A. Shapley [11] proved that if A and B are $m \times n$ real matrices,

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then $|v(A)-v(B)| \leqslant \max_{i,j}|a_{ij}-b_{ij}|$. Gross [6] showed that if $X \times Y$ is the set of all solutions (x, y) of A, then the right and left derivatives of $v(A)$ as a function of a single element a_{ij} of A exist and are given by

$$\frac{dv^+}{da_{ij}} = \max_{x \in X} x_i \min_{y \in Y} y_j,$$

$$\frac{dv^-}{da_{ij}} = \min_{x \in X} x_i \max_{y \in Y} y_j.$$

He observed that when A has a unique solution, then the derivatives all exist (i.e., the left and right derivatives are equal) and $\sum_{i,j} dv/da_{ij} = 1$. Since Bohnenblust, Karlin, and Shapley [2, p. 56] proved that the set of $m \times n$ matrices A which have unique solutions is open and everywhere dense in mn-space, Gross's result implies that the derivatives exist for most A. Raghavan [9, p. 37] showed that if A is $n \times n$ and real, then for $a \in R$, $v(A+aI)$ is a continuous nondecreasing function of a, and $v(A+aI) \to \pm\infty$ as $a \to \pm\infty$. Beyond these few facts, little appears to be known, and even Gross's results, which were never submitted for external publication, do not appear to be widely known.

The derivatives of the value as a function of matrix elements are of applied interest. For an observer of a game in which the matrix elements are estimated from data, the first derivatives indicate the sensitivity of the estimated value to errors in the estimated matrix, and therefore indicate which matrix elements should be estimated with greatest precision. For players attempting to manipulate or influence the matrix of a game in which they are involved, the first derivatives identify the matrix elements that would yield the largest rewards per unit of change, while the second derivatives identify where efforts to change a matrix element would yield (locally) increasing or decreasing returns in the value per unit of change in the matrix element.

Because of the similarity between the value of A and a formula [1]

$$\rho(A) = \max_{x \in P_n^+} \min_{y \in P_n^+} \frac{x^T A y}{x^T y}$$

for the spectral radius of an $n \times n$ matrix $A = (a_{ij})$ with all $a_{ij} \geqslant 0$, where $P_n^+ = \{x \in P_n : x_i > 0,\ i = 1, 2, \dots, n\}$, one might hope for a qualitative similarity between the behavior of $v(A)$ and $\rho(A)$ when A is square and $A \geqslant 0$ (meaning that all $a_{ij} \geqslant 0$).

In fact, the behavior of $v(A)$ and $\rho(A)$ can be dramatically different. For example, let

$$A(a)=\begin{pmatrix} a & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}, \qquad a \geqslant 0.$$

Since $\rho(A) \geqslant \max_i a_{ii}$, $\rho(A(a)) \uparrow \infty$ as $a \uparrow \infty$. On the other hand, using the procedure of Shapley and Snow [12], it requires only elementary calculation to show that $v(A(a)) = 1$, regardless of a, with solution $x^T = (\frac{1}{2}, 0, \frac{1}{2})$, $y^T = (0,1,0)$.

Unlike the spectral radius, which is an analytic function of matrix elements, the derivative dv/da_{ij} of the value $v = v(A)$ as a function of the element a_{ij}, holding all other elements constant, can fail to exist at a finite number of values of a_{ij}. For example, if $a \in R$, then

$$A(a)=\begin{pmatrix} a & 1 \\ 0 & 0 \end{pmatrix}$$

has saddlepoints equal to 0, a, or 1, according as $a \leqslant 0$, $0 < a < 1$, or $1 \leqslant a$. Thus $dv/da_{11} = 0$ for $a < 0$ and $a > 1$, and $dv/da_{11} = 1$ for $0 < a < 1$, but dv/da_{11} fails to exist for $a = 0$ and $a = 1$. This example also shows that v need not be concave or convex in general.

2. GENERAL RESULTS

The matrix A is defined to be *completely mixed* if, for every solution (x, y), no element of x is zero and no element of y is zero. If A is completely mixed, then the solution of A is unique and $m = n$ [7].

In the rest of this section, we shall suppose that $v(A) > 0$. Under this assumption, if A is completely mixed, then A is nonsingular [7].

The first theorem, discovered independently of Gross [6], is an immediate consequence of his results and the fact that a completely mixed matrix has a unique solution. We give an elementary proof, independent of Gross's, that introduces facts useful later.

THEOREM 1. *If A is completely mixed, then, for every i, j, dv/da_{ij} exists and equals $x_i y_i$, $0 < dv/da_{ij} < 1$, and $\sum_{i,j} dv/da_{ij} = 1$.*

Proof. Under the hypotheses, Kaplansky [7] and Shapley and Snow [12] showed that

$$v(A) = 1/\mathbf{1}^T A^{-1}\mathbf{1}, \tag{2.1}$$

$$x^T = \mathbf{1}^T A^{-1} v(A), \tag{2.2}$$

$$y = A^{-1}\mathbf{1} v(A), \tag{2.3}$$

where $\mathbf{1}$ is the n-vector with each element 1. The existence of dv/da_{ij} is immediate from (2.1) and the existence of $d(A^{-1})/da_{ij}$. From (2.2) and (2.3), we have $x^T A y = v(A)$, whence

$$\begin{aligned}
\frac{dv}{da_{ij}} &= \frac{dx^T}{da_{ij}} A y + x^T \frac{dA}{da_{ij}} y + x^T A \frac{dy}{da_{ij}} \\
&= \frac{dx^T}{da_{ij}} \mathbf{1} v + x^T E_{ij} y + \mathbf{1}^T v \frac{dy}{da_{ij}} \\
&= v \frac{d(x^T \mathbf{1})}{da_{ij}} + x_i y_j + v \frac{d(\mathbf{1}^T y)}{da_{ij}} \\
&= v \frac{d1}{da_{ij}} + x_i y_j + v \frac{d1}{da_{ij}} \\
&= x_i y_j,
\end{aligned}$$

where E_{ij} is the $n \times n$ matrix with i, j element 1 and all others 0. This implies $0 < dv/da_{ij} < 1$ and $\Sigma_{i,j} dv/da_{ij} = 1$. ■

Define, for any nonsingular $n \times n$ real matrix A with inverse $A^{-1} = B = (b_{ij})$, for $1 \leqslant j, k \leqslant n$,

$$\phi_{jk}(A) = \sum_{h=1}^{n} b_{jh} \sum_{g=1}^{n} b_{gk} - b_{jk} \sum_{g,h} b_{gh}.$$

THEOREM 2. *Let A be completely mixed. Then for $i, j, k = 1, \dots, n$,*

$$\begin{aligned}
\frac{dx_k}{da_{ij}} &= \frac{x_k x_i y_j}{v(A)} - x_i (A^{-1})_{jk} \\
&= x_i v(A) \phi_{jk}(A), \\
\frac{dy_k}{da_{ij}} &= y_j v(A) \phi_{ki}(A).
\end{aligned}$$

Proof. Assume A is completely mixed. Because we assume $v(A) > 0$, A is nonsingular. From (2.2), $x^T A = \mathbf{1}^T v(A)$, so

$$\mathbf{1}^T \frac{dv}{da_{ij}} = x^T \frac{dA}{da_{ij}} + \frac{dx^T}{da_{ij}} A.$$

Then using $dv/da_{ij} = x_i y_j$ and $dA/da_{ij} = E_{ij}$ gives

$$\begin{aligned}\frac{dx^T}{da_{ij}} &= x_i y_j \mathbf{1}^T A^{-1} - x^T E_{ij} A^{-1}\\ &= x^T \left[\frac{x_i y_j}{v(A)} + A \frac{dA^{-1}}{da_{ij}} \right].\end{aligned}$$

Drawing on (2.1), we let $S = \mathbf{1}^T A^{-1} \mathbf{1} = 1/v(A) = \Sigma_{g,h} (A^{-1})_{gh}$. Then

$$\begin{aligned}\frac{dx_k}{da_{ij}} &= \frac{x_k x_i y_j}{v(A)} - \left[x_i (j\text{th row of } A^{-1}) \right]_k\\ &= \frac{x_k x_i y_j}{v(A)} - x_i (A^{-1})_{jk}\\ &= x_i \left[\frac{(k\text{th col. sum of } A^{-1})(j\text{th row sum of } A^{-1})}{S} - (A^{-1})_{jk} \right]\\ &= x_i v(A) \Bigg[(k\text{th col. sum of } A^{-1})(j\text{th row sum of } A^{-1})\\ &\qquad\qquad - (A^{-1})_{jk} \Big(\sum_{g,h} (A^{-1})_{gh} \Big) \Bigg]\\ &= x_i v(A) \phi_{jk}(A).\end{aligned}$$

The derivation of dy_k/da_{ij} is similar. ■

THEOREM 3. *Let A be a completely mixed $n \times n$ matrix. Then for $g, h, i, j = 1, \ldots, n$, $d^2 v/da_{gh} da_{ij}$ exists and*

$$\frac{d^2 v}{da_{gh} da_{ij}} = v(A) \left[x_i y_h \phi_{jg}(A) + x_g y_j \phi_{hi}(A) \right]. \tag{2.4}$$

In particular,

$$\frac{d^2v}{da_{ij}^2} = 2v(A)x_i y_j \phi_{ji}(A). \tag{2.5}$$

Proof. The derivatives (2.4) and (2.5) exist because $d^2(A^{-1})/da_{gh}\,da_{ij}$ exists whenever A^{-1} exists, and in this case v is given by (2.1). But A^{-1} exists because we assume $v(A) > 0$. Then, by Theorem 1, $d^2v/da_{gh}\,da_{ij} = x_i(dy_j/da_{gh}) + (dx_i/da_{gh})y_j$. Substituting the formulas of Theorem 2 gives (2.4) and (2.5). ■

In light of the central position of $\phi_{ij}(A)$ in Theorems 2 and 3, it will be useful to have equivalent forms of $\phi_{ij}(A)$. For $1 \leqslant p \leqslant n$ and $1 \leqslant i_1 < i_2 < \cdots < i_p \leqslant n$ and $1 \leqslant j_1 < j_2 < \cdots < j_p \leqslant n$, we let $A[i_1,\ldots,i_p|j_1,\ldots,j_p]$ denote the $p \times p$ submatrix of the $n \times n$ matrix A formed from the intersection of rows $i_1,\ldots,i_p$ and columns $j_1,\ldots,j_p$. We let $A(i_1,\ldots,i_p|j_1,\ldots,j_p)$ denote the $(n-p)\times(n-p)$ submatrix of A that remains after striking out rows $i_1,\ldots,i_p$ and columns $j_1,\ldots,j_p$, provided $p < n$, while we define $A(1,\ldots,n|1,\ldots,n) = 1$.

THEOREM 4. *Let A be a nonsingular $n \times n$ matrix and let $B = (b_{ij}) = A^{-1}$. Then for $1 \leqslant j, k \leqslant n$*

$$\phi_{jk}(A) = \sum_{g \neq j} b_{gk} \sum_{h \neq k} b_{jh} - b_{jk} \sum_{g \neq j,\, h \neq k} b_{gh}, \tag{2.6}$$

$$\phi_{jk}(A) = \sum_{g \neq j} \sum_{h \neq k} \left[b_{gk} b_{jh} - b_{jk} b_{gh} \right], \tag{2.7}$$

$$\begin{aligned} \phi_{jk}(A) = &-(\det A)^{-1}(-1)^{j+k} \\ &\times \Bigg[\sum_{g<j} \sum_{h<k} \det A(h,k|g,j)(-1)^{g+h} \\ &\quad + \sum_{g<j} \sum_{h>k} \det A(k,h|g,j)(-1)^{g+h} \\ &\quad + \sum_{g>j} \sum_{h<k} \det A(h,k|j,g)(-1)^{g+h} \\ &\quad + \sum_{g>j} \sum_{h>k} \det A(k,h|j,g)(-1)^{g+h} \Bigg]. \end{aligned} \tag{2.8}$$

Proof. Canceling the terms common to $\Sigma_{g=1}^{n} b_{gk} \Sigma_{h=1}^{n} b_{jh}$ and $b_{jk} \Sigma_{g,h} b_{gh}$ in the definition of $\phi_{jk}(A)$ gives (2.6). Collecting terms in (2.6) with common indices of summation gives (2.7). Specializing a formula found in, e.g., [5, 1:21, (33)],

$$\det B\left[i_1,\dots,i_p | j_1,\dots,j_p\right]$$
$$= (\det A)^{-1}(-1)^{\Sigma i_s + \Sigma j_s} \det A(j_1,\dots,j_p | i_1,\dots,i_p), \quad (2.9)$$

to the summands on the right of (2.7) gives (2.8). ■

3. M-MATRICES

An $n \times n$ real matrix A is called an M-matrix if $A = sI - M$, where I is the $n \times n$ identity matrix, M has nonnegative elements, and $s \geqslant \rho M$. Clearly, A is singular if $s = \rho M$, nonsingular if $s > \rho M$. Nonsingular M-matrices are special cases of Minkowski-Leontief matrices, which Karlin [8, p. 52] showed to be completely mixed with positive value. (See Bohnenblust, Karlin, and Shapley [2, pp. 68–69] for a precursor of Karlin's [8] result, and Raghavan [9] for the special case of M-matrices.)

THEOREM 5. *If A is an M-matrix, then*

$$0 \leqslant v(A) \leqslant s - \rho M,$$

and $0 = v(A)$ if and only if A is singular.

Proof. Since $M \geqslant 0$, there exist $x, y \in P_n$, not necessarily unique, such that $x^T M = (\rho M) x^T$ and $My = (\rho M) y$. Hence $Ay = (s - \rho M)y$ and $s - \rho M \geqslant 0$. Therefore, by Theorem 1 of Cohen and Friedland [3], $v(A) \leqslant s - \rho M$.

If A is singular, $s - \rho M = 0$, so $x^T A = \mathbf{0}^T$ and $Ay = \mathbf{0}$, where $\mathbf{0}$ is the n-vector with each element 0. Therefore (x, y) is a solution of A, and A has value 0.

If A is nonsingular, then $s > \rho M$, so $B = (b_{ij}) = A^{-1} \geqslant 0$ (e.g. [5, 2:66]). Moreover, $S = \Sigma_{i,j} b_{ij} > 0$, since otherwise B would be singular. Theorem 1 of Raghavan [10], rediscovered as Corollary 5 of [3], asserts that if A is a nonsingular matrix with $A^{-1} \geqslant 0$, then $v(A) = S^{-1}$, so $v > 0$. ■

THEOREM 6. *Let A be an $n\times n$ nonsingular M-matrix. Then for $i, j = 1,\dots,n$,*

$$\frac{d^2v}{da_{ii}^2}<0, \tag{3.1}$$

$$\frac{dx_j}{da_{ij}}<0, \tag{3.2}$$

$$\frac{dy_j}{da_{ji}}<0. \tag{3.3}$$

Proof. By Theorems 2 and 3, (3.1), (3.2), and (3.3) are equivalent to $\phi_{ii}(A)<0$ for $i=1,\dots,n$, since $x>0$, $y>0$, and $v(A)>0$. It entails no loss in generality, and is notationally convenient, to relabel the rows and columns of A in the same way so that $i=1$. By Theorem 4, (2.8),

$$\phi_{11} = -(\det A)^{-1}\sum_{g=2}^{n}\sum_{h=2}^{n}(-1)^{g+h}\det A(1,h|1,g).$$

Now $H=A(1|1)=sI_{n-1}-M(1|1)$ is also a nonsingular M-matrix, because $s>\rho M\geqslant \rho M(1|1)$. Therefore $C=(c_{gh})=H^{-1}\geqslant 0$. Also $\Sigma_h c_{gh}>0$, because if $\Sigma_h c_{gh}=0$, then C would be singular. But, by (2.9),

$$c_{gh}=(-1)^{g+h}\frac{\det H(h|g)}{\det H}=(-1)^{g+h}\frac{\det A(1,h|1,g)}{\det A(1|1)}.$$

Now every principal minor of a nonsingular M-matrix is positive [5, 2:70]. Thus

$$\sum_{h=2}^{n}(-1)^{g+h}\det A(1,h|1,g)=\det A(1|1)\sum_h c_{gh}>0,$$

and since $\det A>0$, we have $\phi_{11}(A)<0$. ∎

If A is a nonsingular M-matrix, no generalization about d^2v/da_{ij}^2 $(i\neq j)$ that is as simple as (3.1) holds, at least without some additional hypothesis. For example, let M be the 3×3 matrix with all elements 0 except m_{12}, where $m_{12}=m\geqslant 0$. Then $\rho M=0$ for all $m\geqslant 0$, so $A=I-M$ is a nonsingular

M-matrix with inverse $B = A^{-1} = I + M$. Consequently, $\phi_{12}(A) = 1 - m$, so $\phi_{12}(A) > 0$ or $\phi_{12}(A) < 0$ as $m < 1$ or $m > 1$.

The concavity of $v(A)$ as a function of each diagonal element a_{ii} of A established in (3.1) raises the question whether $v(A)$ is a concave function of all the diagonal elements of A considered jointly. More precisely, if D is a real $n \times n$ diagonal matrix and A, $A + D$, and $A + 2D$ are all nonsingular M-matrices, is $2v(A + D) > v(A) + v(A + 2D)$?

To see that the answer can be no, let $A = I - M$ be the 3×3 matrix in the next to last paragraph with $m = 100$ (i.e., $a_{ii} = 1$, $i = 1,2,3$; $a_{12} = -100$; and all other elements of A are 0). Let D be the 3×3 diagonal matrix with diagonal elements $d_{11} = 2$, $d_{22} = 1$, and $d_{33} = 0$. It is easy to see that A, $A + D$, and $A + 2D$ are all nonsingular M-matrices. It is not hard to show that $v(A) = \frac{1}{103}$, $v(A + D) = \frac{2}{37}$, and $v(A + 2D) = \frac{5}{41}$, from which it follows that $2v(A + D) < v(A) + v(A + 2D)$.

4. OPEN PROBLEM

For finite matrix games that are not completely mixed, find an efficient procedure for deciding when the derivatives of the value as a function of a given matrix element exist, and find formulas for computing those derivatives when they exist. Lloyd Shapley (personal communication, 11 October 1984) observed that the nonexistence of any derivatives as a function of a given matrix element should turn up as degeneracies in the linear-programming solution of the game. As quoted in Section 1, Gross [6] gave formulas for the first derivative of the value in the general case, but these formulas have not been extended yet to higher derivatives of the value or to derivatives of the solutions.

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