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The isometry group of the Urysohn space as a Lévy group

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Abstract

We prove that the isometry group $\text{Iso}(\mathbb{U})$ of the universal Urysohn metric space \mathbb{U} equipped with the natural Polish topology is a Lévy group in the sense of Gromov and Milman, that is, admits an approximating chain of compact (in fact, finite) subgroups, exhibiting the phenomenon of concentration of measure. This strengthens an earlier result by Vershik stating that $\text{Iso}(\mathbb{U})$ has a dense locally finite subgroup.

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1. Introduction

The following concept, introduced by P.S. Urysohn [29,30], has generated a considerable and steadily growing interest over the past two to three decades.

Definition 1.1. The *Urysohn metric space* \mathbb{U} is defined by three conditions:

- (1) \mathbb{U} is a complete separable metric space;
- (2) \mathbb{U} is ultrahomogeneous, that is, every isometry between two finite metric subspaces of \mathbb{U} extends to a global isometry of \mathbb{U} onto itself;
- (3) \mathbb{U} is universal, that is, contains an isometric copy of every separable metric space.

An equivalent property distinguishing \mathbb{U} among complete separable metric spaces is *finite injectivity*: if X is a metric subspace of a finite metric space Y , then every metric embedding of X into \mathbb{U} extends to a metric embedding $Y \hookrightarrow \mathbb{U}$. Establishing an equivalence between this description and Definition 1.1 is an enjoyable exercise.

Such a metric space \mathbb{U} exists and is unique up to an isometry, and in addition to the original proof by Urysohn, there are presently several known alternative proofs of this result, most notably those in [13,36–38].

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At the same time, there is still no known concrete realization (model) of the Urysohn space, and finding such a model is one of the most interesting open problems of the theory, mentioned by such mathematicians as Fréchet [2, p. 100] and P.S. Alexandroff (cf. his commentaries to the article [31]), and presently being advertised by Vershik. The only bit of constructive knowledge about the structure of the Urysohn space currently available is that \mathbb{U} is homeomorphic to the Hilbert space ℓ^2 (Uspenskij [35]).

A “poor man’s version” of the Urysohn space \mathbb{U} , the so-called *random graph* R (discovered much later than the Urysohn space, see e.g. [27]), has a model (in fact, more than one, cf. [1]). The random graph can be viewed as a version of the universal Urysohn metric space whose metric only takes values 0, 1, 2, which fact offers some hope that a model for \mathbb{U} can also be found.

One can in fact study a variety of versions of the Urysohn space defined by a restriction on the collection of values that the distance is allowed to take. Among them, one of the most interesting and useful objects is the *Urysohn space of diameter one*, \mathbb{U}_1 . Here the diameter of a metric space X is defined as $\sup\{d(x, y) : x, y \in X\}$, the space \mathbb{U}_1 has diameter 1, and possesses the same properties as the space \mathbb{U} with the exception that it is universal for all separable metric spaces having diameter one. It is easy to show that the space \mathbb{U}_1 is isometric to the sphere of radius 1/2 around any point in \mathbb{U} .

An interesting approach to the Urysohn space was proposed by Vershik who regards \mathbb{U} as a generic, or random, metric space. Here is one of his results. Denote by M the Polish space of all metrics on a countably infinite set ω . Let $P(M)$ denote the Polish space of all probability measures on M . Then, for a generic measure $\mu \in P(M)$ (in the sense of Baire category), the completion of the metric space (X, d) is isometric to the Urysohn space \mathbb{U} μ -almost surely in $d \in M$. We refer the reader to a very interesting theory developed in [36,37] and especially [38]. Cf. also [39].

The group of all isometries of the Urysohn space \mathbb{U} onto itself, equipped with the topology of simple convergence (or the compact-open topology, which happens to be the same), is a Polish (separable completely metrizable) topological group. It possesses the following remarkable property, discovered by Uspenskij.

Theorem 1.2. (Cf. Uspenskij [33]; also [7], 3.11. $\frac{2}{3}_+$.) *The Polish group $\text{Iso}(\mathbb{U})$ is a universal second-countable topological group. In other words, every second-countable topological group G embeds into $\text{Iso}(\mathbb{U})$ as a topological subgroup.*

The same property is shared by the topological group $\text{Iso}(\mathbb{U}_1)$. Other known results include the following.

Theorem 1.3. (See Uspenskij [34].) *The group $\text{Iso}(\mathbb{U}_1)$ is topologically simple (contains no non-trivial closed normal subgroups) and minimal (admits no strictly coarser Hausdorff group topology).*

One can deduce from the above fact some straightforward but still interesting corollaries which, to this author’s knowledge, have never been stated by anyone explicitly. For example, $\text{Iso}(\mathbb{U}_1)$ admits no non-trivial (different from the identity) continuous unitary representations. In fact, a stronger result holds.

Corollary 1.4. *The topological group $\text{Iso}(\mathbb{U}_1)$ admits no non-trivial continuous representations by isometries in reflexive Banach spaces.*

Proof. According to Megrelishvili [18], the group $\text{Homeo}_+[0, 1]$ consisting of all orientation-preserving self-homeomorphisms of the closed unit interval and equipped with the compact-open topology, admits no non-trivial continuous representations by isometries in reflexive Banach spaces. By Uspenskij’s theorem, $\text{Homeo}_+[0, 1]$ embeds into $\text{Iso}(\mathbb{U}_1)$ as a topological subgroup. If now π is a continuous representation of $\text{Iso}(\mathbb{U}_1)$ in a reflexive Banach space E by isometries, that is, a continuous homomorphism $\pi : \text{Iso}(\mathbb{U}_1) \rightarrow \text{Iso}(E)$ where the latter group is equipped with the strong operator topology, then, by force of Theorem 1.3, the kernel $\ker \pi$ is either $\{e\}$ or all of $\text{Iso}(\mathbb{U}_1)$. In the former case, the restriction of π to a copy of $\text{Homeo}_+[0, 1]$ must be a continuous faithful representation by isometries in a reflexive Banach space, which is ruled out by Megrelishvili’s theorem. We conclude that $\ker \pi = \text{Iso}(\mathbb{U}_1)$, that is, the representation π is trivial (assigns the identity operator to every element of the group). \square

Modulo a result independently obtained by Megrelishvili [17] and Shtern [28], this implies:

Corollary 1.5. *Every continuous weakly almost periodic function on $\text{Iso}(\mathbb{U}_1)$ is constant.*

An action of a topological group G on a finite measure space (X, μ) is called *measurable*, or a *near-action*, if for every $g \in G$ the motion $X \ni x \mapsto gx \in X$ is a bi-measurable map defined μ -almost everywhere, and for every measurable set $A \subseteq X$ the function $G \ni g \mapsto \mu(gA \Delta A) \in \mathbb{R}$ is continuous. In addition, the identities $g(hx) = (gh)x$ and $ex = x$ hold for μ -a.e. $x \in X$ and every $g, h \in G$. Such an action is *measure class preserving* if for every measurable subset $A \subseteq X$ and every $g \in G$, the set $g \cdot A$, defined up to a μ -null set, has measure $\mu(g \cdot A) > 0$ if and only if $\mu(A) > 0$. Finally, we say that an action as above is *trivial* if the set of G -fixed points has full measure.

Corollary 1.6. *The topological group $\text{Iso}(\mathbb{U}_1)$ admits no non-trivial measurable action on a measure space, preserving the measure class.*

Proof. Indeed, every such action leads to a non-trivial strongly continuous unitary representation via the standard construction of the quasi-regular representation in the space $L^2(X, \mu)$, given by the formula

$${}^g f(x) = \left(\frac{d(\mu \circ g^{-1})}{d\mu} \right)^{1/2} f(g^{-1}x),$$

where $d/d\mu$ is the Radon–Nykodim derivative. \square

We do not know if the analogues of Theorem 1.3 and Corollaries 1.4, 1.5, 1.6 hold for the Polish group $\text{Iso}(\mathbb{U})$.

Another example of a universal Polish group was also previously discovered by Uspenskij [32]: the group $\text{Homeo}(Q)$ of self-homeomorphisms of the Hilbert cube $Q = \mathbb{I}^{\mathbb{N}_0}$ equipped with the compact-open topology. Apparently, $\text{Homeo}(Q)$ and $\text{Iso}(\mathbb{U})$ remain to date, essentially, the only known examples of universal Polish groups (if one discounts the modifications of the latter such as $\text{Iso}(\mathbb{U}_1)$). As pointed out in [23], these two topological groups are not isomorphic between themselves. Indeed, the Hilbert cube is topologically homogeneous, that is, the action of $\text{Homeo}(Q)$ on the compact space Q is transitive and therefore fixed point-free, cf. e.g. [19]. At the same time, the dynamic behavior of the groups such as $\text{Iso}(\mathbb{U})$ is markedly different.

Definition 1.7. One says that a topological group G is *extremely amenable*, or has the *fixed point on compacta property*, if every continuous action of G on a compact space X admits a fixed point: for some $\xi \in X$ and all $g \in G$, one has $g\xi = \xi$.

As first noted by Granirer and Lau [6], no locally compact group different from the trivial group $\{e\}$ is extremely amenable. In fact, until an example was constructed by Herer and Christensen in [11], the very existence of extremely amenable topological groups remained in doubt. However, since Gromov and Milman [8] proved that the unitary group $U(\ell^2)$ of a separable Hilbert space equipped with the strong operator topology is extremely amenable, it gradually became clear that the property is rather common among the concrete “infinite-dimensional” topological groups. We refer the reader to two recent articles [14,3] which together cover most of examples of extremely amenable groups known to date.

The present author had shown in [24] that the topological group $\text{Iso}(\mathbb{U})$ is extremely amenable. Consequently, it is non-isomorphic, as a topological group, to $\text{Homeo}(Q)$. The same is true of $\text{Iso}(\mathbb{U}_1)$.

Vershik has demonstrated in [40] that the group $\text{Iso}(\mathbb{U})$ contains a locally finite everywhere dense subgroup. We will give an alternative proof of this result below in Section 2. This proof is a step towards Theorem 2.13 which is the main result of our article. Before stating this result, we need to remind some concepts introduced by Gromov and Milman [8] and linking topological dynamics of “large” groups with asymptotic geometric analysis [21].

The *phenomenon of concentration of measure on high-dimensional structures* says, intuitively speaking, that the geometric structures—such as the Euclidean spheres—of high finite dimension typically have the property that an overwhelming proportion of points are very close to every set containing at least half of the points. Technically, the phenomenon is dealt with in the following framework.

Definition 1.8. (See Gromov and Milman [8].) A *space with metric and measure*, or an *mm-space*, is a triple, (X, d, μ) , consisting of a set X , a metric d on X , and a probability Borel measure μ on the metric space (X, d) .

For a subset A of a metric space X and an $\varepsilon > 0$, denote by A_ε the ε -neighborhood of A in X .

Definition 1.9. (ibid.) A family $\mathcal{X} = (X_n, d_n, \mu_n)_{n \in \mathbb{N}}$ of mm -spaces is called a *Lévy family* if, whenever Borel subsets $A_n \subseteq X_n$ satisfy

$$\liminf_{n \rightarrow \infty} \mu_n(A_n) > 0,$$

one has for every $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} \mu_n((A_n)_\varepsilon) = 1.$$

The concept of a Lévy family can be reformulated in many equivalent ways. For example, it is not difficult to see that a family \mathcal{X} as above is Lévy if and only if for every $\varepsilon > 0$, whenever A_n, B_n are Borel subsets of X_n satisfying

$$\mu_n(A_n) \geq \varepsilon, \quad \mu_n(B_n) \geq \varepsilon,$$

one has $d(A_n, B_n) \rightarrow 0$ as $n \rightarrow \infty$.

This is formalized using the notion of *separation distance*, proposed by Gromov ([7], Section 3 $\frac{1}{2}$.30). Given numbers $\kappa_0, \kappa_1, \dots, \kappa_N > 0$, one defines the invariant

$$\text{Sep}(X; \kappa_0, \kappa_1, \dots, \kappa_N)$$

as the supremum of all δ such that X contains Borel subsets $X_i, i = 0, 1, \dots, N$, with $\mu(X_i) \geq \kappa_i$, every two of which are at a distance $\geq \delta$ from each other. Now a family $\mathcal{X} = (X_n, d_n, \mu_n)_{n \in \mathbb{N}}$ is a Lévy family if and only if for every $0 < \varepsilon < \frac{1}{2}$, one has

$$\text{Sep}(X_n; \varepsilon, \varepsilon) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

The reader should consult Ch. 3 $\frac{1}{2}$ in [7] for numerous other characterizations of Lévy families of mm -spaces.

We will state just one more such reformulation. It is an easy exercise to show that in Definition 1.9 of a Lévy family it is enough to assume that the values $\mu_n(A_n)$ are bounded away from zero by $1/2$ (or by any other fixed constant strictly between zero and one). In other words, a family \mathcal{X} is a Lévy family if and only if, whenever Borel subsets $A_n \subseteq X_n$ satisfy $\mu_n(A_n) \geq 1/2$, one has for every $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} \mu_n(A_n)_\varepsilon = 1.$$

This leads to the following concept [20,22], providing convenient quantitative bounds on the rate of convergence of $\mu_n(A_n)_\varepsilon$ to one.

Definition 1.10. Let (X, d, μ) be a space with metric and measure. The *concentration function* of X , denoted by $\alpha_X(\varepsilon)$, is a real-valued function on the positive axis $\mathbb{R}_+ = [0, \infty)$, defined by letting $\alpha(0) = 1/2$ and for all $\varepsilon > 0$

$$\alpha_X(\varepsilon) = 1 - \inf \left\{ \mu(B_\varepsilon) : B \subseteq X, \mu(B) \geq \frac{1}{2} \right\}.$$

Thus, a family $\mathcal{X} = (X_n, d_n, \mu_n)_{n \in \mathbb{N}}$ of mm -spaces is a Lévy family if and only if

$$\alpha_{X_n} \rightarrow 0 \text{ pointwise on } (0, +\infty) \quad \text{as } n \rightarrow \infty.$$

A Lévy family is called *normal* if for suitable constants $C_1, C_2 > 0$,

$$\alpha_{X_n}(\varepsilon) \leq C_1 e^{-C_2 \varepsilon^2 n}.$$

Example 1.11. The Euclidean spheres $\mathbb{S}^n, n \in \mathbb{N}_+$, of unit radius, equipped with the Haar measure (translation-invariant probability measure) and Euclidean (or geodesic) distance, form a normal Lévy family.

Definition 1.12. (See Gromov and Milman [8].) A metrizable topological group G is called a *Lévy group* if it contains an increasing chain of compact subgroups

$$G_1 < G_2 < \dots < G_n < \dots,$$

having an everywhere dense union in G and such that for some right-invariant compatible metric d on G the groups G_n , equipped with the normalized Haar measures and the restrictions of the metric d , form a Lévy family.

The above concept admits a number of generalizations, in particular, it makes perfect sense for non-metrizable, non-separable topological groups as well. In fact, in the definition of a Lévy group it is the uniform structure on G that matters rather than a metric. Namely, one can easily prove the following.

Proposition 1.13. *Let G be a metrizable topological group containing an increasing chain of compact subgroups (G_n) with everywhere dense union. The subgroups (G_n) form a Lévy family with regard to the normalized Haar measures and the restrictions of some right-invariant metric d on G if and only if for every neighborhood of identity, V , in G and every collection of Borel subsets $A_n \subseteq G_n$ with the property $\mu_n(A_n) \geq 1/2$ one has*

$$\lim_{n \rightarrow \infty} \mu_n(V A_n) = 1.$$

Examples of presently known Lévy groups can be found in [8,22,4,23,15,3,5].

The following result had been also established in [8], and one can give numerous alternative proofs to it, cf. e.g. [4,23,3,26].

Theorem 1.14. *Every Lévy group is extremely amenable.*

The concept of a Lévy group is stronger than that of an extremely amenable group. Typically, examples of extremely amenable groups coming from combinatorics as groups of automorphisms of infinite Fraïssé order structures [14] are not Lévy groups, because they contain no compact subgroups whatsoever. Even the dynamical behavior of Lévy groups has been shown by Glasner et al. [5] to differ considerably from that of the rest of extremely amenable groups.

The main theorem of this article (Theorem 2.13) states that the group $\text{Iso}(\mathbb{U})$ is a Lévy group rather than merely an extremely amenable one.

The monograph [25] by the present author provides an introduction to the theory of extremely amenable groups and its links with geometric functional analysis and combinatorics. The Urysohn metric space can also be found there. In fact, the book also contains Theorem 2.13 (cf. Section 3.4.3). The reason to have this theorem published in the present Proceedings is twofold: firstly, the book [25], published in Brazil, is not readily available, and secondly, the proof of Theorem 2.13 as presented there is not sufficiently accurate.

2. Approximating $\text{Iso}(\mathbb{U})$ with finite subgroups

Let $\Gamma = (V, E)$ be an (undirected, simple) graph, where V is the set of vertices and E is the set of edges. A *weight* on Γ is an assignment of a non-negative real number to every edge, that is, a function $w : E \rightarrow \mathbb{R}_+$. The pair (Γ, w) forms a *weighted graph*. The *path pseudometric* on a connected weighted graph (Γ, w) is the maximal pseudometric on Γ with the property $d(x, y) = w(x, y)$ for any pair of adjacent vertices x, y . Equivalently, the value of $\rho(x, y)$ is given for each $x, y \in V$ by

$$\rho(x, y) = \inf \sum_{i=0}^{N-1} d(a_i, a_{i+1}), \tag{1}$$

where the infimum is taken over all positive natural N and all finite sequences of vertices $x = a_0, a_1, \dots, a_{N-1}, a_N = y$, with the property that a_i and a_{i+1} are adjacent for all i . Notice that here we allow for sequences of length one, in which case the sum above is empty and returns value zero, the distance from a vertex to itself.

In particular, if every edge is assigned the weight one, the corresponding path pseudometric is a metric, called the *path metric* on Γ .

Let G be a group, and V a generating subset of G . Assume that V is symmetric ($V = V^{-1}$) and contains the identity. The *Cayley graph* associated to the pair (G, V) has all elements of the group G as vertices, and two of them, $x, y \in G, x \neq y$, are adjacent if and only if $x^{-1}y \in V$. The Cayley graph is connected. The corresponding path metric on G is called the *word distance* with regard to the generating set V .

If V is an arbitrary generating subset of G , then the word distance with regard to V is defined as that with regard to $V \cup V^{-1} \cup \{e\}$. The value of the word distance between e and an element x is called the *reduced length* of x with regard to the generating set V , and denoted $\ell_V(x)$. It is simply the smallest integer n such that x can be written as a product of $\leq n$ elements of V and their inverses. Since the identity e of the group G is represented, as usual, by an empty word, one has $V^0 = \{e\}$ and $\ell_V(e) = 0$.

Lemma 2.1. *Let G be a group equipped with a left-invariant pseudometric, d . Let V be a finite generating subset of G containing the identity. Then there is the maximal pseudometric, ρ , among all left-invariant pseudometrics on G , whose restriction to V is majorized by d . The restrictions of ρ and d to V coincide. If $d|_V$ is a metric on V , then ρ is a metric as well, and for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $\ell_V(x) \geq N$ implies $\rho(e, x) \geq \varepsilon$.*

Proof. Make the Cayley graph Γ associated to the pair $(G, V^{-1}V)$ into a weighted graph, by assigning to every edge (x, y) , $x^{-1}y \in V^{-1}V$, the value $d(x, y) \equiv d(x^{-1}y, e)$. Denote by ρ the corresponding path pseudometric on the weighted graph Γ . To prove the left-invariance of ρ , let $x, y, z \in G$. Consider any sequence of elements of G ,

$$x = a_0, a_1, \dots, a_{N-1}, a_N = y, \tag{2}$$

where $N \in \mathbb{N}$ and $a_i^{-1}a_{i+1} \in V^{-1}V$, $i = 0, 1, \dots, n - 1$. Since for all i the elements za_i, za_{i+1} are adjacent in the Cayley graph ($(za_i)^{-1}za_{i+1} = a_i^{-1}a_{i+1} \in V^{-1}V$), one has

$$d(zx, zy) \leq \sum_{i=0}^{n-1} d(za_i, za_{i+1}) = \sum_{i=0}^{n-1} d(a_i, a_{i+1}),$$

and taking the infimum over all sequences as in Eq. (2) on both sides, one concludes $d(zx, zy) \leq d(x, y)$, which of course implies the equality.

For every $x, y \in V$, one has $x^{-1}y \in V^{-1}V$ and consequently $\rho(x, y) = \rho(x^{-1}y, e) \leq d(x^{-1}y, e) = d(x, y)$. Now let ζ be any left-invariant pseudometric on G whose restriction to V is majorized by d . If $a, b \in G$ are such that $a^{-1}b \in V^{-1}V$, then for some $c, d \in V$ one has $a^{-1}b = c^{-1}d$, and

$$\zeta(a, b) = \zeta(a^{-1}b, e) = \zeta(c^{-1}d, e) = \zeta(c, d) \leq d(c, d) = d(c^{-1}d, e) = d(a^{-1}b, e) = d(a, b).$$

For every sequence as in Eq. (2), one now has

$$\zeta(x, y) \leq \sum_{i=0}^{n-1} \zeta(a_i, a_{i+1}) \leq \sum_{i=0}^{n-1} d(a_i, a_{i+1}),$$

and by taking the infimum over all such finite sequences on both sides, one concludes

$$\zeta(x, y) \leq \rho(x, y),$$

that is, ρ is maximal among all left-invariant pseudometrics whose restriction to V is majorized by d . In particular, $\rho \geq d$, which implies $\rho|_V = d|_V$.

Assuming that $d|_V$ is a metric, all the weights on the Cayley graph Γ as above assume strictly positive values, and consequently ρ is a metric. As we have already noticed, for every $x, y \in G$ with the property $x^{-1}y \in V^{-1}V$, the value $d(x, y)$ is of the form $d(a, b)$ for suitable $a, b \in V$. Consequently, there exists the smallest value taken by d between pairs of distinct elements $x, y \in G$ with the property $x^{-1}y \in V^{-1}V$, and it is strictly positive. Denote this value by δ . Clearly, for every $x \in G$ one has $\rho(e, x) \geq \delta \ell_{V^{-1}V}(x) \geq (\delta/2)\ell_V(x)$, and the proof is finished. \square

Next we are going to get rid of the restrictions on V . The price to pay is to agree that all pseudometrics will be bounded by 1. In the following lemma, $\ell_V(x)$ will denote the word length of x with regard to V if x is contained in the subgroup generated by V , and ∞ otherwise.

Lemma 2.2. *Let G be a group equipped with a left-invariant pseudometric, d , whose values are bounded by 1. Let V be a finite subset of G . Then there is the maximal pseudometric, ρ , among all left-invariant pseudometrics on G , bounded by one and whose restriction to V is majorized by d . The restrictions of ρ and d to V coincide. If $d|_V$ is a metric on V , then ρ is a metric on G .*

Proof. The set Ψ of all left-invariant pseudometrics on G bounded by one and whose restrictions to V are majorized by d is nonempty ($d \in \Psi$), and contains the maximal element, ρ , given by $\rho(x, y) = \sup_{\zeta \in \Psi} \zeta(x, y)$. Obviously, $\rho|_V = d|_V$. To verify the last assertion, let δ be the smallest strictly positive value of the form $d(x, y)$, $x, y \in V$, $x \neq y$. Let ζ now denote the metric on G taking values 0 and δ . Denote by $\langle V \rangle$ the subgroup of G generated by V . According to Lemma 2.1, there exists the maximal metric ζ_1 on the subgroup $\langle V \rangle$ of G generated by V whose restriction to $V \cup V^{-1} \cup \{e\}$ only takes the values 0 or δ . Define a pseudometric ζ_2 on all of G by the rule

$$\zeta_2(x, y) = \begin{cases} \min\{1, \zeta_1(x, y)\}, & \text{if } x^{-1}y \in \langle V \rangle, \\ 1, & \text{otherwise.} \end{cases}$$

Since $\zeta_1|_V \leq d|_V$, it follows that $\zeta_2|_V \leq \rho$, and thus ρ is a metric. \square

Lemma 2.3. *Let ρ be a left-invariant pseudometric on a group G , and let $H \triangleleft G$ be a normal subgroup. The formula*

$$\bar{\rho}(xH, yH) := \inf_{h_1, h_2 \in H} \rho(xh_1, yh_2) \equiv \inf_{h_1, h_2 \in H} \rho(h_1x, h_2y) \equiv \inf_{h \in H} \rho(hx, y) \tag{3}$$

defines a left-invariant pseudometric on the factor-group G/H . This is the largest pseudometric on G/H with respect to which the quotient homomorphism $G \rightarrow G/H$ is 1-Lipschitz.

Proof. The triangle inequality follows from the fact that, for all $h' \in H$,

$$\begin{aligned} \bar{\rho}(xH, yH) &= \inf_{h \in H} \rho(hx, y) \leq \inf_{h \in H} [\rho(hx, h'z) + \rho(h'z, y)] = \inf_{h \in H} \rho(hx, h'z) + \rho(h'z, y) \\ &= \inf_{h \in H} \rho(h'^{-1}hx, z) + \rho(h'z, y) = \bar{\rho}(xH, zH) + \rho(h'z, y), \end{aligned}$$

and the infimum of the right-hand side taken over all $h' \in H$ equals $\bar{\rho}(xH, zH) + \bar{\rho}(zH, yH)$. Left-invariance of $\bar{\rho}$ is obvious. If d is a pseudometric on G/H making the quotient homomorphism into a 1-Lipschitz map, then $d(xH, yH) \leq \rho(xh_1, yh_2)$ for all $x, y \in G$, $h_1, h_2 \in H$, and therefore $d(xH, yH) \leq \bar{\rho}(xH, yH)$. \square

We will make a distinction between the notion of a *distance-preserving map* $f : X \rightarrow Y$ between two pseudometric spaces, which has the property $d_Y(fx, fy) = d_X(x, y)$ for all $x, y \in X$, and an *isometry*, that is, a distance-preserving bijection.

Let G be a group. For every left-invariant bounded pseudometric d on G , denote $H_d = \{x \in G : d(x, e) = 0\}$, and let \hat{d} be the metric on the left coset space G/H_d given by $\hat{d}(xH_d, yH_d) = d(x, y)$. The metric \hat{d} is invariant under left translations by elements of G . We will denote the metric space $(G/H_d, \hat{d})$, equipped with the left action of G by isometries, simply by G/d .

A distance-preserving map need not be an isometry. For instance, if d is a left-invariant pseudometric on a group G , then the natural map $G \rightarrow G/d$ is distance-preserving, onto, but not necessarily an injection.

A group G is *residually finite* if it admits a separating family of homomorphisms into finite groups, or, equivalently, if for every $x \in G$, $x \neq e$, there exists a normal subgroup $H \triangleleft G$ of finite index such that $x \notin H$. Every free group is residually finite (cf. e.g. [16]), and the free product of two residually finite groups is residually finite [9].

Lemma 2.4. *Let G be a residually finite group equipped with a left-invariant pseudometric $d \leq 1$, and let $V \subseteq G$ be a finite subset. Suppose the restriction $d|_V$ is a metric, and let ρ be the maximal left-invariant metric on G bounded by one with $\rho|_V = d|_V$. Then there exists a normal subgroup $H \triangleleft G$ of finite index with the property that the restriction of the quotient homomorphism $G \rightarrow G/H$ to V is an isometry with regard to ρ and the quotient pseudometric $\bar{\rho}$ (which is in fact a metric).*

Proof. Let $\delta > 0$ be the smallest distance between any pair of distinct elements of V . Let $N \in \mathbb{N}_+$ be even and such that $(N - 2)\delta \geq 1$. The subset formed by all words of length $\leq N$ in V is finite, and, since the intersection of finitely many subgroups of finite index has finite index (Poincaré’s theorem), one can choose a normal subgroup $H \triangleleft G$ of finite index containing no words of V -length $\leq N$ other than e . Let $x, y \in V$ and $h \in H, h \neq e$. If $y^{-1}hx \notin \langle V \rangle$, then $\rho(hx, y) = 1$. If $y^{-1}hx \in \langle V \rangle$, then the reduced V -length of the word $y^{-1}hx$ is $\geq N - 2$, and consequently

$$\rho(hx, y) = \rho(y^{-1}hx, e) \geq \min\{1, (N - 2)\delta\} \geq 1.$$

(Otherwise there would exist a representation

$$y^{-1}hx = v_1v_2 \dots v_k$$

with $v_i \in V$ and $d(e, v_1) + \sum_{i=1}^{k-1} d(v_i, v_{i+1}) < (N - 2)\delta$, that is, $k < N - 2$.)

In either case, the distance $\bar{\rho}(xH, yH)$ between cosets is realized on the representatives x, y :

$$\bar{\rho}(xH, yH) = \rho(x, y).$$

The factor-pseudometric $\bar{\rho}$ on G/H is, according to Lemma 2.3, the largest pseudometric making the factor-map π 1-Lipschitz. We claim that $\bar{\rho}$ is the largest left-invariant pseudometric on G/H , bounded by one, whose restriction to V coincides with the metric on V . Indeed, denoting such a pseudometric by ζ , one sees that $\zeta \circ \pi$ is a left-invariant pseudometric on G , bounded by one, and whose restriction to V equals $d_\xi|_V$. It follows that $\zeta \circ \pi \leq \rho$, thence $\zeta \leq \bar{\rho}$ and the two coincide. Now Lemma 2.2 tells us that $\bar{\rho}$ is a metric. \square

The following concept, along with the two subsequent results, forms a powerful tool in the theory of the Urysohn space.

Definition 2.5. (See Uspenskij [34].) One says that a metric subspace Y is *g-embedded* into a metric space X if there exists an embedding of topological groups $e: \text{Iso}(Y) \hookrightarrow \text{Iso}(X)$ with the property that for every $h \in \text{Iso}(Y)$ the isometry $e(h): X \rightarrow X$ is an extension of h :

$$e(h)|_X = h.$$

Proposition 2.6. (See Uspenskij [33,34].) *Each separable metric space X admits a g-embedding into the complete separable Urysohn metric space \mathbb{U} .*

Every isometry between two compact metric subspaces of the Urysohn space \mathbb{U} extends to a global self-isometry of \mathbb{U} (it was first established in [12]). Together with Proposition 2.6, this fact immediately leads to the following result.

Proposition 2.7. *Each isometric embedding of a compact metric space into \mathbb{U} is a g-embedding.*

Recall that an action of a group G on a set X is *free* if for all $g \in G, g \neq e$ and all $x \in X$, one has $g \cdot x \neq x$. Here comes the main technical result of this paper.

Lemma 2.8. *Let X be a finite subset of the Urysohn space \mathbb{U} , and let a finite group G act on X freely by isometries. Let f be an isometry of \mathbb{U} , and let $\varepsilon > 0$. There exist a finite group \tilde{G} containing G as a subgroup, an element $\tilde{f} \in \tilde{G}$, and a finite metric space $Y, X \subseteq Y \subseteq \mathbb{U}$, upon which \tilde{G} acts freely by isometries, extending the original action of G on X and so that for all $x \in X$ one has $d(fx, \tilde{f}x) < \varepsilon$.*

Proof. Without loss in generality, one can assume that the image $f(X)$ does not meet X , by replacing f , if necessary, with an isometry f' such that the image $f'(X)$ does not intersect X , and yet for every $x \in X$ one has $d_{\mathbb{U}}(f(x), f'(x)) < \varepsilon$. By renormalizing the distance if necessary, we will further assume that the diameter of the set $X \cup f(X)$ does not exceed 1.

Since every compact subset of \mathbb{U} such as X is *g-embedded* into the Urysohn space (Proposition 2.7), one can extend the action of G by isometries from X to all of \mathbb{U} .

Choose any element $\xi \in \mathbb{U}$ at a distance 1 from every element of $X \cup f(X)$. Let $\Theta = X/G$ denote the set of G -orbits of X . For each $\theta \in \Theta$, choose an element $x_\theta \in \theta$ and an isometry f_θ of \mathbb{U} in such a way that $f_\theta(\xi) = x_\theta$. Let $n = |\Theta|$, and let F_n be the free group on n generators which we will denote likewise f_θ , $\theta \in \Theta$.

Finally, denote by f a generator of the group \mathbb{Z} , and let $F = G * F_n * \mathbb{Z}$ be the free product of three groups.

There is a unique homomorphism $F \rightarrow \text{Iso}(\mathbb{U})$, which sends all elements of $G \cup \{f_\theta: \theta \in \Theta\} \cup \{f\}$ to the corresponding self-isometries of \mathbb{U} . In this way, F acts on \mathbb{U} by isometries. Denote

$$V = \{g \circ f_\theta: g \in G, \theta \in \Theta\} \cup \{f \circ g \circ f_\theta: g \in G, \theta \in \Theta\}.$$

The formula

$$d_\xi(g, h) := \min\{1, d_{\mathbb{U}}(g(\xi), h(\xi))\}, \quad g, h \in F,$$

defines a left-invariant pseudometric d_ξ on the group F , bounded by 1.

Denote by $\text{ev}: F \rightarrow \mathbb{U}$ the evaluation map $\phi \mapsto \phi(\xi)$. The restriction $\text{ev}|_V$ is an isometry between V , equipped with the restriction of the pseudometric d_ξ , and $X \cup f(X)$. Also notice that the restriction $\text{ev}|_{\{g \circ f_\theta: g \in G, \theta \in \Theta\}}$ establishes an isomorphism of G -spaces between the latter set (upon which G acts by left multiplication in the group F) and X . Both properties take into account the freeness of the action of G on X .

The restriction of the pseudometric d_ξ to V is a metric. Let ρ be the maximal left-invariant metric on F bounded by 1 such that $\rho|_V = d_\xi|_V$ (Lemma 2.2).

The group F , being the free product of three residually finite groups, is residually finite, and so we are under the assumptions of Lemma 2.4. Choose a normal subgroup $H \triangleleft F$ of finite index in such a way that if the finite factor-group F/H is equipped with the factor-pseudometric $\bar{\rho}$, then the restriction of the factor-homomorphism $\pi: F \rightarrow F/H$ to V is an isometry. This $\bar{\rho}$ is then a metric. In addition, by replacing H with a smaller normal subgroup of finite index if necessary, one can clearly choose H so that $H \cap G = \{e\}$, and thus $\pi|_G$ is a monomorphism.

The finite group $\tilde{G} = F/H$ acts on itself by left translations, and this action is a free action by isometries on the finite metric space $Y = (F/H, \bar{\rho})$. The metric space $X \cup f(X)$ embeds into Y as a metric subspace through the isometry $\pi \circ \text{ev}$, and $\tilde{f}|_X = f|_X$. Finally, G is a subgroup of \tilde{G} , and X is contained inside Y as a G -space.

Finally, the embedding of $X \cup f(X)$ (considered as a subspace of Y) can be extended over Y , so we can view Y as a metric subspace of \mathbb{U} , and it is a g -embedding by Proposition 2.7. \square

Now we are ready to give an alternative proof of the following result of Vershik. Recall that a group G is *locally finite* if every finitely generated subgroup of G is finite. A countable group is locally finite if and only if it is the union of an increasing chain of finite subgroups.

Theorem 2.9. (See Vershik [40].) *The isometry group $\text{Iso}(\mathbb{U})$ of the Urysohn space, equipped with the standard Polish topology, contains an everywhere dense locally finite countable subgroup.*

Proof. Choose an everywhere dense subset $F = \{f_i: i \in \mathbb{N}_+\}$ of $\text{Iso}(\mathbb{U})$ and a point $x_1 \in \mathbb{U}$.

Let $G_1 = \{e\}$ be a trivial group, trivially acting on \mathbb{U} by isometries. Clearly, the restriction of this action on the G_1 -orbit of $\{x_1\}$ is free.

Assume that for an $n \in \mathbb{N}$ one has chosen recursively a finite group G_n , an action σ_n by isometries on \mathbb{U} , and a collection of points $\{x_1, \dots, x_{2^n}\}$ in such a way that the restriction of the action σ_n to the G_n -orbit of $\{x_1, x_2, \dots, x_{2^n}\}$ is free.

Using Lemma 2.8, choose a finite group G_{n+1} containing (an isomorphic copy) of G_n , an element $\tilde{f}_n \in G_{n+1}$ and an action σ_{n+1} of G_{n+1} on \mathbb{U} by isometries such that for every $j = 1, 2, \dots, 2^n$ and each $g \in G_n$ one has

$$\sigma_n(g)x_j = \sigma_{n+1}(g)x_j,$$

the elements $x_{2^n+j} = \tilde{f}_n(x_j)$, $j = 1, 2, \dots, 2^n$, are all distinct from any of x_i , $i \leq 2^n$, the restriction of the action of G_{n+1} on the G_{n+1} -orbit of $\{x_0, x_1, \dots, x_{2^{n+1}}\}$ is free, and

$$d_{\mathbb{U}}(f_n(x_j), \tilde{f}_n(x_j)) < 2^{-n}, \quad j \leq 2^n.$$

The subset $X = \{x_i: i \in \mathbb{N}_+\}$ is everywhere dense in \mathbb{U} . Indeed, for each $n \in \mathbb{N}$ the subset $\{f_i(x_n): i \geq n\}$ is everywhere dense in \mathbb{U} , and since it is contained in the 2^{-n} -neighborhood of $\{\tilde{f}_i(x_n): i \geq n\} \subset X$, the statement follows.

The group $G = \bigcup_{i=1}^{\infty} G_n$ is locally finite. Now let $g \in G$. For every $i \in \mathbb{N}_+$, the value $g \cdot x_i$ is well-defined as the limit of an eventually constant sequence, and determines an isometry from an everywhere dense subset $X \subset \mathbb{U}$ into \mathbb{U} . Consequently, it extends uniquely to an isometry from \mathbb{U} into itself. If $g, h \in G$, then the isometry determined by gh is the composition of isometries determined by g and h : every $x \in X$ has the property $(gh)(x) = g(h(x))$, once $x = x_i, i \leq N$, and $g, h \in G_N$, and this property extends over all of \mathbb{U} . Thus, G acts on \mathbb{U} by isometries (which are therefore onto).

Finally, notice that G is everywhere dense in $\text{Iso}(\mathbb{U})$. It is enough to consider the basic open sets of the form

$$\{f \in \text{Iso}(\mathbb{U}): d(f(x_i), g(x_i)) < \varepsilon, i = 1, 2, \dots, n\},$$

where $g \in \text{Iso}(\mathbb{U}), n \in \mathbb{N}$, and $\varepsilon > 0$. Since F is everywhere dense in $\text{Iso}(\mathbb{U})$, there is an $m \in \mathbb{N}$ with $n \leq 2^{m-1}, 2^{-m} < \varepsilon/2$, and $d(f_m(x_i), g(x_i)) < \varepsilon/2$ for all $i = 1, 2, \dots, n$. One concludes: $d(\tilde{f}_m(x_i), g(x_i)) < \varepsilon$ for $i = 1, 2, \dots, n$, and $\tilde{f}_m \in G_m \subset G$, which settles the claim. \square

A further refinement of our argument leads to another approximation Theorem 2.13, which states that $\text{Iso}(\mathbb{U})$ is a Lévy group and forms the central result of the present paper. The proof will interlace the recursion steps in the proof of Theorem 2.9 with an adaptation of an idea used in the proof of the following result to obtain, historically, the second ever example of a Lévy group, after $U(\ell^2)$.

Theorem 2.10. (See Glasner [4]; Furstenberg and Weiss (unpublished).) *Let G be a compact metric group, and let d be an invariant metric on G . The group $L^1([0, 1]; G)$ of all equivalence classes of Borel maps from the unit interval $[0, 1]$ to G , equipped with the metric $d_1(f, g) = \int_0^1 d(f(x), g(x)) dx$ (cf. [10]), is a Lévy group.*

The following well-known and important result is being established using the probabilistic techniques (martingales). (Cf. the more general Theorem 7.8 in [22] or Theorem 4.2 in [15].)

Theorem 2.11. *Let $(X_i, d_i, \mu_i), i = 1, 2, \dots, n$, be metric spaces with measure, each having finite diameter a_i . Equip the product $X(i) = \prod_{i=1}^n X_i$ with the product measure $\otimes_{i=1}^n \mu_i$ and the ℓ_1 -type (Hamming) metric*

$$d(x, y) = \sum_{i=1}^n d_i(x_i, y_i).$$

Then the concentration function of X satisfies

$$\alpha_X(\varepsilon) \leq 2e^{-\varepsilon^2/8 \sum_{i=1}^n a_i^2}.$$

Let us consider the following particular case. Let (X, d) be a finite metric space, and let Z be a finite set equipped with the normalized counting measure μ_{\sharp} , that is, $\mu_{\sharp}(A) = |A|/|Z|$. We will equip the collection X^Z of all maps from Z to X with the $L_1(\mu_{\sharp})$ -metric:

$$d_1(f, g) = \int_Z d(f(z), g(z)) d\mu_{\sharp}(z).$$

This is just the ℓ_1 -metric normalized:

$$d_1(f, g) = \frac{1}{|Z|} \sum_{z \in Z} d(f(z), g(z)).$$

It is also known as the (generalized) *normalized Hamming distance*. In particular, if $a = \text{diam}(Z)$ is the diameter of Z , then the diameter of every “factor” of the form $\{z\} \times Z$ is a/n , and Theorem 2.11 gives the following.

Corollary 2.12. *Let (X, d) is a finite metric space of diameter a and let $n \in \mathbb{N}$. Let the metric space X^n be equipped with the normalized counting measure and the normalized Hamming distance. Then the concentration function of m -space X^n satisfies*

$$\alpha_{X^n}(\varepsilon) \leq 2e^{-n\varepsilon^2/8a^2}.$$

Notice that X^n with the above metric contains an isometric copy of X , consisting of all constant functions.

If a finite group G acts on a finite metric space X by isometries, then this action naturally extends to an action of G^n on X^n by isometries, where the latter set is equipped with the normalized Hamming, or $L_1(\mu_{\sharp})$, metric. If the action of G on X is free, then so is the action of G^n on X^n .

Theorem 2.13. *The isometry group $\text{Iso}(\mathbb{U})$ of the Urysohn space, equipped with the standard Polish topology, is a Lévy group. Moreover, the groups in the approximating Lévy family can be chosen finite.*

Proof. As in the proof of Theorem 2.9, choose an everywhere dense subset $F = \{f_i : i \in \mathbb{N}_+\}$ of $\text{Iso}(\mathbb{U})$ and a point $x_1 \in \mathbb{U}$. Set $G_1 = \{e\}$ and $X_1 = \{x_1\}$. Assume that for an $n \in \mathbb{N}_+$ a finite group G_n , an action σ_n by isometries on \mathbb{U} , and a finite G_n -invariant subset $X_n \subset \mathbb{U}$ have been chosen. Also assume that G_n acts on X_n freely. Let a_n be the diameter of X_n . Choose $m_n \in \mathbb{N}$ so that

$$m_n \geq 8a_n^2 n. \tag{4}$$

The finite metric space $\tilde{X}_n = X_n^{m_n}$ (with the $L_1(\mu_{\sharp})$ -metric) contains X_n as a subspace of constant functions, therefore one can embed \tilde{X}_n into \mathbb{U} so as to extend the embedding $X_n \hookrightarrow \mathbb{U}$ (the finite injectivity of \mathbb{U}).

The group $\tilde{G}_n = G_n^{m_n}$ acts on the metric space \tilde{X}_n freely by isometries. Since every embedding of a compact subspace into \mathbb{U} is a g -embedding, one can simultaneously extend the action of \tilde{G}_n to a global action, $\tilde{\sigma}_n$, on \mathbb{U} by isometries. Now construct the group G_{n+1} and its action σ_{n+1} by isometries exactly as in the proof of Theorem 2.9, but beginning with \tilde{G}_n instead of G_n and \tilde{X}_n instead of $\{x_1, \dots, x_{2^n}\}$. Namely, using Lemma 2.8, choose a finite group G_{n+1} containing (an isomorphic copy) of \tilde{G}_n , an element $f_n \in G_{n+1}$ and an action σ_{n+1} of G_{n+1} on \mathbb{U} by isometries such that for every $x \in \tilde{X}_n$ and each $g \in G_n$ one has

$$\sigma_n(g)x = \sigma_{n+1}(g)x,$$

the sets $f_n(\tilde{X}_n)$ and \tilde{X}_n are disjoint, the restriction of the action of G_{n+1} on the G_{n+1} -orbit of \tilde{X}_n is free, and

$$d_{\mathbb{U}}(f_n(x), \tilde{f}_n(x)) < 2^{-n} \quad \text{for all } x \in \tilde{X}_n.$$

Denote $X_{n+1} = G_{n+1} \cdot \tilde{X}_n$. The step of recursion is accomplished.

The union $G = \bigcup_{i=1}^{\infty} G_i = \bigcup_{i=1}^{\infty} \tilde{G}_i$ is, like in the proof of Theorem 2.9, an everywhere dense locally finite subgroup of $\text{Iso}(\mathbb{U})$, and it only remains to show that the groups \tilde{G}_n , $n \in \mathbb{N}_+$, form a Lévy family with regard to the uniform structure inherited from $\text{Iso}(\mathbb{U})$.

First, consider the groups $\tilde{G}_n = G_n^{m_n}$ equipped with the $L_1(\mu_{\sharp})$ -metric formed with regard to the *discrete* (that is, $\{0, 1\}$ -valued) metric on G_n . If V_{ε} is the ε -neighborhood of the identity, then for every $g \in V_{\varepsilon}$ and each $x \in \tilde{X}_n = X_n^{m_n}$ one has $d_1(g \cdot x, x) < \varepsilon \cdot a_n$, where $a_n = \text{diam } X_n$. Consequently, if $g \in V_{\varepsilon/a_n}$, then $d_1(g \cdot x, x) < \varepsilon$.

Now let us turn to the group topology induced from $\text{Iso}(\mathbb{U})$. Let

$$V[x_1, \dots, x_t; \varepsilon] = \{f \in \text{Iso}(\mathbb{U}) : \forall i = 1, 2, \dots, t, d_{\mathbb{U}}(x_i, f(x_i)) < \varepsilon\}$$

be a standard neighborhood of the identity in $\text{Iso}(\mathbb{U})$. Here one can assume without loss of generality, that $x_i \in \bigcup_{n=1}^{\infty} X_n$, $i = 1, 2, \dots, t$, because the union of X_n 's is everywhere dense in \mathbb{U} . Let $k \in \mathbb{N}$ be such that $x_1, x_2, \dots, x_t \in X_k$. For all $n \geq k$, if $A \subseteq \tilde{G}_n$ contains at least half of all elements, the set $V_{\varepsilon/a_n} A$ is of Haar measure (taken in \tilde{G}_n) at least $1 - 2e^{-m_n \varepsilon / 8a_n^2}$, according to Theorem 2.11. The set $V[x_1, \dots, x_t; \varepsilon] \cdot A$ contains $V_{\varepsilon/a_n} A$ and so the measure of its intersection with \tilde{G}_n is at least as big. According to the choice of numbers m_n (Eq. (4)),

$$\mu_n(\tilde{G}_n \cap (V[x_1, \dots, x_t; \varepsilon] \cdot A)) \geq 1 - e^{-n\varepsilon^2}.$$

By Proposition 1.13, the family of groups \tilde{G}_n is Lévy. \square

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