Available online at www.sciencedirect.com

SCIENCE DIRECT

Advances in Mathematics 194 (2005) 332–344

ADVANCES IN Mathematics

www.elsevier.com/locate/aim

h-Vectors of Gorenstein* simplicial posets

Mikiya Masuda

Department of Mathematics, Graduate School of Science, Osaka City University, Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan

Received 20 November 2003; accepted 28 June 2004

Communicated by Michael Hopkins
Available online 13 August 2004

Dedicated to Professor Yukio Matsumoto for his 60th birthday

Abstract

As is well known, h-vectors of simplicial convex polytopes are characterized. Those h-vectors satisfy Dehn–Sommerville equations and some inequalities conjectured by P. McMullen and first proved by R. Stanley using toric geometry. The boundary of a simplicial convex polytope determines a Gorenstein* simplicial poset but there are many Gorenstein* simplicial posets which do not arise this way. However, it is known that h-vectors of Gorenstein* simplicial posets still satisfy Dehn–Sommerville equations and that every component in the h-vectors is non-negative. In this paper we prove that h-vectors of Gorenstein* simplicial posets must satisfy one more subtle condition conjectured by R. Stanley and complete the characterization of h-vectors of Gorenstein* simplicial posets. Our proof is purely algebraic but the idea of the proof stems from topology.

MSC: 05A15; 52B05; 57S15

Keywords: Face ring; Simplicial poset; Face number

1. Introduction

A *simplicial poset P* (also called a *boolean poset* and a *poset of boolean type*) is a finite poset with a smallest element $\hat{0}$ such that every interval $[\hat{0}, y]$ for $y \in P$ is a boolean algebra, i.e., $[\hat{0}, y]$ is isomorphic to the set of all subsets of a finite

E-mail address: masuda@sci.osaka-cu.ac.jp.

doi:10.1016/j.aim.2004.06.009

set, ordered by inclusion. The set of all faces of a (finite) simplicial complex with empty set added forms a simplicial poset ordered by inclusion, where the empty set is the smallest element. Such a simplicial poset is called the *face poset* of a simplicial complex, and two simplicial complexes are isomorphic if and only if their face posets are isomorphic. Therefore, a simplicial poset can be thought of as a generalization of a simplicial complex.

Although a simplicial poset P is not necessarily the face poset of a simplicial complex, it is always the face poset of a CW-complex $\Gamma(P)$. In fact, to each $y \in P\backslash\{\hat{0}\} = \overline{P}$, we assign a (geometrical) simplex whose face poset is $[\hat{0}, y]$ and glue those geometrical simplices according to the order relation in P. Then we get the CW-complex $\Gamma(P)$ such that all the attaching maps are inclusions. For instance, if two simplices of a same dimension are identified on their boundaries via the identity map, then it is not a simplicial complex but a CW-complex obtained from a simplicial poset. The CW-complex $\Gamma(P)$ has a well-defined barycentric subdivision which is isomorphic to the order complex $\Delta(\overline{P})$ of the poset $\overline{P}$. Here, $\Delta(\overline{P})$ is a simplicial complex on the vertex set $\overline{P}$ whose faces are the chains of $\overline{P}$.

We say that $y \in P$ has rank i if the interval $[\hat{0}, y]$ is isomorphic to the boolean algebra of rank i (in other words, the face poset of an $(i-1)$-simplex), and the rank of P is defined to be the maximum of ranks of all elements in P. Let $d = \operatorname{rank} P$. In exact analogy to simplicial complexes, the f-vector of the simplicial poset P, $(f_0, f_1, \dots, f_{d-1})$, is defined by

$$f_i = f_i(P) = \#\{y \in P \mid \operatorname{rank} y = i+1\}$$

and the h-vector of P, $(h_0, h_1, \dots, h_d)$, is defined by the following identity:

$$\sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} = \sum_{i=0}^{d} h_i t^{d-i},$$

where $f_{-1} = 1$, so $h_0 = 1$. Note that the number of facets of P, that is f_{d-1}, is related to h-vectors as follows:

$$f_{d-1} = \sum_{i=0}^{d} h_i. \tag{1.1}$$

When P is the face poset of a simplicial complex Σ, the f- and h-vector of P coincide with the classical f- and h-vector of the simplicial complex Σ, respectively.

f- and h-vectors have equivalent information, but h-vectors are often easier than f-vectors. Stanley [8] discussed characterization of h-vectors for certain classes of simplicial posets. For example, he proved that a vector $(h_0, h_1, \dots, h_d)$ of integers with $h_0 = 1$ is the h-vector of a Cohen–Macaulay simplicial poset of rank d if and only if $h_i \geqslant 0$ for all i. Gorenstein* simplicial posets are more special than Cohen–Macaulay simplicial posets. If the CW-complex $\Gamma(P)$ is homeomorphic to a sphere of dimension $d-1$, then the simplicial poset P of rank d is Gorenstein* (see Section 5 for more details). It is known that h-vectors of Gorenstein* simplicial posets satisfy Dehn–Sommerville equations $h_i = h_{d-i}$ for all i, in addition to the non-negativity conditions $h_i \geqslant 0$. In this paper, we will prove that h-vectors of Gorenstein* simplicial posets must satisfy one more subtle condition conjectured by Stanley [8], see [1,5,8] for partial results.

Theorem 1.1. *If P is a Gorenstein* simplicial poset of rank d and $h_i(P) = 0$ for some i between 0 and d, then $\sum_{i=0}^{d} h_i(P)$, that is the number of facets of P by* (1.1), *is even.*

Combining this with Theorem 4.3 in [8], one completes characterization of h-vectors of Gorenstein* simplicial posets.

Corollary 1.2. *Let $(h_0, h_1, \ldots, h_d)$ be a vector of non-negative integers with $h_i = h_{d-i}$ for all i and $h_0 = 1$. There is a Gorenstein* simplicial poset P of rank d with $h_i(P) = h_i$ for all i if and only if either $h_i > 0$ for all i, or else $\sum_{i=0}^{d} h_i$ is even.*

Our proof of Theorem 1.1 is purely algebraic but the idea stems from topology, so we will explain how our proof is related to topology in Section 2. A main tool to study the h-vector of a simplicial poset P is a (generalized) face ring A_P introduced in [8] of the poset P. In Section 3, we discuss restriction maps from A_P to polynomial rings. In Section 4, we construct a map called an index map from A_P to a polynomial ring. Theorem 1.1 is proven in Section 5.

2. Relation to topology

In the toric geometry, simplicial convex polytopes are closely related to *toric* manifolds or orbifolds (see [2]). Similarly, to this, Gorenstein* simplicial posets, which contain the boundary complexes of simplicial polytopes as examples, are closely related to objects (in topology) called *torus* manifolds or orbifolds (see [4,5]), and the proof of Theorem 1.1 is motivated by a topological observation described in this section. Here, a torus manifold (resp., orbifold) means a closed smooth manifold (resp. orbifold) of dimension $2d$ with an effective smooth action of a d-dimensional torus group having at least one fixed point.

We shall illustrate relations between combinatorics and topology with simple examples. In the following, T will denote the product of d copies of the circle group consisting of complex numbers with unit length, i.e., T is a d-dimensional torus group.

Example 2.1. A complex projective space $\mathbb{C}P^d$ has a T-action defined in the homogeneous coordinates by

$$(t_1, \ldots, t_d) \cdot (z_0 : z_1 : \cdots : z_d) = (z_0 : t_1 z_1 : \cdots : t_d z_d).$$

The orbit space $\mathbb{C}P^d/T$ has a natural face structure. Its facets are the images of (real) codimension two submanifolds $z_i = 0$ $(i = 0, 1, \ldots, d)$ under the quotient map $\mathbb{C}P^d \to \mathbb{C}P^d/T$. The map (called a moment map)

$$(z_0 : z_1 : \cdots : z_d) \mapsto \frac{1}{\sum_{i=0}^{d} |z_i|^2}(|z_1|^2, \ldots, |z_d|^2)$$

induces a face preserving homeomorphism from the orbit space $\mathbb{C}P^d/T$ to a standard d-simplex. The face poset of $\mathbb{C}P^d/T$ ordered by reverse inclusion (so $\mathbb{C}P^d/T$ itself is the smallest element) is the face poset of a simplicial complex of dimension $d-1$ and Gorenstein*.

Similarly, the product of d copies of $\mathbb{C}P^1$ admits a T-action, the orbit space $(\mathbb{C}P^1)^d/T$ is homeomorphic to a d-cube, and the face poset of $(\mathbb{C}P^1)^d/T$ ordered by reverse inclusion is also the face poset of a simplicial complex of dimension $d-1$ and Gorenstein*.

In any case, the orbit space is a simple convex polytope and its polar is a simplicial convex polytope. The Gorenstein* simplicial complex is the *boundary* complex of the simplicial convex polytope.

Example 2.2. Let S^{2d} be the $2d$-sphere identified with the following subset in $\mathbb{C}^d \times \mathbb{R}$:

$$\left\{(z_1, \ldots, z_d, y) \in \mathbb{C}^d \times \mathbb{R} \mid |z_1|^2 + \cdots + |z_d|^2 + y^2 = 1\right\}$$

and define a T-action on S^{2d} by

$$(t_1, \ldots, t_d)(z_1, \ldots, z_d, y) = (t_1 z_1, \ldots, t_d z_d, y).$$

The facets in the orbit space S^{2d}/T are the images of codimension two submanifolds $z_i = 0$ $(i = 1, \ldots, d)$ under the quotient map $S^{2d} \to S^{2d}/T$, and the map

$$(z_1, \ldots, z_d, y) \to (|z_1|, \ldots, |z_d|, y)$$

induces a face preserving homeomorphism from S^{2d}/T to the following subset of the d-sphere:

$$\{(x_1, \ldots, x_d, y) \in \mathbb{R}^{d+1} \mid x_1^2 + \cdots + x_d^2 + y^2 = 1,\ \ x_1 \geqslant 0, \ldots, x_d \geqslant 0\}.$$

The orbit space S^{2d}/T is not (isomorphic to) a simple convex polytope because the intersection of d facets consists of two points, but it is a manifold with corners and every face (even S^{2d}/T itself) is acyclic. The face poset of S^{2d}/T ordered by reverse inclusion is not the face poset of a simplicial complex when $d \geqslant 2$. However, it is a simplicial poset and Gorenstein*. The geometric realization of the face poset of S^{2d}/T is formed from two $(d-1)$-simplices by gluing their boundaries via the identity map.

More generally, it is proved in [5] that if a torus manifold M has vanishing odd degree cohomology, then the orbit space M/T is a manifold with corners and every face (even M/T itself) is acyclic; so the face poset of M/T ordered by reverse inclusion is a Gorenstein* simplicial poset, say P. Moreover, $h_i(P)$ agrees with the $2i$th betti number $b_{2i}(M)$ of M and the equivariant cohomology ring $H^*_T(M;\mathbb{Z})$ of M is isomorphic to the face ring A_P of P (defined over $\mathbb{Z}$). Here $H^*_T(M;\mathbf{k})$ for a ring $\mathbf{k}$ is defined as

$$H^*_T(M;\mathbf{k}) := H^*(ET\times_T M;\mathbf{k}),$$

where ET is the total space of the universal principal T-bundle (on which T acts freely) and $ET\times_T M$ is the orbit space of the product $ET\times M$ by the diagonal T-action.

A projective toric orbifold is related to a simplicial convex polytope as in Example 2.1, and the h-vector of the simplicial convex polytope agrees with the (even degree) betti numbers of the toric orbifold. Noting this fact, Stanley [7] deduced constraints on the h-vector by applying the hard Lefschetz theorem to the toric orbifold and completed the characterization of h-vectors of simplicial convex polytopes. In some sense our proof of Theorem 1.1 is on this line. The topological argument developed below in this section is not complete but would be helpful for the reader to understand what is done in subsequent sections.

Let P be a Gorenstein* simplicial poset of dimension $d-1$. Looking at the result in [5] mentioned above, it is likely that there exists a torus orbifold M which have the following properties:

Properties.

(1) $H^{\mathrm{odd}}(M;\mathbb{Q})=0$,
(2) $h_i(P)=b_{2i}(M)$,
(3) $H^*_T(M;\mathbb{Q})$ is isomorphic to A_P (defined over $\mathbb{Q}$).

What we will use to deduce the necessity in Theorem 1.1 is the *index map* (or evaluation map) in equivariant cohomology

$$\mathrm{Ind}_T\colon H^*_T(M;\mathbb{Q})\to H^{*-2d}_T(pt;\mathbb{Q})=H^{*-2d}(BT;\mathbb{Q}),$$

where $BT=ET/T$ is the classifying space of principal T-bundles. The index map is nothing but the Gysin homomorphism in equivariant cohomology induced from the collapsing map $\pi\colon M\to pt$. As is well known, BT is the product of d copies of $\mathbb{C}P^\infty$ (up to homotopy) and $H^*(BT;\mathbb{Q})$ is a polynomial ring in d variables of degree two. The index map Ind_T decreases cohomological degrees by $2d$ because the dimension of M is $2d$. Moreover, $H^*_T(M;\mathbb{Q})$ is a module over $H^*(BT;\mathbb{Q})$ through $\pi^*\colon H^*(BT;\mathbb{Q})=H^*_T(pt;\mathbb{Q})\to H^*_T(M;\mathbb{Q})$ and Ind_T is an $H^*(BT;\mathbb{Q})$-module map. Since $H^{\mathrm{odd}}(M;\mathbb{Q})=0$ and $H^*(BT;\mathbb{Q})$ is a polynomial ring in d variables, say $t_1,\dots,t_d$, the quotient ring of $H^*_T(M;\mathbb{Q})$ by the ideal generated by

$\pi^*(t_1), \ldots, \pi^*(t_d)$ agrees with the ordinary cohomology $H^*(M;\mathbb{Q})$. Similarly, the quotient ring of $H_T^*(pt;\mathbb{Q}) = H^*(BT;\mathbb{Q})$ by the ideal generated by $t_1, \ldots, t_d$ agrees with $H^*(pt;\mathbb{Q})$. Therefore, the index map in equivariant cohomology induces the index map in ordinary cohomology:

$$\mathrm{Ind}\colon H^*(M;\mathbb{Q}) \to H^{*-2d}(pt;\mathbb{Q}).$$

This map agrees with the Gysin homomorphism in ordinary cohomology induced from the collapsing map π, so it is the evaluation map on a fundamental class of M. Thus, we have a commutative diagram:

$$\begin{tikzcd}
H_T^{2d}(M;\mathbb{Q}) \arrow[r, "\mathrm{Ind}_T"] \arrow[d] & H_T^0(pt;\mathbb{Q}) = H^0(BT;\mathbb{Q}) = \mathbb{Q} \arrow[d] \\
H^{2d}(M;\mathbb{Q}) \arrow[r, "\mathrm{Ind}"] & H^0(pt;\mathbb{Q}) = \mathbb{Q},
\end{tikzcd}$$

where the right vertical map is the identity.

A key thing is to find an element ω_T in $H_T^{2d}(M;\mathbb{Q})$ such that

(i) ω_T is a polynomial in elements of $H_T^2(M;\mathbb{Q})$,
(ii) $\mathrm{Ind}_T(\omega_T)$ is an integer and $\mathrm{Ind}_T(\omega_T) \equiv \chi(M) \pmod 2$, where $\chi(M)$ is the Euler characteristic of M.

We may think of ω_T as a “lifting" of the equivariant top Stiefel–Whitney class $w_{2d}^T(M) \in H_T^{2d}(M;\mathbb{Z}/2)$ of M. If we find such an element ω_T, then it follows from the commutativity of the above diagram that

$$\mathrm{Ind}_T(\omega_T) = \mathrm{Ind}(\omega), \tag{2.1}$$

where ω is the image of ω_T under the left vertical map in the above diagram.

Now suppose $h_i(P) = 0$ for some $1 \le i \le d-1$. Then the $2i$th betti number $b_{2i}(M)$ of M is zero by property (2) and the element ω vanishes because it is a polynomial in degree two elements by (i) above, so the right-hand side of (2.1) is zero and $\chi(M)$ is even by (ii) above. On the other hand, it follows from properties (1) and (2) that

$$\chi(M) = \sum_{i=0}^{d} b_{2i}(M) = \sum_{i=0}^{d} h_i(P).$$

These prove that $\sum_{i=0}^{d} h_i(P)$ is even.

It turns out that the argument developed above works without assuming the existence of the torus orbifold M. In fact, the face ring A_P takes the place of $H_T^*(M;\mathbb{Q})$ by property (3) and an l.s.o.p. for A_P plays the role of $\pi^*(t_1), \ldots, \pi^*(t_d)$ so that the polynomial ring generated by the l.s.o.p. corresponds to the polynomial ring $\pi^*(H^*(BT;\mathbb{Q}))$ (or $H^*(BT;\mathbb{Q})$ since π^* is injective). The index map Ind_T has an expression (so-called Lefschetz fixed point formula) in terms of local data around T-fixed points of M, and since the formula is purely algebraic, one can use it to define an “index map" from A_P. To carry out this idea, we need to study restriction maps from A_P to polynomial rings because restriction maps to T-fixed points in equivariant cohomology are involved in the Lefschetz fixed point formula. We will discuss such restriction maps in Section 3 and construct the index map from A_P in Section 4.

3. Restriction maps

In this and next sections, we consider rings over $\mathbb{Q}$. A main tool to study the h-vector of a (finite) simplicial poset P is the face ring A_P of the poset P introduced by Stanley in [8]. We recall it first.

Definition. Let P be a simplicial poset of rank d with elements $\hat{0} = y_0, y_1, \dots, y_p$. Let $A = \mathbb{Q}[y_0, y_1, \dots, y_p]$ be the polynomial ring over $\mathbb{Q}$ in the variables y_i and define $\mathcal{I}_P$ to be the ideal of A generated by the following elements:

$$y_i y_j - (y_i \wedge y_j)\left(\sum_z z\right), \quad y_0 - 1,$$

where $y_i \wedge y_j$ is the greatest lower bound of y_i and y_j, z ranges over all minimal upper bounds of y_i and y_j, and we understand $\sum_z z = 0$ if y_i and y_j have no common upper bound. Then the face ring A_P of the simplicial poset P is defined as the quotient ring $A/\mathcal{I}_P$ and made graded

$$A_P = (A_P)_0 \oplus (A_P)_1 \oplus \cdots$$

by defining $\deg y_i = \operatorname{rank} y_i$. The ring A_P reduces to a classical Stanley–Reisner face ring when P is the face poset of a simplicial complex.

We denote by P_s the subset of P consisting of elements of rank s. Elements in P_1 will be denoted by $x_1, \dots, x_n$ and called *atoms* in P. The set $\{x_1, \dots, x_n\}$ is a basis of $(A_P)_1$.

Suppose that y is an element of P_d. Then the interval $[\hat{0}, y]$ is a boolean algebra of rank d and $A_{[\hat{0},y]}$ is a polynomial ring in d variables. Sending all elements in P which are not lower than y to zero, we obtain an epimorphism

$$\iota_y : A_P \to A_{[\hat{0},y]}.$$

Since $\mathbb{Q}$ is a field with infinitely many elements, A_P admits an l.s.o.p. $\theta_1, \dots, \theta_d$ (see the proof of Theorem 3.10 in [8]). In the following we fix the l.s.o.p. and denote by

Θ the vector space of dimension d spanned by $\theta_1, \dots, \theta_d$ over $\mathbb{Q}$, and by $\mathbb{Q}[\Theta]$ the polynomial ring generated by $\theta_1, \dots, \theta_d$. Note that Θ is a vector subspace of $(A_P)_1$ and $\mathbb{Q}[\Theta]$ is a subring of A_P.

Lemma 3.1. *The restriction of ι_y to $\mathbb{Q}[\Theta]$ is an isomorphism onto $A_{[\hat{0},y]}$.*

Proof. Since A_P is finitely generated as a $\mathbb{Q}[\Theta]$-module, so is $A_{[\hat{0},y]}$. This implies that ι_y maps the vector space Θ isomorphically onto the vector space spanned by d elements of degree one generating the polynomial ring $A_{[\hat{0},y]}$, thus the lemma follows. □

Henceforth, we identify $A_{[\hat{0},y]}$ with $\mathbb{Q}[\Theta]$ via ι_y, and think of ι_y as a map to $\mathbb{Q}[\Theta]$, i.e.,

$$\iota_y : A_P \to \mathbb{Q}[\Theta].$$

Note that ι_y is the identity on the subring $\mathbb{Q}[\Theta]$ and a $\mathbb{Q}[\Theta]$-module map.

For $w \in P_s$, we set

$$\mathcal{A}(w) := \{i \in \{1, \dots, n\} \mid x_i \text{ is an atom lower than } w\}.$$

The cardinality of $\mathcal{A}(w)$ is s. Let $y \in P_d$. By definition of ι_y,

$$\iota_y(x_i) = 0 \text{ whenever } i \notin \mathcal{A}(y). \tag{3.1}$$

We set

$$\theta_i(y) := \iota_y(x_i) \quad \text{for } i \in \mathcal{A}(y). \tag{3.2}$$

Since $\iota_y : (A_P)_1 \to \Theta$ is surjective and the cardinality of $\mathcal{A}(y)$, that is d, agrees with the dimension of Θ, the set $\{\theta_i(y) \mid i \in \mathcal{A}(y)\}$ is a basis of Θ.

Let $z \in P_{d-1}$. Let y be an element in P_d above z and define $\ell \in \{1, \dots, n\}$ by

$$\mathcal{A}(y) \backslash \mathcal{A}(z) = \{\ell\}.$$

The canonical map $A_{[\hat{0},y]} = \mathbb{Q}[\Theta] \to A_{[\hat{0},z]}$ is surjective and $A_{[\hat{0},z]}$ can canonically be identified with $\mathbb{Q}[\Theta]/(\theta_\ell(y))$. Let y' be another element in P_d above z and define $\ell' \in \{1, \dots, n\}$ similarly to ℓ. It may happen that $\ell = \ell'$. Since

$$\mathbb{Q}[\Theta]/(\theta_\ell(y)) = A_{[\hat{0},z]} = \mathbb{Q}[\Theta]/(\theta_{\ell'}(y')), \tag{3.3}$$

$\theta_\ell(y)$ and $\theta_{\ell'}(y')$ are same up to a non-zero scalar multiple; so the following lemma makes sense.

Lemma 3.2. $\iota_y(\alpha) \equiv \iota_{y'}(\alpha) \bmod \theta_\ell(y)$ *for any* $\alpha \in A_P$. *In particular,* $\theta_i(y) \equiv \theta_i(y') \bmod \theta_\ell(y)$ *for* $i \in \mathcal{A}(z) (= \mathcal{A}(y) \setminus \{\ell\} = \mathcal{A}(y') \setminus \{\ell'\})$.

Proof. We have canonical surjections $A_P \to A_{[\hat{0},y]} \to A_{[\hat{0},z]}$ and $A_P \to A_{[\hat{0},y']} \to A_{[\hat{0},z]}$, whose composite surjections $A_P \to A_{[\hat{0},z]}$ are the same. Therefore the lemma follows from (3.3). $\square$

4. Index maps

In this section, we define an “index map" from A_P to the polynomial ring $\mathbb{Q}[\Theta]$, which corresponds to the index map Ind_T in Section 2. It is a $\mathbb{Q}[\Theta]$-module map, so it induces a homomorphism from the quotient $A_P/(\Theta)$ modulo the linear system of parameters $\theta_1, \dots, \theta_d$ to $\mathbb{Q}$. This induced map corresponds to the index map Ind in Section 2.

We shall make some observations needed later before we define the index map. Let $z \in P_{d-1}$ and let $y, y' \in P_d$ lie above z as before. Give an orientation on Θ determined by an ordered basis $(\theta_1, \dots, \theta_d)$ and choose an order of the basis $\{\theta_i(y) \mid i \in \mathcal{A}(y)\}$ whose induced orientation on Θ agrees with the given orientation. We then define $m(y)$ to be the determinant of a matrix sending the ordered basis $\{\theta_i(y) \mid i \in \mathcal{A}(y)\}$ to the ordered basis $(\theta_1, \dots, \theta_d)$. Note that $m(y)$ is positive. It follows from the latter statement in Lemma 3.2 that

$$m(y)\theta_\ell(y) = \lambda(y, y')m(y')\theta_{\ell'}(y'), \tag{4.1}$$

where $\lambda(y, y') = \pm 1$. If $\mathcal{A}(y) = \mathcal{A}(y')$, then $\ell = \ell'$ and both $\theta_\ell(y)$ and $\theta_{\ell'}(y')$ restrict to the element x_ℓ in $A_{[\hat{0},x_\ell]}$. Therefore

$$m(y) = m(y') \text{ (and } \lambda(y, y') = 1) \text{ if } \mathcal{A}(y) = \mathcal{A}(y'). \tag{4.2}$$

The order of the basis $\{\theta_i(y) \mid i \in \mathcal{A}(y)\}$ determines an order of atoms x_i $(i \in \mathcal{A}(y))$ and then determines an orientation on the $(d-1)$-simplex with those atoms as vertices. The oriented $(d-1)$-simplex obtained in this way is denoted by $\langle y \rangle$. Then, the boundaries $\partial\langle y \rangle$ and $\partial\langle y' \rangle$ of $\langle y \rangle$ and $\langle y' \rangle$ have opposite orientations on the $(d-2)$-simplex $[z]$ corresponding to z (in other words, $[z]$ does not appear in $\partial\langle y \rangle + \partial\langle y' \rangle$) if and only if $\lambda(y, y') = -1$.

Now we pose the following assumption, which we shall see in Section 5 is satisfied by all Gorenstein* simplicial posets.

Assumption.

(1) For any $z \in P_{d-1}$, there are exactly two elements in P_d above z.
(2) One can assign a sign $\varepsilon(y) \in \{\pm 1\}$ to each $y \in P_d$ so that $\sum_{y \in P_d} \varepsilon(y)\langle y \rangle$ is a cycle (hence defines a fundamental class in $H_{d-1}(\Gamma(P); \mathbb{Z})$ where $\Gamma(P)$ denotes the CW-complex explained in the Introduction).

When $\langle y\rangle$ and $\langle y'\rangle$ share a $(d-2)$-simplex $[z]$, it follows from the above assumption that $[z]$ does not appear in $\partial(\varepsilon(y)\langle y\rangle)+\partial(\varepsilon(y')\langle y'\rangle)$. Therefore,

$$\lambda(y, y') \text{ and } \varepsilon(y)\varepsilon(y') \text{ have opposite signs} \tag{4.3}$$

by the remark mentioned above the assumption.

Definition. For a simplicial poset P which satisfies the assumption above, we define the *index map* by

$$\mathrm{Ind}_T(\alpha) := \sum_{y\in P_d} \frac{\varepsilon(y)\iota_y(\alpha)}{m(y)\prod_{i\in\mathcal{A}(y)}\theta_i(y)} \quad \text{for } \alpha\in A_P. \tag{4.4}$$

Apparently, $\mathrm{Ind}_T(\alpha)$ lies in the quotient field of $\mathbb{Q}[\Theta]$, but we have

Theorem 4.1. $\mathrm{Ind}_T(\alpha)\in\mathbb{Q}[\Theta]$ *for any* $\alpha\in A_P$.

Remark. The proof given below is essentially same as that of Theorem 2.2 in [3]. A similar result can be found in [4, Section 8].

Proof. The right-hand side of (4.4) can be expressed as

$$\frac{g}{\prod_{j=1}^N f_j} \tag{4.5}$$

with $g\in\mathbb{Q}[\Theta]$ and $f_j\in\Theta\subset\mathbb{Q}[\Theta]$ such that any two of $f_1,\dots,f_N$ are linearly independent. It suffices to show that f_1 divides g.

Let Q be the set of $y\in P_d$ such that $\theta_i(y)$ is not a scalar multiple of f_1 for every $i\in\mathcal{A}(y)$, and let Q^{c} be the complement of Q in P_d. In (4.4), the sum of terms for elements in Q reduces to

$$\sum_{y\in Q}\frac{\varepsilon(y)\iota_y(\alpha)}{m(y)\prod_{i\in\mathcal{A}(y)}\theta_i(y)} = \frac{g_1}{\prod_{j=2}^N f_j} \tag{4.6}$$

with $g_1\in\mathbb{Q}[\Theta]$, so that f_1 does not appear in the denominator.

On the other hand, if $y\in Q^{\mathrm{c}}$, then it follows from the definition of Q that there is an element $\ell\in\mathcal{A}(y)$ such that

$$\theta_\ell(y) = cf_1 \qquad (0\neq c\in\mathbb{Q}), \tag{4.7}$$

and there is a unique element $z\in P_{d-1}$ such that z is lower than y and $\mathcal{A}(z)=\mathcal{A}(y)\backslash\{\ell\}$. By assumption, there is a unique element in P_d which lies above z and is different from y. We denote it by y'. Now we are in the same situation as before. It follows from (4.1) and (4.7) that y' is also an element in Q^c. Noting that $\mathcal{A}(y) = \mathcal{A}(z) \cup \{\ell\}$ and $\mathcal{A}(y') = \mathcal{A}(z) \cup \{\ell'\}$ and using (4.1), we combine the two terms in (4.4) for y and y' to get

$$\begin{aligned}&\frac{\varepsilon(y)\imath_y(\alpha)}{m(y)\prod_{i\in\mathcal{A}(y)}\theta_i(y)} + \frac{\varepsilon(y')\imath_{y'}(\alpha)}{m(y')\prod_{i\in\mathcal{A}(y')}\theta_i(y')} \\ &\quad = \frac{\varepsilon(y)\imath_y(\alpha)\prod_{i\in\mathcal{A}(z)}\theta_i(y') + \lambda(y,y')\varepsilon(y')\imath_{y'}(\alpha)\prod_{i\in\mathcal{A}(z)}\theta_i(y)}{m(y)\theta_\ell(y)\prod_{i\in\mathcal{A}(z)}\theta_i(y)\prod_{i\in\mathcal{A}(z)}\theta_i(y')}. \qquad (4.8)\end{aligned}$$

Here

$$\imath_y(\alpha)\prod_{i\in\mathcal{A}(z)}\theta_i(y') \equiv \imath_{y'}(\alpha)\prod_{i\in\mathcal{A}(z)}\theta_i(y) \bmod \theta_\ell(y)$$

by Lemma 3.2, and

$$\varepsilon(y) + \lambda(y,y')\varepsilon(y') = 0$$

by (4.3), so the numerator of the right-hand side of the identity (4.8) is divisible by $\theta_\ell(y) = cf_1$. This means that we can arrange the left-hand side of (4.8) with a common denominator in which f_1 does not appear as a factor. Since elements in Q^c appear pairwise like this, one has

$$\sum_{y\in Q^c}\frac{\varepsilon(y)\imath_y(\alpha)}{m(y)\prod_{i\in A(y)}\theta_i(y)} = \frac{g_2}{\prod_{j=2}^N f_j}$$

with $g_2 \in \mathbb{Q}[\Theta]$. This together with (4.6) implies that the numerator g in (4.5) is divisible by f_1. $\square$

Since $\imath_y$ is a $\mathbb{Q}[\Theta]$-module map, so is Ind_T. Therefore

$$\mathrm{Ind}_T\colon A_P \to \mathbb{Q}[\Theta]$$

induces a homomorphism

$$\mathrm{Ind}\colon A_P/(\Theta) \to \mathbb{Q}. \qquad (4.9)$$

This map decreases degrees by d because Ind_T does.

5. Gorenstein* simplicial posets

We shall prove Theorem 1.1 in this section. Let $\mathbf{k}$ be an arbitrary field. Suppose that a simplicial poset P is Gorenstein* over $\mathbf{k}$, i.e., the order complex $\Delta(\overline{P})$ of $\overline{P} = P - \{\hat{0}\}$, which is a simplicial complex, is Gorenstein* over $\mathbf{k}$. According to Theorem II.5.1 in [9], a simplicial complex Δ of dimension $d-1$ is Gorenstein* over $\mathbf{k}$ if and only if for all $p \in |\Delta|$,

$$\widetilde{H}_q(|\Delta|, \mathbf{k}) \cong H_q(|\Delta|, |\Delta| - p; \mathbf{k}) \cong \begin{cases} \mathbf{k}, & q = d-1, \\ 0, & q < d-1. \end{cases}$$

Therefore, it follows from the universal coefficient theorem [6, Corollary 55.2] that if a simplicial poset P is Gorenstein* over $\mathbf{k}$, then it is Gorenstein* over $\mathbb{Q}$. In the sequel we may assume $\mathbf{k} = \mathbb{Q}$. According to Theorem II.5.1 in [9] again, $\Delta(\overline{P})$ is an orientable pseudomanifold, so the assumption in Section 4 is satisfied for the Gorenstein* simplicial poset P because $\Delta(\overline{P})$ is the barycentric subdivision of the CW-complex $\Gamma(P)$.

Since a Gorenstein* simplicial poset is Cohen–Macaulay, $h_i = h_i(P)$ agrees with the dimension of the homogeneous part of degree i in $A_P/(\Theta)$, see the proof of Theorem 3.10 [8]. Therefore, if $h_i = 0$ for some i $(1 \le i \le d-1)$, then a product of d elements in $(A_P)_1$ vanishes in $A_P/(\Theta)$, in particular, the product is zero when evaluated by the index map in (4.9).

We take a subset I of $\{1, \dots, n\}$ with cardinality d such that $I = \mathcal{A}(y)$ for some $y \in P_d$. If $\mathcal{A}(y) = \mathcal{A}(y')(= I)$, then $m(y) = m(y')$ by (4.2). Therefore we may write $m(y)$ as m_I. Since

$$\iota_y\left(\prod_{i\in I} x_i\right) = \begin{cases} \prod_{i\in\mathcal{A}(y)} \theta_i(y) & \text{if } \mathcal{A}(y) = I, \\ 0 & \text{otherwise} \end{cases}$$

by (3.1) and (3.2), we have

$$\mathrm{Ind}_T(m_I \prod_{i\in I} x_i) = \sum_{\mathcal{A}(y)=I} \varepsilon(y) \in \mathbb{Q}$$

by (4.4). Hence, if we regard $m_I \prod_{i\in I} x_i$ as an element in $A_P/(\Theta)$, then we have

$$\mathrm{Ind}\left(m_I \prod_{i\in I} x_i\right) = \sum_{\mathcal{A}(y)=I} \varepsilon(y). \tag{5.1}$$

Now suppose that $h_i = 0$ for some i $(1 \le i \le d-1)$. Then the left-hand side of (5.1) is zero as remarked above. This means that (since $\varepsilon(y) = \pm 1$) there must be an even number of elements $y \in P_d$ with $\mathcal{A}(y) = I$ at the right-hand side of (5.1). Since

I is arbitrary, we conclude that f_{d-1} (the number of elements in P_d) is even. This together with (1.1) completes the proof of Theorem 1.1. $\square$

Remark. An element corresponding to ω_T in Section 2 is $\sum_I m_I \prod_{i \in I} x_i$, where I runs over all subsets of $\{1, \ldots, n\}$ with cardinality d and m_I is understood to be zero if there is no $y \in P_d$ such that $I = \mathcal{A}(y)$.

Acknowledgments

I am grateful to Takayuki Hibi for informing me of the above problem and for his interest. I am also grateful to Akio Hattori and Taras Panov for the successful collaborations [4,5] from which the idea used in this paper originates. Finally I am grateful to Ezra Miller for his comments on an earlier version of the paper, which were very helpful to improve the paper.

References

[1] A.M. Duval, A combinatorial decomposition of simplicial complexes, Israel J. Math. 87 (1994) 77–87.
[2] W. Fulton, An Introduction to Toric Varieties, Annals of Mathematical Studies, vol. 113, Princeton University Press, Princeton, NJ, 1993.
[3] V. Guillemin, C. Zara, Equivariant de Rham theory and graphs, Asian J. Math. 3 (1999) 49–76.
[4] A. Hattori, M. Masuda, Theory of multi-fans, Osaka J. Math. 40 (2003) 1–68.
[5] M. Masuda, T.E. Panov, On the cohomology of torus manifolds, preprint.
[6] J.R. Munkres, Elements of Algebraic Topology, Addison-Wesley, Reading MA, 1984.
[7] R.P. Stanley, The number of faces of a simplicial convex polytope, Adv. Math. 35 (1980) 236–238.
[8] R.P. Stanley, *f*-vectors and *h*-vectors of simplicial posets, J. Pure Appl. Algebra 71 (1991) 319–331.
[9] R.P. Stanley, Combinatorics and Commutative Algebra, second ed., Progress in Mathematics, vol. 41, Birkhäuser, Boston, 1996.