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A matrix subadditivity inequality for $f(A+B)$ and $f(A)+f(B)$

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Abstract

In 1999 Ando and Zhan proved a subadditivity inequality for *operator* concave functions. We extend it to *all* concave functions: Given positive semidefinite matrices A, B and a non-negative concave function f on $[0, \infty)$,

$$\|f(A+B)\| \leqslant \|f(A)+f(B)\|$$

for all symmetric norms (in particular for all Schatten p-norms). The case $f(t)=\sqrt{t}$ is connected to some block-matrix inequalities, for instance the operator norm inequality

$$\left\| \begin{pmatrix} A & X^* \\ X & B \end{pmatrix} \right\|_\infty \leqslant \max\{\| |A| + |X| \|_\infty ; \| |B| + |X^*| \|_\infty\}$$

for any partitioned Hermitian matrix.

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1. A subadditivity inequality

Capital letters $A, B, \ldots, Z$ mean n-by-n complex matrices, or operators on an n-dimensional Hilbert space $\mathscr{H}$. If A is positive semidefinite, respectively, positive definite, we write $A \geqslant 0$, respectively, $A > 0$. Recall that a symmetric (or unitarily invariant) norm $\|\cdot\|$ satisfies $\|A\| = \|UAV\|$ for all A and all unitaries U, V. We will prove.

Theorem 1.1. *Let $A, B \geqslant 0$ and let $f : [0, \infty) \longrightarrow [0, \infty)$ be a concave function. Then, for all symmetric norms,*

$$\|f(A+B)\| \leqslant \|f(A) + f(B)\|.$$

For the trace norm Theorem 1.1 is a classical inequality. In case of the operator norm, Kosem [7] recently gave a three-line proof! But the general case is much more difficult. When f is operator concave, Theorem 1.1 has been proved by Ando and Zhan [1]. Their proof is not elementary and makes use of integral representations of operator concave functions. By a quite ingenious process, Kosem [7] derived from Ando–Zhan's result a related superadditive inequality:

Theorem 1.2. *Let $A, B \geqslant 0$ and let $g : [0, \infty) \longrightarrow [0, \infty)$ be a convex function with $g(0) = 0$. Then, for all symmetric norms,*

$$\|g(A) + g(B)\| \leqslant \|g(A+B)\|.$$

The special case $g(t) = t^m$, $m = 1, 2, \ldots$ is due to Bhatia–Kittaneh [4]. The general case has been conjectured by Aujla and Silva [3].

In this note we first give a simple proof of these two theorems. Our method is elementary: we only use a simple inequality for operator convex functions and some basic facts about symmetric norms and majorization. For background we refer to [9] and references herein.

Next, we consider some inequalities for block-matrices inspired by the observation that Theorem 1.1 for $f(t) = \sqrt{t}$ can be written as $\|\sqrt{A^2 + B^2}\| \leqslant \|A + B\|$, or equivalently

$$\left\| \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} \right\| \leqslant \|A + B\|.$$

We naturally asked if a similar result holds when the zeros are replaced by arbitrary positive matrices. We got proofs from T. Ando, E. Ricard and X. Zhan. We thank them for their collaboration.

2. Proof of Theorems 1.1–1.2 and related results

First we sketch the simple proof from [8] for Theorem 1.1 in the operator concave case (Ando–Zhan's inequality). Let us recall some basic facts about operator convex/concave functions on an interval $[a, b]$. If g is operator convex and A is Hermitian, $a \geqslant A \geqslant b$, then for all subspaces $\mathscr{S} \subset \mathscr{H}$, Davis' Inequality holds for compressions onto $\mathscr{S}$,

$$g(A_{\mathscr{S}}) \leqslant g(A)_{\mathscr{S}}. \tag{1}$$

Assuming $0 \in [a, b]$, $g(0) \leqslant 0$, one can derive Hansen's Inequality: Z being any contraction,

$$g(Z^*AZ) \leqslant Z^*g(A)Z.$$

Of course, for an operator concave function f on $[a, b]$ with $f(0) \geqslant 0$, the reverse inequality holds. For such an f on the positive half-line and $A, B > 0$ we then have

$$f(A) \geqslant A^{1/2}(A+B)^{-1/2}f(A+B)(A+B)^{-1/2}A^{1/2}$$

since $Z = (A+B)^{-1/2}A^{1/2}$ is a contraction and $A = Z^*(A+B)Z$. Similarly

$$f(B) \geqslant B^{1/2}(A+B)^{-1/2}f(A+B)(A+B)^{-1/2}B^{1/2}.$$

Consequently

$$f(A) + f(B) \geqslant A^{1/2}\frac{f(A+B)}{(A+B)}A^{1/2} + B^{1/2}\frac{f(A+B)}{(A+B)}B^{1/2}. \tag{2}$$

Next, the main observation of [8] can be stated as

Proposition 2.1. *Let $A, B \geqslant 0$ and let $g : [0,\infty) \longrightarrow [0,\infty)$. If $g(t)$ decreases and $tg(t)$ increases, then for all symmetric norms,*

$$\|(A+B)g(A+B)\| \leqslant \|A^{1/2}g(A+B)A^{1/2} + B^{1/2}g(A+B)B^{1/2}\|.$$

Combining (2) and Proposition 2.1 with $g(t) = f(t)/t$ yields the Ando–Zhan Inequality [1]:

Corollary 2.2. *Theorem 1.1 holds when f is operator concave.*

This means that the eigenvalues of $f(A+B)$ are weakly majorised by those of $f(A) + f(B)$. Suppose now that f is onto, thus $f(0) = 0$, $f(\infty) = \infty$ and its inverse function g is convex, increasing. Therefore the eigenvalues of $g(f(A+B)) = A + B$ are weakly majorised by those of $g(f(A) + f(B))$. Replacing A and B by $g(A)$ and $g(B)$ respectively, we get the second Ando–Zhan Inequality [1]:

Corollary 2.3. *Let $A, B \geqslant 0$ and let $g : [0,\infty) \longrightarrow [0,\infty)$ be a one to one function whose inverse function is operator concave. Then, for all symmetric norms,*

$$\|g(A) + g(B)\| \leqslant \|g(A+B)\|.$$

Now we turn to a quite simple proof of Theorem 1.2. It suffices to consider Ky Fan k-norms $\|\cdot\|_k$. Suppose that f and g both satisfy Theorem 1.2. Using the triangle inequality and the fact that f and g are non-decreasing,

$$\begin{aligned}\|(f+g)(A) + (f+g)(B)\|_k &\leqslant \|f(A) + f(B)\|_k + \|g(A) + g(B)\|_k \\ &\leqslant \|f(A+B)\|_k + \|g(A+B)\|_k = \|(f+g)(A+B)\|_k,\end{aligned}$$

hence the set of functions satisfying Theorem 1.2 is a cone. It is also closed for pointwise convergence. Since any positive convex function vanishing at 0 can be approached by a positive combination of angle functions at $a > 0$,

$$\gamma(t) = \tfrac{1}{2}\{|t-a| + t - a\},$$

it suffices to prove Theorem 1.2 for such a γ. By Corollary 2.3 it suffices to approach γ by functions whose inverses are operator concave. We take (with $r > 0$)

$$h_r(t) = \tfrac{1}{2}\left\{\sqrt{(t-a)^2 + r} + t - \sqrt{a^2 + r}\right\},$$

whose inverse

$$t - \frac{r/2}{2t + \sqrt{a^2+r} - a} + \frac{\sqrt{a^2+r} + a}{2}$$

is operator concave since $1/t$ is operator convex on the positive half-line (inequality (1) is then a basic fact of Linear Algebra). Clearly, as $r \to 0$, $h_r(t)$ converges uniformly to γ.

From Theorem 1.2 we can derive Theorem 1.1:

Proof of Theorem 1.1. It suffices to prove the theorem for the Ky Fan k-norms $\|\cdot\|_k$. This shows that we may assume $f(0) = 0$. Note that f is necessarily non-decreasing. Hence, there exists a rank k spectral projection E for $A + B$, corresponding to the k-largest eigenvalues $\lambda_1(A + B), \dots, \lambda_k(A + B)$ of $A + B$, such that

$$\|f(A+B)\|_k = \sum_{j=1}^{k} \lambda_j(f(A+B)) = \mathrm{Tr} E f(A+B) E.$$

Therefore, using a well-known property of Ky Fan norms, it suffices to show that

$$\mathrm{Tr} E f(A+B) E \leqslant \mathrm{Tr} E(f(A) + f(B))E.$$

This is the same as requiring that

$$\mathrm{Tr} E(g(A) + g(B))E \leqslant \mathrm{Tr} E g(A+B) E \tag{3}$$

for all non-positive convex functions g on $[0, \infty)$ with $g(0) = 0$. Any such function can be approached by a combination of the type

$$g(t) = \lambda t + h(t)$$

for a scalar $\lambda < 0$ and some non-negative convex function h vanishing at 0. Hence, it suffices to show that (3) holds for $h(t)$. We have

$$\begin{aligned}
\mathrm{Tr} E(h(A) + h(B))E &= \sum_{j=1}^{k} \lambda_j(E(h(A) + h(B))E) \\
&\leqslant \sum_{j=1}^{k} \lambda_j(h(A) + h(B)) \\
&\leqslant \sum_{j=1}^{k} \lambda_j(h(A+B)) \quad \text{(by Theorem 1.2)} \\
&= \sum_{j=1}^{k} \lambda_j(E h(A+B) E) \\
&= \mathrm{Tr} E h(A+B) E,
\end{aligned}$$

where the second equality follows from the fact that h is non-decreasing and hence E is also a spectral projection of $h(A + B)$ corresponding to the k largest eigenvalues. $\square$

The above proof is inspired by a part of the proof of the following result [5]:

Theorem 2.4. *Let $f : [0, \infty) \longrightarrow [0, \infty)$ be a concave function. Let $A \geqslant 0$ and let Z be expansive. Then, for all symmetric norms,*

$$\|f(Z^*AZ)\| \leqslant \|Z^* f(A) Z\|.$$

Here Z expansive means $Z^*Z \geqslant I$, the identity operator. Besides such symmetric norms inequalities, there also exist interesting inequalities involving unitary congruences. For instance [2]:

Theorem 2.5. *Let $A, B \geqslant 0$ and let $f : [0, \infty) \longrightarrow [0, \infty)$ be a concave function. Then, there exist unitaries U, V such that*

$$f(A + B) \leqslant Uf(A)U^* + Vf(B)V^*.$$

This implies that $\lambda_{j+k+1} f(A + B) \leqslant \lambda_{j+1} f(A) + \lambda_{k+1} f(B)$ for all integers $j, k \geqslant 0$.

Combining Theorem 2.5 with Thompson's triangle inequality we get

Corollary 2.6. *For any A, B and any non-negative concave function f on $[0, \infty)$,*

$$f(|A + B|) \leqslant Uf(|A|)U^* + Vf(|B|)V^*$$

for some unitaries U, V.

Therefore, we recapture a result from [8]: the map $X \longrightarrow \|f(X)\|$ is subadditive. For the trace norm, this is Rotfel'd Theorem.

A remark may be added about Theorem 1.1: It can be stated for a family $\{A_i\}_{i=1}^m$ of positive operators,

$$\|f(A_1 + \cdots + A_m)\| \leqslant \|f(A_1) + \cdots + f(A_m)\|.$$

Indeed, Proposition 2.1 can be stated for a suitable family $A, B, \ldots$.

3. Inequalities for block-matrices

In Section 1 we noted an inequality involving a partitioned matrix. The following two theorems are generalizations due to T. Ando, E. Ricard and X. Zhan (private communications). The symbol $\|\cdot\|_\infty$ means the operator norm.

Theorem 3.1. *For all block-matrices whose entries are normal matrices of same size and for all symmetric norms,*

$$\left\| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\| \leqslant \| |A| + |B| + |C| + |D| \|.$$

Theorem 3.2. *For all block-matrices whose entries are normal matrices of same size,*

$$\left\| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\|_\infty \leqslant \max\{\| |A| + |B| \|_\infty; \| |C| + |D| \|_\infty; \| |A| + |C| \|_\infty; \| |B| + |D| \|_\infty\}$$

Proof. Let A_1, A_2, B_1, B_2 be positive and let C_1, C_2 be contractions. Note that

$$A_1C_1B_1 + A_2C_2B_2 = \begin{pmatrix} A_1 & A_2 \end{pmatrix} \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$

Applying the Cauchy–Shwarz inequality $\|XY\| \leqslant \|X^*X\|^{1/2}\|YY^*\|^{1/2}$ and using $\|ST\| \leqslant \|S\|_\infty\|T\|$ show

$$\|A_1C_1B_1 + A_2C_2B_2\| \leqslant \|A_1^2 + A_2^2\|^{1/2}\|B_1^2 + B_2^2\|^{1/2}.$$

Considering polar decompositions $A = |A^*|^{1/2}U|A|^{1/2}$ and $B = |B^*|^{1/2}V|B|^{1/2}$ then shows that

$$\|A + B\| \leqslant \||A| + |B|\|^{1/2}\||A^*| + |B^*|\|^{1/2} \tag{4}$$

for all A, B. Replacing A and B in (4) by

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

and using normality of A, B, C, D yield

$$\left\|\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right\| \leqslant \left\|\begin{pmatrix} |A|+|C| & 0 \\ 0 & |B|+|D| \end{pmatrix}\right\|^{1/2} \left\|\begin{pmatrix} |A|+|B| & 0 \\ 0 & |C|+|D| \end{pmatrix}\right\|^{1/2}.$$

This proves Theorem 3.2. This also proves Theorem 3.1 by using the fact that

$$\left\|\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}\right\| \leqslant \|X + Y\|$$

for all $X, Y \geqslant 0$. □

Corollary 3.3. *For any partitioned Hermitian matrix,*

$$\left\|\begin{pmatrix} A & X^* \\ X & B \end{pmatrix}\right\|_\infty \leqslant \max\{\||A| + |X|\|_\infty; \||B| + |X^*|\|_\infty\}.$$

Proof. We consider the block-matrix

$$\begin{pmatrix} 0 & 0 & 0 & X \\ 0 & A & X^* & 0 \\ 0 & X & B & 0 \\ X^* & 0 & 0 & 0 \end{pmatrix}$$

with normal blocks

$$\begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}$$

and we apply Theorem 3.2. □

A special case of (4) is the following statement:

Proposition 3.4. *Let A, B be normal. Then, for all symmetric norms,*

$$\|A + B\| \leqslant \||A| + |B|\|.$$

This can be regarded as a triangle inequality for normal operators. In case of Hermitian operators, a stronger triangle inequality holds [6].

Proposition 3.5. *Let S, T be Hermitian. Then, for some unitaries U, V,*

$$|S + T| \leqslant \frac{1}{2}\{U(|S| + |T|)U^* + V(|S| + |T|)V^*\}.$$

Proposition 3.5 implies Proposition 3.4 by letting

$$S = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & B^* \\ B & 0 \end{pmatrix}.$$

Proposition 3.4 shows that, given $A, B \geqslant 0$ and any complex number z,

$$\|A + zB\| \leqslant \|A + |z|B\|$$

so that for all integers $m = 1, 2 \ldots,$

$$\|(A + zB)^m\| \leqslant \|(A + |z|B)^m\|. \tag{5}$$

This was observed by Bhatia and Kittaneh. They also noted the identity

$$A^m + B^m = \frac{1}{m} \sum_{j=0}^{m-1} (A + wB)^j, \tag{6}$$

where w is the primitive mth root of the unit. Combining (5) and (6), Bhatia and Kittaneh obtained [4]: *Given* $A, B \geqslant 0$

$$\|A^m + B^m\| \leqslant \|(A + B)^m\|$$

for all $m = 1, 2, \ldots$ *and all symmetric norms*. This result was the starting point of superadditive or subadditive inequalities for symmetric norms.

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