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Abstract and Generic Rigidity in the Plane

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We consider the concept of abstract two-dimensional rigidity and provide necessary and sufficient conditions for a matroid to be an abstract rigidity matroid of a complete graph. This characterization is a natural extension of the characterization of graphic matroids due to Graver or Sachs. We also give an example of an abstract rigidity matroid which is not infinitesimal. © 1994 Academic Press, Inc.

1. INTRODUCTION

Let $G=(V, K)$ be the complete graph on the vertex set $V=\{1, 2, ..., n\}$. A *two-dimensional framework* is a graph (V, E) together with an embedding p of V into real 2-space, where E is interpreted as the collection of those pairs of vertices whose images under p are to be joined by rigid rods. We may identify the embedding p with a point in $\mathbb{R}^{2n}$, and measure the distance between vertices by evaluating the *rigidity function* ρ: $\mathbb{R}^{2n} \rightarrow \mathbb{R}^{|K|}$ defined by $\rho(p)_{i,j}=(p_i-p_j)^2$ (where the coordinates of $\mathbb{R}^{|K|}$ are indexed by (i, j) in, say, lexicographical order). Clearly ρ is continuously differentiable

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and we define $R(p)$, the rigidity matrix for the embedding p, by $\rho'(p) = 2R(p)$. $R(p)$ is an $\binom{n}{2}$ by $2n$ matrix whose entries are functions of the coordinates of p as a point in $\mathbb{R}^{2n}$. The framework (V, E, p) is called *rigid* if the rows of $R(p)$ corresponding to E span a $2n-3$ dimensional subspace of $\mathbb{R}^{2n}$.

The vertices of a plane structure are in *generic* position if their coordinates are algebraically independent over the rational field. This highly non-mechanical assumption means that the linear dependence of the rows of the rigidity matrix depends only on the underlying graph, and consequently the rigidity of the framework depends on the graph only. A graph G is called *generically rigid* if there is a generic embedding of G which is rigid.

One can use the rigidity matrix to define a matroid on the edge set E of a framework (G, E, p) by calling a subset F of E independent if the rows in $R(p)$ corresponding to the elements in F are independent as vectors over $\mathbb{R}$. Such a matroid is called an *infinitesimal rigidity matroid.* If p is generic, the corresponding matroid is called a *generic rigidity matroid* and is denoted by $\mathscr{G}_2(n)$.

2. Abstract Rigidity in the Plane

Infinitesimal rigidity matroids are special cases of the *abstract rigidity matroids* introduced by Graver in [4]. Let (V, K) be a complete graph, and for $E \subseteq K$ let $V(E)$ denote the vertex set of the subgraph of (V, K) induced by E. If U is a subset of V, let $K(U)$ denote the set of edges having both endpoints in U. A two-dimensional abstract rigidity matroid $\mathscr{A}$ on K is a matroid on K whose closure operator, $\langle \cdot \rangle$, in addition to the four closure axioms of a matroid, satisfies

C5. If $|V(E) \cap V(F)| \leqslant 1$ then $\langle E \cup F \rangle \subseteq K(V(E)) \cup K(V(F))$.

C6. If $\langle E \rangle = K(V(E))$ and $\langle F \rangle = K(V(F))$ and $|V(E) \cap V(F)| \geqslant 2$, then $\langle E \cup F \rangle = K(V(E \cup F))$.

From Axiom C5 alone it follows that a single edge is independent, or $\langle \varnothing \rangle = \varnothing$, and that $\langle E \rangle \subseteq K(V(E))$ for any edge set E. From Axiom C6 we deduce that K_4 is a cycle in $\mathscr{A}$. By induction on n we may show that any edge set in an abstract rigidity matroid which is supported by n vertices has rank at most $2n-3$; i.e., Laman's condition

$$|I'| \leqslant 2|V(I')| - 3 \qquad \text{for all non-empty subsets } I' \text{ of } I$$

must be satisfied for all independent sets I. A useful consequence of this fact is that every independent set of a two-dimensional abstract rigidity matroid contains at least one vertex of valence at most 3, since the average valence of an edge set satisfying Laman's condition is at most $4 - 6/n$.

If an independent set I contains a vertex v of valence 2, we can, using Axioms C5 and C6, delete the edges of E incident with v to obtain a smaller independent set I' with the same rigidity properties as I. This process is reversible; we can enlarge an independent set by attaching a new vertex by two new edges and preserve independence and rigidity properties. I is called a *0-extension* of I'. It follows that every abstract rigidity matroid on the edge set of a complete graph on n vertices has rank $2n-3$.

To simplify notation we write $E-v$ for the edge set obtained by deleting from E all edges with endpoint v. We also use $+$ instead of $\cup$ and omit set brackets around single element sets, so $E+e$ stands for $E\cup\{e\}$, whenever the context is clear.

Suppose that I is an independent set with a vertex v of valence 3, so (v, x), (v, y), and $(v, z)\in I$. Then not all of (x, y), (y, z), and (x, z) belong to $\langle I-v\rangle$, since otherwise, by C6, $(v, x)\in\langle\{(x, y), (y, z), (x, z), (v, y), (v, z)\}\rangle\subseteq\langle I-(v, x)\rangle$, contradicting the independence of I. It follows that given any vertex v of degree 3 in an independent set I, we can find an edge e joining some pair of its neighbors so that $I'=I-v+e$ is also independent. I is called a *1-extension* of I'. Any independent set containing a vertex of degree 3 can be viewed as a 1-extension of a smaller independent set, but a 1-extension of an independent set is not necessarily independent.

An edge set E in $\mathscr{A}$ is called *rigid* if $\langle E\rangle = K(V(E))$.

If in an abstract rigidity matroid $\mathscr{A}$ *every* 1-extension of an independent set is independent, we say that $\mathscr{A}$ satisfies the 1-extendability condition. In [4] it is shown that if V is a finite set and $\mathscr{A}$ is any two-dimensional abstract rigidity matroid for V that satisfies the 1-extendability property, then $E\subseteq K$ is independent in $\mathscr{A}$ if and only if E satisfies Laman's condition for dimension 2. This key observation serves to characterize $\mathscr{G}_2(n)$ combinatorially, and the following theorems are immediate consequences.

THEOREM 1. *Let $\mathscr{A}$ be a two-dimensional abstract rigidity matroid on n vertices. The following are equivalent*:

1. $\mathscr{A}=\mathscr{G}_2(n)$.
2. (Laman [5]) *The independent sets of $\mathscr{A}$ are those sets which satisfy Laman's condition.*
3. *$\mathscr{A}$ is maximal among all abstract rigidity matroids on n vertices.*
4. *All cycles of $\mathscr{A}$ are rigid.*
5. (Graver [4]) *$\mathscr{A}$ has the 1-extendability property.*
6. (Dress [2]) *For any closed set E of $\mathscr{A}$ with maximal cliques* $E_1, ..., E_k$, $r(E)=r(E_1)+\cdots+r(E_k)$.

In dimension 1 there is only one abstract rigidity matroid, namely the generic one, which coincides with the cycle matroid of K_n.

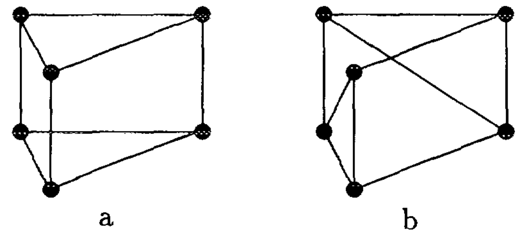

FIG. 1. Prisms.

In dimension 2 there are many abstract rigidity matroids on K_n for $n \geqslant 6$. The following example shows that not all abstract rigidity matroids can be realized as an infinitesimal rigidity matroid.

We define a matroid on the edges of $K_6 = (V, E)$ as follows. Let $\mathscr{B}$ denote the collection of all subsets of E which are bases of $\mathscr{G}_2(6)$ with the exception of those which correspond to subgraphs isomorphic to the prism, illustrated in Fig. 1a. We first show that $\mathscr{B}$ is the collection of bases of a matroid, $\mathscr{A}$. Every proper subset of a prism is contained in a basis other than a prism, see Fig. 2. It follows that a minimal set not contained in an element of $\mathscr{B}$ is either a cycle in $\mathscr{G}_2(6)$ not containing a prism or a prism. We show that the collection $\mathscr{C}$ of minimal sets which are not subsets of elements in $\mathscr{B}$ satisfy the cycle axioms of a matroid. Clearly no element of $\mathscr{C}$ is contained in another. Let $C_1, C_2 \in \mathscr{C}$. If neither are prisms, then $C_1 \cup C_2 - e$ contains a cycle of $\mathscr{G}_2(6)$, hence an element of $\mathscr{C}$. If C_1 is a prism and C_2 is not, then $C_1 \cup C_2$ contains at least 11 edges, since the triangles are the only proper rigid subgraphs of a prism and C_2 is overbraced in $\mathscr{G}_2(6)$. Likewise, if both are prisms, $C_1 \cup C_2 - e$ violates Laman's condition and so contains a cycle of $\mathscr{G}_2(6)$, hence an element of $\mathscr{C}$. To see that the matroid with bases $\mathscr{B}$ is an abstract rigidity matroid we note that the only difference between its closure operator and that of $\mathscr{G}_2(6)$ is that the closure of a prism, or of a prism minus an edge, is that prism, and no such edge set has either a separating vertex or is the union of two rigid subgraphs, so the axioms are satisfied. Let $\mathscr{A}$ denote this matroid.

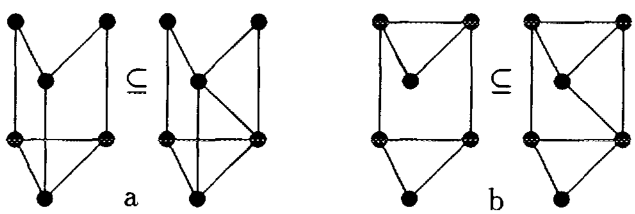

FIG. 2. Subsets of prisms.

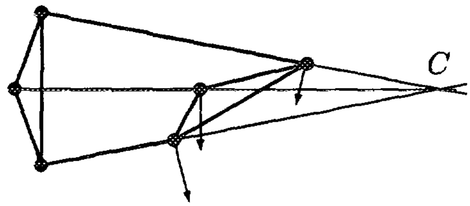

FIG. 3. An infinitesimal motion on a prism.

To see that $\mathscr{A}$ is not an infinitesimal rigidity matroid, we have to show that there is no embedding p of K_6 into $\mathbb{R}^2$, so that the corresponding infinitesimal rigidity matroid is isomorphic to $\mathscr{A}$.

Consider a non-trivial infinitesimal motion of a prism, which we may assume to be zero on one of the triangles. The motion of the other triangle must extend to an infinitesimal isometry of $\mathbb{R}^2$, which is either an infinitesimal translation or rotation, see Fig. 3. If it is a translation the lines connecting the triangles are parallel, while if it is a rotation the lines must all pass through the center of rotation. So, in order for the edge set of a prism to be infinitesimally dependent, the three lines connecting the two triangles must be projectively concurrent. Since the 3 diagonal points of a complete quadrangle are never collinear in the projective plane, there is no general embedding of six points in $\mathbb{R}^2$ such that all prisms are dependent, see [1].

3. Conditions for a Matroid to Be an Abstract Rigidity Matroid

Graver [3] and Welsh [8] proved that a matroid $\mathscr{M}$ is the cycle matroid of a graph if and only if it is binary and has a 2-complete basis of cocircuits, i.e., a basis for the cocircuit space such that each element of $\mathscr{M}$ is contained in at most two members of this basis; see also [7]. For each graphic matroid $\mathscr{M}$ there is a graph $G = (V, E)$ which has $\mathscr{M}$ as its cycle matroid and for which every connected component of G is 2-connected. Then a 2-complete basis of cocircuits consists of all but one of the vertex cocycles $\text{star}(v) = \{e \in E \mid v \text{ is an endpoint of } e\}$.

To generalize the above result to dimension 2 we need an appropriate collection of cocircuits through which we can identify the vertices. The next theorem shows that the stars of vertices minus one edge are cocycles in every two-dimensional abstract rigidity matroid on n vertices if $n > 2$.

THEOREM 2. *Let K_n be the complete graph on n vertices and let $\mathscr{A}$ be any abstract rigidity matroid on K. Let r denote the rank function of $\mathscr{A}$. Then $r(K - v + e) = 2n - 4$ for all $e \in \text{star}(v)$, and if S is any proper subset of $\text{star}(v) - e$ then $r(K - S) = r(K)$.*

Proof. $K - v + e$ is the 1-point union of a complete graph on $n-1$ vertices and an edge, and so by C5 has rank $2n-4$. If v is connected to this K_{n-1} by more than one edge, then by C6 the resulting graph has full rank. ▮

The "vertex cocycles" allows us to find conditions under which a matroid is an adstract rigidity matroid of a complete graph.

THEOREM 3. *A matroid $\mathcal{M}$ on E, $|E| = \binom{n}{2}$, is isomorphic to an abstract rigidity matroid on n vertices if and only if there is a collection of n subsets $\{E_i\}$ of E, with $|E_i \cap E_j| \leq 1$ and each $e \in E$ contained in exactly two of the E_i's, such that the following conditions hold*:

A1. *$E_i - e$ is a cocircuit for each $e \in E_i$.*

A2. *For all non-empty $F \subseteq E$ we have $r(F) \leq 2\sigma(F) - 3$ where $\sigma(F)$ is the number of E_i's with non-empty intersection with F.*

Proof. For each E_i we draw a vertex v_i, and if $e \in E_i \cap E_j$ we say the edge e joins v_i and v_j. Clearly we obtain K_n by this process, so $\mathcal{M}$ is a matroid on the edges of K_n. Since $\sigma(F) = |V(F)|$, A2 can be written to say that $r(F) \leq 2\,|V(F)| - 3$. It remains to show that A1 and A2 imply that $\mathcal{M}$ is an abstract rigidity matroid. Observe that no cycle in $\mathcal{M}$ has a vertex of valence less than 3, since otherwise there would be a cocircuit intersecting this circuit in exactly one edge, which is impossible. It follows that if F is independent in $\mathcal{M}$ and $v \notin V(F)$ then $F + (x, v) + (y, v)$ is independent in $\mathcal{M}$ for all $x, y \in V(F)$. A single edge is independent, since it intersects some vertex cocycle in exactly one edge, and by the above, any sequence of 0-extensions of an edge is independent, hence the rank of $\mathcal{M}$ is $2n-3$. To show that $\mathcal{M}$ satisfies C5, let E and F be subsets of K intersecting in at most one vertex. Without loss of generality we may assume that both E and F are complete. If there were an edge $e \in \langle E \cup F \rangle - E - F$, then we could obtain a basis for $\langle E \cup F \rangle$ by first choosing a basis B for E and then augmenting $B + e$ by connecting the free endpoint of e in $B + e$ to the vertex of intersection of E and F (if E and F do not intersect at all, choose any edge of F incident with e) and then attaching all remaining vertices of F by 0-extensions. The so constructed independent set has rank $2(|V(E \cup F)|) - 3 = r(E) + r(F) + 1$, a contradiction.

To verify that $\mathcal{M}$ also satisfies C6, we take two complete graphs whose intersection contains at least one edge and use 0-extensions on the edge of intersection to show that the union of the two graphs has full rank. ▮

In [6] the above conditions were claimed to characterize $\mathcal{G}_2(n)$, which cannot be the case since there are many two-dimensional abstract rigidity matroids which are not generic, for instance, any infinitesimal rigidity matroid arising from a general but not generic embedding. The mistake in the proof of Theorem 3 in [6] was the tacit assumption that cycles are rigid. The proper setting for the results in the remainder of [6] is the theory of abstract rigidity matroids.

Theorem 1 suggests many ways to add another condition to those of Theorem 3 to give a characterization of the generic rigidity matroid of a complete graph, for instance:

A3. All sets satisfying Laman's condition are independent.

A3′. All cycles are rigid.

A3″. $\mathcal{M}$ has the 1-extendability property.

A3‴. Every proper connected closed set of edges is the complement of the union of sets $E_i - e$.

Note that A3‴ corresponds to Dress's characterization.

To obtain conditions under which a matroid is a generic rigidity matroid of some graph, a careful study of the behavior of $\mathcal{A}$ under restriction is necessary.

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REFERENCES

1. H. S. M. COXETER, "Projective Geometry," 2nd ed. Springer, New York, 1973.
2. A. DRESS, A. DREIDING, AND H. HAEGI, "Classification of Mobile Molecules by Category Theory in Symmetries and Properties of Nonrigid Molecules: A Comprehensive Study" (J. Maruna and J. Serre, Eds.), pp. 39–59, Elsevier, Amsterdam, 1983.
3. J. E. GRAVER, "Lectures on the Theory of Matroids," Univ. of Alberta Press, Alberta, 1966.
4. J. E. GRAVER, Rigidity matroids, *SIAM J. Discrete Math.* **4** (1991), 355–368.
5. G. LAMAN, On graphs and rigidity of plane skeletal structures, *J. Engrg. Math.* **4** (1970), 331–340.
6. B. SERVATIUS, On the two-dimensional generic rigidity matroid and its dual, *J. Combin. Theory Ser. B* **53** (1991), 106–113.
7. D. SACHS, Graphs, matroids, and geometric lattices, *J. Combin. Theory* **9** (1970), 192–199.
8. D. J. A. WELSH, A bound on the number of matroids, *J. Combin. Theory* **6** (1969), 313–316.