Characterizations of strongly countable-dimensional and locally finite-dimensional spaces*

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Abstract


We characterize metrizable strongly countable-dimensional and locally finite-dimensional spaces in terms of partitions, special bases and closed mappings from zero-dimensional (in the sense of dim) metric spaces.

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1. Introduction

The following theorem contains standard characterizations of metrizable countable-dimensional spaces due to Nagata, and Nagami and Roberts (the equivalence of (i), (iii) and (iv) was established in [12], where also a characterization close to (ii) appears; the equivalence of (i) and (ii) was proved independently in [11] and the first (1965) edition of [13]).

Theorem 1.1. For every metrizable space X the following conditions are equivalent:

(i) The space X is countable-dimensional.

(ii) For every sequence \((A_1, B_1), (A_2, B_2), \ldots\) of pairs of disjoint closed subsets of X there exist closed sets \(L_1, L_2, \ldots\) such that \(L_i\) is a partition between \(A_i\) and \(B_i\) and the family \(\{L_i : i = 1, 2, \ldots\}\) is point-finite.

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(iii) The space $X$ has a $\sigma$-discrete base $\mathcal{B}$ such that the family $\{Fr U : U \in \mathcal{B}\}$ is point-finite.

(iv) The space $X$ is the image of a metrizable space $Z$ such that $\dim Z \leq 0$ under a closed mapping of finite order.

Within the last ten years several authors tried to characterize some smaller classes of spaces in a similar manner. In [16] Walker and Wenner found characterizations of locally finite-dimensional and strongly countable-dimensional spaces that correspond to (iv); in [7] Pol and the present author characterized spaces that have large transfinite dimension and separable spaces that have small transfinite dimension in terms similar to (ii) and (iii), respectively; and, finally, Hattori in [8] gave a complete analogue of the above theorem for spaces that have large transfinite dimension. In the present paper we give such analogues for strongly countable-dimensional and locally finite-dimensional metrizable spaces. The proofs are strictly parallel and based on common lemmas. Our counterpart of (iv) in the locally finite-dimensional case coincides with the characterization given in [16], but this is not so in the case of strongly countable-dimensional spaces (see Remark 4.4).

The terminology and notations follow [4-6]. I am grateful to W. Olszewski for his valuable comments on the first draft of this paper.

2. Characterizations of strongly countable-dimensional and locally finite-dimensional spaces in terms of coverings

The first characterization in this section is a restatement of a theorem of Arhangel'skiĭ given in [2] ([6, Theorem 1.18]), and the proof is similar; the second is an obvious counterpart of the first one.

**Lemma 2.1.** If a normal space $X$ is represented as the union of an increasing sequence $F_1 \subseteq F_2 \subseteq \cdots$ of closed subspaces such that $\dim F_n \leq n$ for $n = 1, 2, \ldots$, then every finite open cover $\mathcal{U}$ of the space $X$ has an open refinement $\mathcal{V}$ with the property that $\text{ord} \mathcal{V}|F_n \leq (2^+ 3^+ \cdots + n + 1) - 1$.

**Proof.** For each $n$ the finite open cover $\mathcal{U}|F_n$ of the subspace $F_n$ has a closed shrinking $\mathcal{F}_n$ such that $\text{ord} \mathcal{F}_n \leq n$, and by [4, Theorem 3.1.2] there exists a family $\mathcal{V}_n$ of open subsets of $X$ such that $F_n = \bigcup \mathcal{V}_n$, $\text{ord} \mathcal{V}_n \leq n$ and every member of $\mathcal{V}_n$ is contained in a member of $\mathcal{U}$. One easily checks that the union $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n(X \setminus F_{n-1})$, where $F_0 = \emptyset$, is an open refinement of $\mathcal{U}$ and has the required property. $\square$

**Theorem 2.2.** A normal space $X$ is strongly countable-dimensional if and only if $X$ can be represented as the union of a sequence of closed subsets $X_1, X_2, \ldots$ such that every finite open cover $\mathcal{U}$ of the space $X$ has an open refinement $\mathcal{V}$ with the property that $\text{ord} \mathcal{V}|X_n \leq n$ for $n = 1, 2, \ldots$. 

Proof. Every normal space $X$ that can be thus represented is strongly countable-dimensional, because—obviously—$\dim X_n \leq n$ for $n = 1, 2, \ldots$

If $X$ is a strongly countable-dimensional normal space, then $X = \bigcup_{n=1}^{\infty} F_n$, where $F_1 \subseteq F_2 \subseteq \cdots$ are closed and $\dim F_n \leq n$ for $n = 1, 2, \ldots$. By Lemma 2.1, the sequence $F_1, F_1, F_2, \ldots$, where the set $F_n$ is repeated $n+2$ times, has the required property. \qed

Since for every open cover $\mathcal{V}$ of a topological space $X$ and every subspace $A$ of $X$ we have $\ord \mathcal{V}|_A = \ord \mathcal{V}|_A$, in Theorem 2.2 one can drop the assumption that the subsets $X_n$ are closed.

Let us recall that a normal space $X$ is locally finite-dimensional, if for every point $x \in X$ there exists a neighbourhood $U$ such that $\dim U$ is finite. The class of locally finite-dimensional spaces was first studied in some detail in [17]; in particular it was observed there that every locally finite-dimensional paracompact space is strongly countable-dimensional.

**Theorem 2.3.** A metrizable space $X$ is locally finite-dimensional if and only if $X$ can be represented as the union of a sequence of open subsets $X_1, X_2, \ldots$ such that every finite open cover $\mathcal{U}$ of the space $X$ has an open refinement $\mathcal{V}$ with the property that $\ord \mathcal{V}|X_n \leq n$ for $n = 1, 2, \ldots$.

Proof. Every metrizable space $X$ that can be thus represented is locally finite-dimensional; indeed, we have $\dim X_n \leq n$ for $n = 1, 2, \ldots$, because $\dim F \leq n$ for every closed set $F \subseteq X$ that is contained in $X_n$, and $X_n$ is the union of countably many such sets.

Consider now a locally finite-dimensional metrizable space $X$ and let $W_n$ be the set of all points in $X$ that have a neighbourhood of dimension $\leq n$. From the paracompactness of $W_n$ and the locally finite sum theorem (see [4, Theorem 3.1.10 or 4.1.10]) it follows that $\dim W_n = n$, and from the countable paracompactness and [5, Theorem 5.2.1] there exists an increasing sequence $F_1 \subseteq F_2 \subseteq \cdots$ of closed subsets of $X$ such that $F_n \subseteq W_n$ and $\bigcup_{n=1}^{\infty} \operatorname{Int} F_n = X$. By Lemma 2.1, the sequence $\operatorname{Int} F_1, \operatorname{Int} F_1, \operatorname{Int} F_2, \ldots$, where the set $\operatorname{Int} F_n$ is repeated $n+2$ times, has the required property. \qed

Let us note that—as one easily checks using [4, Theorem 3.1.14] and [5, Exercise 5.3.C(b)]—Theorem 2.3 holds for weakly paracompact perfectly normal spaces.

Let us also note that in Theorems 2.2 and 2.3 one can assume that $\mathcal{V}$ is a shrinking of $\mathcal{U}$.

**Remark 2.4.** One might ask if there exists a characterization of countable-dimensional spaces analogous to Theorems 2.2 and 2.3. The observation which follows Theorem 2.2 shows that it cannot be obtained just by considering sequences...
$X_1, X_2, \ldots$ of arbitrary subsets; a further modification is necessary, and it turns out that the following holds:

**Proposition 2.5.** A metrizable space $X$ is countable-dimensional if and only if $X$ can be represented as the union of a sequence of subsets $X_1, X_2, \ldots$ such that every finite open cover $\mathcal{U}$ of the space $X$ has a finite closed refinement $\mathcal{F}$ with the property that $\text{ord } \mathcal{F}|X_n \leq n$ for $n = 1, 2, \ldots$.

**Proof.** The "only if" part follows immediately from a result of Nagata ([12, Lemma 3.1] or [13, III.7.A]).

To obtain the "if" part it suffices to show that if a subspace $A$ of a metrizable space $X$ has the property that every finite open cover $\mathcal{U}$ of $X$ has a finite closed refinement $\mathcal{F}$ such that $\text{ord } \mathcal{F}|A \leq n$, then $\dim A \leq 2n + 1$. Consider a finite family $\mathcal{W}$ of open subsets of $X$ such that $A \subseteq W = \bigcup \mathcal{W}$, represent $W$ as the union $F_1 \cup F_2 \cup \cdots$ of closed subsets of $X$ such that $F_i \subseteq \text{Int } F_{i+1}$, and define $R_i = F_i \setminus \text{Int } F_{i-1}$ for $i = 1, 2, \ldots$, where $F_0 = \emptyset$. For each $i$ take a finite closed refinement $\mathcal{F}_i$ of the cover $\mathcal{W} \cup \{X \setminus F_i\}$ such that $\text{ord } \mathcal{F}_i|X_{<n} \leq n$; obviously each member of $\mathcal{F}_i|R_i$ is contained in a member of $\mathcal{W}$ and the union $\mathcal{F}_n = \bigcup_{i=1}^{n} \mathcal{F}_i|R_i$ is a locally finite closed cover of $W$. One easily checks that $\text{ord } \mathcal{F}_n|A \leq 2n + 1$, so that $\dim A \leq 2n + 1$. \qed

Let us observe that—as shown by Sitnikov's example [4, Example 1.10.23]: a two-dimensional dense subspace of $I^3$ that has finite closed covers of order 1 by arbitrary small sets—it is not necessarily true that $\dim A \leq n$ for the set $A$ in the last proof. The best evaluation is $\dim A \leq 2n$, and it can be obtained by using Katětov's theorem on metric dimension [4, Problem 1.6.D] and the fact that on every metrizable space $X$ there exists a metric with the property that for each $\varepsilon > 0$ the space $X$ has an open cover by sets of diameter less than $\varepsilon$ which is the union of finitely many discrete families (see [14, Lemma 3.3]).

One easily sees that Theorems 2.2 and 2.3 remain valid if the open refinement $\mathcal{V}$ with the property that $\text{ord } \mathcal{V}|X_n \leq n$ is replaced by a finite closed refinement $\mathcal{F}$ with the property that $\text{ord } \mathcal{F}|X_n \leq n$. Thus the three classes of countable-dimensional, strongly countable-dimensional and locally finite-dimensional spaces can be characterized in strictly parallel ways.

**Remark 2.6.** Lemma 2.1 remains valid if $\mathcal{U}$ is a locally finite open cover of $X$ (in the proof one has then to use [4, Problem 3.1.A] rather than [4, Theorem 3.1.2]). Thus Theorems 2.2 and 2.3 hold if one assumes that $\mathcal{U}$ is a locally finite open cover of $X$; in fact, Theorem 2.3 holds for an arbitrary open cover $\mathcal{U}$, and so does Theorem 2.2 under the assumption that $X$ is paracompact.

Also Proposition 2.5 holds if one assumes that $\mathcal{U}$ is an arbitrary open cover and $\mathcal{F}$ a locally finite closed refinement; the proof of the "only if" part remains unchanged, and the "if" part follows from Proposition 2.5.
3. Lemmas to the main theorem

Lemma 3.1. Let $X$ be a metrizable space and $\mathcal{M}$ a family of subspaces of $X$. If for every sequence $(A_1, B_1)$, $(A_2, B_2), \ldots$ of pairs of disjoint closed subsets of $X$ there exist closed sets $L_i, L_i, \ldots$ such that $L_i$ is a partition between $A_i$ and $B_i$ and ord($\{L_i: i = 1, 2, \ldots\}$) $\leq n_M$ for each $M \in \mathcal{M}$, then the space $X$ has a $\sigma$-discrete base $\mathcal{B}$ such that ord($\{F \cup U: U \in \mathcal{B}\}$) $\leq n_M$ for each $M \in \mathcal{M}$.

Proof. Consider a $\sigma$-discrete base $\{V_s: s \in S\}$ for $X$, where $S = \bigcup_{i=1}^{\infty} S_i, S_i \cap S_j = \emptyset$ whenever $i \neq j$, and the family $\{V_s: s \in S_i\}$ is discrete for $i = 1, 2, \ldots$. For every $s \in S$ let $V_s = \bigcup_{j=1}^{\infty} A_{s,j}$, where the sets $A_{s,j}$ are closed, and for $i, j = 1, 2, \ldots$ consider the disjoint closed sets $A_{s,j} = \bigcup_{s \in S_j} A_{s,j}$ and $B_{s,j} = X \setminus W_i$, where $W_i = \bigcup_{s \in S_i} V_s$. There exist open sets $U_{s,j}$ such that $A_{s,j} \subset U_{s,j} \subset W_i$ for $i, j = 1, 2, \ldots$ and ord($\{M \cap F \cup U_{s,j}: i, j = 1, 2, \ldots\}$) $\leq n_M$ for each $M \in \mathcal{M}$. The family $\mathcal{B}_{s,i} = \{U_{s,j}: s \in S_i\}$, where $U_{s,j} = V_s \cap U_{s,j}$, is discrete, and $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_{s,i}$ is a base for $X$. The family $\{F \cup U_{s,j}: s \in S_i\}$ also is discrete, and—since $\bigcup \mathcal{B}_{s,i} = U_{s,i}$—we have $\bigcup \{F \cup U_{s,j}: s \in S_i\} = F \cup U_{s,i}$; so that if a point belongs to $F \cup U_{s,i}$, it belongs to exactly one set $F \cup U_{s,j}$ with $s \in S_i$. Hence ord($\{F \cup U: U \in \mathcal{B}\}$) $\leq n_M$ for each $M \in \mathcal{M}$. □

Lemma 3.2. For every $\sigma$-discrete base $\mathcal{B} = \{U_s: s \in S\}$ of a metrizable space $X$, where $S = \bigcup_{i=1}^{\infty} S_i, S_i \cap S_j = \emptyset$ whenever $i \neq j$, and the family $\{U_s: s \in S_i\}$ is discrete for $i = 1, 2, \ldots$, there exists a sequence of locally finite closed covers $\mathcal{F}_1, \mathcal{F}_2, \ldots$, where $\mathcal{F}_i = \{F_{t_1, t_2, \ldots, t_i}: t_1, t_2, \ldots, t_i \in T\}$, such that

(a) $F_{t_1, t_2, \ldots, t_i} = \bigcup \{F_{t_1, t_2, \ldots, t_i}: t \in T\}$ for all $t_1, t_2, \ldots, t_i \in T$ and $i = 1, 2, \ldots$.

(b) For every point $x \in X$ and any neighbourhood $U$ of this point there exists a natural number $i$ such that $\text{St}(x, \mathcal{F}_i) \subset U$.

(c) For $i = 1, 2, \ldots$ each point $x \in X$ such that $|\{s \in \bigcup_{j=1}^{\infty} S_j: x \in F_{t_1, t_2, \ldots, t_i}\}| \leq n$ belongs to at most $2^n$ members of $\mathcal{F}_i$.

Proof. Let $T = S \cup \{t_0\}$, where $t_0 \not\in S$. For $i = 1, 2, \ldots$ consider the closed cover $\mathcal{B}_i = \{B_t: t \in T\}$ of the space $X$, where $B_{t_0} = \emptyset$, $B_{t_1} = \emptyset$ for $s \in S_i$, $B_{t_1} = \emptyset$ for $s \in S \setminus S_i$, and $B_{t_1} = X \setminus \bigcup_{s \in S_i} U_s$; clearly, $\mathcal{B}_i$ is locally finite and each point $x \in X$ belongs to at most two members of $\mathcal{B}_i$.

The sequence $\mathcal{F}_1, \mathcal{F}_2, \ldots$ shall be defined inductively. Let $\mathcal{F}_1 = \mathcal{B}_1$ and—assuming that $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{i-1}$ are already defined—$\mathcal{F}_i = \{F_{t_1, t_2, \ldots, t_i}: t_1, t_2, \ldots, t_i \in T\}$, where

$$F_{t_1, t_2, \ldots, t_i} = F_{t_1, t_2, \ldots, t_{i-1}} \cap B_{t_i}. \tag{2}$$

From the definition it follows directly that $\mathcal{F}_1, \mathcal{F}_2, \ldots$ are locally finite closed covers of $X$ and that (a) is satisfied.

Consider a point $x \in X$ and a neighbourhood $U$ of this point. There exists an $s \in S$ such that $x \in U \subset \bar{U} \subset U$ and a natural number $i$ such that $s \in S_i$. Since $\bar{U}$ is the only member of $\mathcal{B}_i$ that contains $x$, it follows from (2) that $\text{St}(x, \mathcal{F}_i) \subset \bar{U} \subset U$. 

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It remains to show that (c) is satisfied. We shall proceed by induction. For \( i = 1 \) this follows from (1) and the definition of \( \mathcal{F}_1 \). Assume that

\[
\left| \left\{ s \in \bigcup_{j=1}^{i-1} S_j : x \in \text{Fr } U_s \right\} \right| \leq m,
\]

then \( |\{ F \in \mathcal{F}_{i-1} : x \in F \}| \leq 2^m \), \hspace{1cm} (3)

and consider an \( x \in X \) such that \( |\{ s \in \bigcup_{j=1}^{i-1} S_j : x \in \text{Fr } U_s \}| \leq n \). If \( x \notin \text{Fr } U_s \) with \( s \in S_i \), then \( x \) belongs to exactly one member of \( \mathcal{F}_i \), and since \( |\{ F \in \mathcal{F}_{i-1} : x \in F \}| \leq 2^n \) by (3), it follows from (2) that \( x \) belongs to at most \( 2^n \) members of \( \mathcal{F} \). If \( x \in \text{Fr } U_s \) for some \( s \in S_i \), then \( |\{ s \in \bigcup_{j=1}^{i-1} S_j : x \in \text{Fr } U_s \}| \leq n - 1 \), so that \( |\{ F \in \mathcal{F}_{i-1} : x \in F \}| \leq 2^{n-1} \) by (3), and from (1) and (2) it follows that \( x \) belongs to at most \( 2^m \) members of \( \mathcal{F} \). \( \square \)

As the reader certainly noticed, the proof of Lemma 3.1 repeats a standard argument. The proof of Lemma 3.2 is a slight modification of arguments that appear in [1, proof of Theorem 6, Ch. 10, Section 2] and in [3, proof of Theorem 1.11].

4. The main theorem

To begin, we have to define two notions that shall replace the point-finiteness in the characterizations of strongly countable-dimensional and locally finite-dimensional spaces analogous to those in Theorem 1.1 from our Introduction.

**Definition 4.1.** A family \( \{ A_s : s \in S \} \) of subsets of a topological space \( X \) is closedly point-finite (openly point-finite) if \( X \) can be represented as the union of a sequence \( X_1, X_2, \ldots \) of closed (open) subsets such that \( \text{ord}\{ A_n \cap X_n : s \in S \} < \infty \) for \( n = 1, 2, \ldots \).

One easily sees that each locally finite family is openly point-finite, each closedly point-finite family is point-finite, and in a countably paracompact normal space each openly point-finite family is closedly point-finite (see [5, Theorem 5.2.3]).

None of the above implications can be reversed: The family of all isolated points in \( X = \{0, 1, 1/2, 1/3, \ldots \} \) is openly point-finite without being locally finite; in the same space \( X \) the family of all sets \( \{1/n, 1/n + 1, \ldots, 1/2n\} \) is closedly point-finite without being openly point-finite; and the family of all sets \( \{x_n, x_{n+1}, \ldots, x_{2n}\} \), where \( \{x_1, x_2, \ldots\} \) is a countable dense subset (with \( x_i \neq x_j \) whenever \( i \neq j \)) of \( X = [0, 1] \) is point-finite without being—as follows from the Baire category theorem—closedly point-finite.

**Definition 4.2.** A mapping \( f : Z \to X \) is of closedly finite (openly finite) order if \( X \) can be represented as the union of a sequence \( X_1, X_2, \ldots \) of closed (open) subsets such that \( \text{ord } f|_{f^{-1}(X_n) < \infty} \) for \( n = 1, 2, \ldots \), i.e., that there exists a natural number \( m \) with the property that all fibers of the restriction \( f|_{f^{-1}(X_n)} \) have cardinality less than \( m \).
Our main theorem is formulated simultaneously for strongly countable-dimensional and locally finite-dimensional spaces:

**Theorem 4.3.** For every metrizable space $X$ the following conditions are equivalent:

(i) The space $X$ is strongly countable-dimensional (locally finite-dimensional).

(ii) For every sequence $(A_1, B_1), (A_2, B_2), \ldots$ of pairs of disjoint closed subsets of $X$ there exist closed sets $L_1, L_2, \ldots$ such that $L_i$ is a partition between $A_i$ and $B_i$ and the family $\{L_i : i = 1, 2, \ldots\}$ is closedly (openly) point-finite.

(iii) The space $X$ has a $\sigma$-discrete base $\mathcal{B}$ such that the family $\{F \setminus U : U \in \mathcal{B}\}$ is closedly (openly) point-finite.

(iv) The space $X$ is the image of a metrizable space $Z$ such that $\dim Z \leq 0$ under a closed mapping of closedly (openly) finite order.

**Proof.** We begin with showing that (i) $\Rightarrow$ (ii). Let us represent $X$ as the union of closed (open) subsets $X_1, X_2, \ldots$ in accordance with Theorem 2.2 (Theorem 2.3).

The first step consists in showing that for every finite sequence $(A_1, B_1), (A_2, B_2), \ldots, (A_m, B_m)$ of pairs of disjoint closed subsets of $X$ there exist open sets $V_1, V_2, \ldots, V_m$ and $W_1, W_2, \ldots, W_m$ such that

$$A_i \subset V_i \subset \bar{V}_i \subset W_i \subset X \setminus B_i \quad \text{for} \quad i = 1, 2, \ldots, m \quad (4)$$

and

$$\text{ord}\{(\bar{W}_i \setminus V_i) \cap X_n : i = 1, 2, \ldots, m\} \leq n \quad \text{for} \quad n = 1, 2, \ldots. \quad (5)$$

Let $\mathcal{F}$ be the family of all finite subsets of $\{1, 2, \ldots, m\}$, and for each $T \in \mathcal{F}$ let

$$G_T = \bigcap\{X \setminus B_i : i \in T\} \cap \bigcap\{X \setminus A_i : i \in T\}. \quad (6)$$

The family $\{G_T : T \in \mathcal{F}\}$, being a finite open cover of $X$, has an open shrinking $\{H_T : T \in \mathcal{F}\}$ such that $\text{ord}\{(H_T \cap X_n : T \in \mathcal{F}\} \leq n$ for $n = 1, 2, \ldots$. Take a closed shrinking $\{K_T : T \in \mathcal{F}\}$ of the cover $\{H_T : T \in \mathcal{F}\}$ and observe that

if $K_T \cap A_i \neq \emptyset$, then $i \in T. \quad (7)$

For each $T \in \mathcal{F}$ and $i \in T$ we can find open sets $V_{T,i}$ and $W_{T,i}$ such that

$$K_T \subset V_{T,i} \subset \bar{V}_{T,i} \subset W_{T,i} \subset \bar{W}_{T,i} \subset H_T \quad \text{for} \quad i \in T \in \mathcal{F} \quad (8)$$

and

if $i, i' \in T$ and $i < i'$, then $\bar{W}_{T,i'} \subset V_{T,i}. \quad (9)$

Consider the open sets

$$V_i = \bigcup\{V_{T,i} : i \in T \in \mathcal{F}\} \quad \text{and} \quad W_i = \bigcup\{W_{T,i} : i \in T \in \mathcal{F}\}.$$

From (8) and (6) it follows that $V_i \subset W_i \subset \bar{V}_i \subset X \setminus B_i$ for $i = 1, 2, \ldots, m$; now, for every point $x \in A_i$, there exists a $T$ such that $x \in K_T$, and since (7) implies that $i \in T$, we have $x \in V_{T,i}$—thus $A_i \subset V_i$ for $i = 1, 2, \ldots, m$ and (4) is established.
To establish (5) consider $i_1 < i_2 < \cdots < i_k \leq m$ such that $\bigcap_{j=1}^{i_k} (\overline{W}_{i_j} \setminus V_{i_j}) \cap X_n \neq \emptyset$. There exist sets $T_1, T_2, \ldots, T_k \in \mathcal{T}$ such that $i_j \in T_j$ and $\bigcap_{j=1}^{i_k} (\overline{W}_{T_{i_j}} \setminus V_{T_{i_j}}) \cap X_n \neq \emptyset$, where by virtue of (9), $T_j \neq T_{j'}$ for $j \neq j'$. The inclusion $\overline{W}_{T_{i_j}} \subset H_{T_j}$ and the inequality $\operatorname{ord}(H_T \cap X_n : T \in \mathcal{T}) \leq n$ imply that $k \leq n + 1$, i.e., that (5) holds.

Now, consider a sequence $(A_1, B_1), (A_2, B_2), \ldots$ of pairs of disjoint closed subsets of $X$. By the first step in our proof we can define inductively for $j = 1, 2, \ldots$ open sets $V_{j,1}, V_{j,2}, \ldots, V_{j,j}$ and $W_{j,1}, W_{j,2}, \ldots, W_{j,j}$ such that

$$A_j \subset V_{j,j} \subset W_{j,j} \subset X \setminus B_j,$$

$$W_{j-1,j} \subset V_{j,j} \subset W_{j,j} \subset W_{j-1,i} \quad \text{for } i < j,$$  

and

$$\operatorname{ord}((\overline{W}_{i,j} \setminus V_{i,j}) \cap X_n : i = 1, 2, \ldots, j) \leq n \quad \text{for } n = 1, 2, \ldots.$$  

By virtue of (10) and (11) the open sets $V_i = \bigcup \{V_{i,j} : j = i, i+1, \ldots\}$ satisfy the inclusion $A_i \subset V_i \subset V_i \subset X \setminus B_i$ for $i = 1, 2, \ldots$, so that the set $L_i = \overline{V}_i \setminus V_i$ is a partition between $A_i$ and $B_i$ for $i = 1, 2, \ldots$. Since $L_i = \overline{V}_i \setminus V_i \setminus W_{i,j}$ for $j \geq i$, it follows from (12) that $\operatorname{ord}(L_i \cap X_n : i = 1, 2, \ldots) \leq n$ for $n = 1, 2, \ldots$, i.e., that the family $\{L_i : i = 1, 2, \ldots\}$ is closedly (openly) point-finite.

The implication (ii) $\Rightarrow$ (iii) follows from Lemma 3.1. We shall now show that (iii) $\Rightarrow$ (iv). Consider a $\sigma$-discrete base $\mathcal{B} = \{V_s : s \in S\}$, where $S = \bigcup_{i=1}^{\infty} S_i$, $S_i \cap S_j = \emptyset$ whenever $i \neq j$ and the family $\{U_s : s \in S_i\}$ is discrete for $i = 1, 2, \ldots$, such that $\operatorname{ord}(\overline{F}_s \cap X_n : s \in S) \leq n$ for a representation of $X$ as the union of closed (open) subsets $X_1, X_2, \ldots$. Applying Lemma 3.2 we obtain a sequence of locally finite closed covers $\mathcal{F}_1, \mathcal{F}_2, \ldots$, where $\mathcal{F}_i = \{F_{t_1, t_2, \ldots} : t_1, t_2, \ldots, t_i \in T\}$, satisfying Lemma 3.2(a)-(c). In this situation the required mapping is obtained in a standard way (cf. [13, Theorem III.8.1]):

In the Cartesian power $T^{\aleph_0}$, where $T$ is endowed with the discrete topology, we consider the subspace

$$Z = \left\{ (t_1, t_2, \ldots) : \bigcap_{i=1}^{\infty} F_{t_1, t_2, \ldots} \neq \emptyset \right\};$$

by virtue of Lemma 3.2(b), for each $z = (t_1, t_2, \ldots) \in Z$ the intersection $\bigcap_{i=1}^{\infty} F_{t_1, t_2, \ldots}$ consists of exactly one point of $X$, and assigning this point to $z$ defines a function $f$ from $Z$ to $X$. Applying standard arguments one checks that $f : Z \to X$ is a closed mapping of $Z$ onto $X$. If $x \in X_n$, then $\|\{s \in S : x \in \overline{F}_s\}\| \leq n + 1$, so that by Lemma 3.2(c), $x$ belongs to at most $2^{n+1}$ members of $\mathcal{F}_i$ for $i = 1, 2, \ldots$, and this implies that $\operatorname{ord}(f^{-1}(X_n)) \leq 2^{n+1}$. Thus $f$ is of closedly (openly) finite order.

To conclude the proof it remains to show that (iv) $\Rightarrow$ (i), but this is an immediate consequence of the theorem on dimension-raising mappings [4, Theorem 4.3.3].

Let us observe that to obtain (i) from (iii) one does not have to pass necessarily through (iv): the implication (iii) $\Rightarrow$ (i) follows easily from a characterization of $n$-dimensional spaces due to Morita [9, Theorem 8.7] or [4, Theorem 4.2.2].
The validity of the implication (i) ⇒ (ii) for strongly countable-dimensional spaces follows from [9, Theorem 9.1], and in our proof of this implication we applied Morita's arguments. The validity of the implication (i) ⇒ (iv) for strongly countable-dimensional spaces is implicit in [10] (see the proof of [10, Theorem 3.6]).

The validity of the equivalence (i) ⇔ (iv) for locally finite-dimensional spaces was established in [16]. But—as observed by Pasynkov in [15]—this is a rather easy consequence of the corresponding characterization of n-dimensional spaces [13, Theorem III.8]. Also (iii) for locally finite-dimensional spaces can be obtained from the corresponding characterization of n-dimensional spaces [4, Theorem 4.2.2] by decomposing X into open finite-dimensional "rings" \( R_1, R_2, \ldots \) in such a way that \( R_i \cap R_j = \emptyset \) whenever \( |i - j| > 1 \).

**Remark 4.4.** In [16] it is proved that a metrizable space X is locally finite-dimensional if and only if it is the image of a metrizable space Z such that \( \dim Z = 0 \) under a closed mapping \( f \) with finite fibers that has strong local order, i.e., the property that for each \( z \in Z \) there exists a neighbourhood \( U \) such that \( \operatorname{ord} f|U < \infty \). It is not hard to see, however, that the class of closed mappings with finite fibers that have strong local order coincide with the class of closed mappings of openly finite order, so that our characterization of locally finite-dimensional metrizable spaces by condition (iv) coincides with the characterization from [16].

On the other hand, our characterization of strongly countable-dimensional spaces by condition (iv) does not seem equivalent to the characterization in [16, Theorem 2], stating that a metrizable space X is strongly countable-dimensional if and only if it is the image of a metrizable space Z such that \( \dim Z = 0 \) under a closed mapping \( f \) with finite fibers that has weak local order, i.e., the property that for each \( x \in X \) there exist a \( z \in f^{-1}(x) \) and a neighbourhood \( U \) of \( z \) such that \( \operatorname{ord} f|U < \infty \). That characterization, though, can be easily obtained from ours (as shown by Olszewski):

Consider a closed mapping \( f_0: Z_0 \to X \) of closedly finite order of a metrizable space \( Z_0 \) with \( \dim Z_0 = 0 \) onto our space X and let \( X_n = n - 1, 2, \ldots \) be a closed cover of X such that \( \operatorname{ord} f_0|f_0^{-1}(X_n) < \infty \) for \( n = 1, 2, \ldots \). One can assume that \( X_1 \subset X_2 \subset \cdots \); then by replacing each \( X_n \setminus X_{n-1} \), with \( n > 2 \) by countably many closed “rings around \( X_{n-1} \)” intersected with \( X_n \) one obtains a closed cover \( \{ K_i : i = 1, 2, \ldots \} \) of order \( \leq 1 \) such that \( \operatorname{ord} f_0|f_0^{-1}(K_i) < \infty \) for \( i = 1, 2, \ldots \). Now consider the mapping \( f_i = f_0 \times \text{id}_Y: Z_0 \times Y \to X \times Y \), where \( Y = \{0, 1, 1/2, 1/3, \ldots \} \), and the closed subspace \( X_1 = (X \times \{0\}) \cup \bigcup_{i=1}^\infty (K_i \times \{1/i\}) \) of \( X \times Y \). It is not hard to verify that the space \( Z = f_1^{-1}(X_1) \subset Z_0 \times Y \) and the mapping \( f = p(f_1|Z): Z \to X \), where \( p: X_1 \to X \) is the projection, have the required properties.

Let us note that the proof of [16, Theorem 2] contains—as it seems—a gap (in the last paragraph one should consider an arbitrary fiber restricted to \( U \), and not just \( f_1^{-1}(x) \)); the theorem is (implicitly) proved in [8] (the “only if” part of the proof of [8, Theorem 3.5]).
Remark 4.5. As one easily sees, the proofs of implications (ii) ⇒ (iii) ⇒ (iv) ⇒ (i) in Theorem 4.3 work also in the case of the corresponding implications in Theorem 1.1 from our Introduction. Thus, to obtain the full characterization of countable-dimensional spaces given in that theorem it suffices to use the following lemma due to Nagata (see [12, Lemma 2.1] or [13, III.4.A]):

Lemma 4.6. If $X_1, X_2, \ldots$ is a sequence of subspaces of a metrizable space $X$ with $\dim X_n \leq 0$ for $n = 1, 2, \ldots$, then for every sequence $(A_1, B_1), (A_2, B_2), \ldots$ of pairs of disjoint closed subsets of $X$ there exist closed sets $L_1, L_2, \ldots$ such that $L_i$ is a partition between $A_i$ and $B_i$ and $\text{ord}\{L_i \cap X_1 : i = 1, 2, \ldots\} < n$ for $n = 1, 2, \ldots$.

Let us note, too, that the proofs of implications (ii) ⇒ (iii) ⇒ (iv) in Theorem 4.3 work also in the case of the corresponding implications in Hattori’s theorem mentioned in the Introduction (the implication which corresponds to (i) ⇒ (ii) follows from Lemma 4.6, and that which corresponds to (iv) ⇒ (i) is obvious).

References