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Lyapunov Functions for Nonlinear Time-Varying Systems

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For a quite general class of dynamic systems having a single memoryless time-varying nonlinearity in the feedback path some frequency domain stability criteria are developed using Lyapunov's second method. Four classes of nonlinearities are considered, and it is seen that as the behaviour of the nonlinearity is restricted, the stability conditions are relaxed. For the first three classes of nonlinearities, the results for time invariant systems are well known, and for two of the classes the result has previously been extended to apply to time-varying situations. The result concerning a significant new class of nonlinearities as well as the extension of the third previously known time-invariant result to cover time-varying systems are original with this paper.

I. INTRODUCTION

Recently V. M. Popov (1961) developed the first frequency domain criterion for nonlinear time-invariant systems of the form considered in this paper utilizing Lyapunov's second method. Basically, if the system is characterized by a forward loop transfer function $G(s)$ and a single first and third quadrant continuous nonlinearity $[\sigma f(\sigma) > 0$ all $\sigma \neq 0, f(0) = 0]$ in the feedback path, then if $\alpha > 0$ exists such that $G(s) \cdot (s + \alpha)$ is positive real,[1] the system is absolutely stable.

Since this classic work was presented, considerable effort has been expended to produce stability criteria with more general frequency domain multipliers than $(s + \alpha)$; both Lyapunov's second method and the passive operator approach have been fruitful. As would be expected, it is necessary to add constraints to the nonlinearity in order to relax conditions on the linear portion of the system. Brockett and Willems (1965) first considered monotonic nonlinearities, i.e., $df(\sigma)/d\sigma \geqq 0$ for all σ. Further contributions of this type were made by Narendra and

[1] Henceforth denoted "$G(s) \cdot (s + \alpha) =$ p.r.". If $\operatorname{Re} H(j\omega) > 0$ strictly, then it will be denoted "$H(s) =$ s.p.r." (strictly positive real).

Neuman (5/1966), Narendra and Cho (1967), and Thathachar, Srinath and Ramapriyan (1967).

This paper presents a unified approach to this type of problem, which yields not only all of the frequency domain multipliers obtained in previous works, but some novel results for time-varying nonlinear systems [$k(t)f(\sigma_0)$ in the feedback path] in the form of upper bounds on $1/k\ dk/dt$. A new class of nonlinearities is also introduced which allows significant further relaxations of the constraints on $G(s)$ and/or higher upper bounds on $1/k\ dk/dt$. A summary of all results is detailed in Table I.

Since the proof of the stability theorem for all nonlinearity classes is quite complex, the theorem is first simply stated in Section IV, then seven lemmas central to the theorem proof are listed in Section VI, and finally the proof is carried out for the first two classes and outlined for the remainder in Section VII.

II. SYSTEM REPRESENTATION

The systems considered in this paper are assumed to be characterized by the n dimensional vector state equation

$$\begin{aligned} \dot{x} &= Ax - bk(t)f(\sigma_0) \\ \sigma_0 &= h^T x; \end{aligned} \tag{2.1}$$

TABLE I

A Synopsis of Results

Multiplier form	System class			
	F/F-TV	FM/FM-TV	FMO/FMO-TV	FMOS/FMOS-TV
Popov ($s + \lambda_0$)	P/P	*	*	*
Z_{FM}	N.A.	P/P	*	*
Z_{FMO}	N.A.	N.A.	P/O	*/O
Z_{FMOS}	N.A.	N.A.	N.A.	O/O

Symbols:

N.A.–frequency domain multiplier form not applicable to the nonlinearity class

P –previous result

O –original result

* –multiplier for broader class of systems applies with no change

*/O –as (*) except for a new larger upper bound on $1/k\ dk/dt$.

TV –time-varying; nonlinearity classes defined in text.

for time invariant nonlinear systems $k(t) = 1$ and for time-varying linear systems $f(\sigma_0) = \sigma_0$ with no loss of generality.

This is equivalent to a linear plant in the forward path with transfer function

$$G(s) = h^T(sI - A)^{-1}b \tag{2.2}$$

with a single memoryless nonlinearity (which may or may not be time-varying) in the feedback portion of the loop.

It is assumed that the plant is completely controllable, completely observable and asymptotically stable [i.e., all eigenvalues of A have negative real parts]. Due to its controllability, the phase-variable canonical form can be used with no loss of generality; c.f. Johnson and Wonham (1964):

$$h^T = [h_1, h_2, \cdots h_n] \qquad A = \left[\begin{array}{c|c} 0 & I \\ \hline -a_1 & -a_2, \cdots - a_n \end{array}\right] \qquad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{2.3}$$

By inspection,

$$G(s) = \frac{h_n s^{n-1} + h_{n-1} s^{n-2} + \cdots + h_2 s + h_1}{s^n + a_n s^{n-1} + \cdots + a_2 s + a_1} \tag{2.4}$$

This transfer function is assumed to have both real zeros at $s = -\eta_i$, $i = 1, 2, \cdots m_1$ and complex zeros at $s = -\lambda_i \pm j\mu_i$, $i = 1, 2, \cdots, m_2$ where $(m_1 + 2m_2) \leqq (n - 1)$. The complex zeros are the roots of $s^2 + \pi_i s + \rho_i$, so

$$\begin{aligned} \pi_i &= 2\lambda_i \\ \rho_i &= \lambda_i^2 + \mu_i^2 \end{aligned} \tag{2.5}$$

and

$$(\rho_i - \tfrac{1}{4}\pi_i^2) \geqq 0 \tag{2.6}$$

A sector S_n of the s-plane is defined by $\lambda \geqq (1/n)\mu$; thus a zero is said to lie in S_2 if its real part is greater than one half its imaginary part. In the polar coordinate θ the condition is

$$S_n \equiv \{\theta : (\pi - \tan^{-1} n) < \theta < (\pi + \tan^{-1} n)\} \tag{2.7}$$

Whenever the real zeros are ordered in one or more groups, it will be assumed that their magnitudes increase with increasing index in each group, e.g.,

$$\begin{gathered} \eta_1 < \eta_2 < \cdots < \eta_{n_1} \\ \eta_{n_1+1} < \eta_{n_1+2} \cdots < \eta_{m_1} \end{gathered} \tag{2.8}$$

No relationship between η_i's of different groups are assumed. As can be demonstrated, when the real or complex zeros are divided into two groups (as in some multipliers; see Z_{FMO} and Z_{FMOS} to follow) there is no loss of generality in assuming that no zero appears in both groups.

The time-varying gain $k(t)$ is always assumed to be nonnegative and bounded, i.e., $0 \leqq k(t) \leqq \bar{K} < \infty$. It is also absolutely continuous, thus ensuring the existence of dk/dt.

In this paper, four classes N of nonlinearities $f(\sigma)$ will be considered (and referred to by abbreviation):

(i) First and third quadrant nonlinearities (F); any continuous function is allowed as long as $f(0) = 0$ and $0 < f(\sigma)/\sigma \leqq \bar{F} < \infty$ for all finite nonzero values of σ.

(ii) Monotonic nonlinearities (FM); this is a subclass of (i) in which it is assumed that $df(\sigma)/d\sigma \geqq 0$ for all σ.

(iii) Monotonic odd nonlinearities (FMO); this is a subclass of (ii) in which it is further assumed that $f(-\sigma) = -f(\sigma)$ for all σ.

(iv) Monotonic odd saturating nonlinearities (FMOS); this is a subclass of (iii) in which $\sigma\, d^2f/d\sigma^2 \leqq 0$ all σ, i.e., the slope $df/d\sigma$ is never increasing as $|\sigma|$ increases.

As a measure of nonlinearity, it is useful to introduce the parameter $F_{\min}$:

$$F_{\min} \equiv \{\min_{\sigma} F(\sigma)\} \equiv \min_{\sigma} \left\{ \frac{\sigma f(\sigma)}{\int_0^{\sigma} f(z)\, dz} \right\} \tag{2.9}$$

By studying the form of each class, it can be seen that the following ranges for $F_{\min}$ are permitted:

(i) F: $0 < F_{\min} < \infty$
(ii) FM and FMO: $1 < F_{\min} < \infty$
(ii) FMOS: $1 < F_{\min} \leqq 2$

By inspection, for linear systems $F_{\min} = 2$.

III. FORM OF LYAPUNOV FUNCTION

The form of the Lyapunov function used in this report is an extension of that used by Popov, viz.:

$$V(x, t) = \frac{1}{2} x^T P x + \sum_{i=0}^{m} \beta_i \, k(t) \int_0^{\sigma_i} f(z) \, dz \tag{3.1}$$

where P is an $n \times n$ symmetric positive definite matrix $[P = P^T > 0]$, all $\beta_i > 0$ and the signals used in the upper limits of the integral terms are derived from the state vector by the relations

$$\sigma_i = r_i^T x; \qquad r_0 \equiv h. \tag{3.2}$$

It is obvious that $V(x, t)$ is a positive definite decrescent form since $k(t)$ and $f(\sigma)/\sigma$ are nonnegative and bounded. In fact

$$0 < \tfrac{1}{2} x^T P x \leqq V(x, t) \leqq \tfrac{1}{2} x^T [P + \bar{K}\bar{F} \sum_{i=0}^{m} \beta_i r_i r_i^T] x.$$

It can be seen that in the case of linear time-varying systems this form reduces to

$$V(x, t) = \tfrac{1}{2}[x^T P x + k(t) x^T M x] \tag{3.3}$$

where

$$M = M^T = \sum_{i=0}^{m} \beta_i r_i r_i^T \geqq 0 \tag{3.4}$$

M can be positive definite (>0) only if the vectors r_i satisfy

$$\operatorname{rank} \left[r_0 \vdots r_1 \vdots \cdots \vdots r_m \right] = n;$$

otherwise M is only semidefinite ($\geqq 0$).

The existence of this class (3.3) of quadratic Lyapunov function as a necessary and sufficient condition for the asymptotic stability of a linear time-invariant feedback system for all values of the feedback gain parameter $0 < k < \bar{K}$ was conjectured by Narendra and Neuman (1966) and subsequently proven by Thathachar and Srinath (1967).

Making no further assumptions about the signals $r_i^T x$ other than $r_0 = h$, one sees that

$$\begin{aligned} \dot{V} = {} & \tfrac{1}{2} x^T (PA + A^T P) x - k(t) f(\sigma_0) x^T [Pb - \beta_0(\lambda_0 h + A^T h)] \\ & - \beta_0 h^T b \, [k(t) f(\sigma_0)]^2 - \beta_0 \lambda_0 k(t) \sigma_0 f(\sigma_0) \\ & + \sum_{i=1}^{m} \beta_i \, k(t) f(\sigma_i) r_i^T [Ax - bk(t) f(\sigma_0)] + \sum_{i=0}^{m} \beta_i \frac{dk}{dt} \int_0^{\sigma_i} f(z) \, dz \end{aligned} \tag{3.5}$$

To obtain this expression, $\beta_0\lambda_0 k(t)\sigma_0 f(\sigma_0)$ is added to and subtracted from the formal time derivative of (3.1). This term is positive semi-definite for all classes of nonlinearities considered here.

The strategy of Lyapunov's second method is to show that for a given set of conditions, $\dot{V}$ is negative definite. Establishing that the total time derivative of a generalized norm V is always negative except at the null state $x \equiv 0$ where $\dot{V} = 0$ is sufficient to ensure the absolute stability of the system under consideration. For a more detailed and rigorous discussion of this method, c.f. Lefschetz (1965).

IV. STABILITY THEOREM

In the theorem that follows, it will be seen that the stability conditions for a nonlinear time-varying system of the type considered in this paper are dependent upon the existence of a frequency domain multiplier $Z(s)$.

THEOREM. *If in the system described by Eq.* (2.1), $G(s)$, $f(\sigma)$ *and* $k(t)$ *satisfy the properties of Section* II, *and in addition a multiplier* $Z(s) \in Z_N(s)$ *exists such that*

(i) $$G(s)Z(s) = \text{s.p.r.} \tag{4.1}$$

(ii) $$Z(s - \Lambda_N) \in Z_N(s) \tag{4.2}$$

(iii) $$f(\sigma) \in N \quad \textit{and} \quad 1/k\, dk/dt \leqq \Lambda_N F_{\min}, \tag{4.3}$$

the system is absolutely stable. The four classes N of nonlinearities are defined in Section II; *the corresponding multiplier classes are defined subsequently.*

A discussion of this theorem is deferred until these definitions are made:

(1) $f(\sigma) \in \mathrm{F}$: for this general class of nonlinearities, one is restricted to the use of the Popov multiplier;

$$Z_F(s) = s + \lambda_0 \tag{4.4}$$

where $\lambda_0 \geqq 0$. Clearly $\Lambda_F = \lambda_0$.

(2) $f(\sigma) \in \mathrm{FM}$: by constraining $f(\sigma)$ to be monotonic, it is possible to use a more general RL multiplier;

$$Z_{\mathrm{FM}}(s) = \beta_0(s + \lambda_0) + \sum_{i=1}^{m1} \gamma_i \frac{s + \lambda_0}{s + \eta_i} \tag{4.5}$$

where $0 \leqq \lambda_0 < \eta_1$. It can be shown that any general RL impedance with poles at $s = -\eta_i$ may be expanded into this form, even though it might seem restrictive to have the numerator of each term be $(s + \lambda_0)$.

The phase of this multiplier, as that of the Popov multiplier, must be in the range (0, 90°), but it no longer needs to increase monotonically with frequency as in the case of $(s + \lambda_0)$. Again it is evident that $\Lambda_{FM} = \lambda_0$.

(3) $f(\sigma) \in$ FMO: the additional condition that $f(\sigma)$ be odd allows the use of a much more general RLC multiplier:

$$\begin{aligned} Z_{FMO}(s) = \beta_0(s + \lambda_0) &+ \sum_{i=1}^{n1} \gamma_i \frac{s + \lambda_0}{s + \eta_i} + \sum_{i=n_1+1}^{m1} \gamma_i \frac{s + \tau_i \eta_i}{s + \eta_i} \\ &+ \sum_{i=1}^{n2} \delta_i \frac{s + \zeta_i}{s^2 + \pi_i s + \rho_i} + \sum_{i=n_2+1}^{m2} \kappa_i \frac{s^2 + \phi_i s + \psi_i}{s^2 + \pi_i s + \rho_i} \end{aligned} \tag{4.6}$$

All parameters must be nonnegative. As above, the first two terms represent an RL impedance and $0 \leqq \lambda_0 < \eta_1$ is required. The third term is the expansion of a special RC function, since it is required that $1 \leqq \tau_i \leqq 2$. In the fourth term, $\zeta_i < \pi_i$, and in the fifth, $\phi_i < \pi_i$ and $\psi_i < \rho_i$ are required. In addition, it is necessary to define the following parameters for the last two terms:

$$0 < \nu_i \equiv \begin{cases} \left[\dfrac{\delta_i}{2\beta_i}\left(\left(1 + \dfrac{(\zeta_i - \lambda_i)^2}{\mu_i^2}\right)^{1/2} + 1\right)\right]^{1/2} & i = 1, 2, \cdots n_2 \\ \left[\dfrac{\kappa_i(\pi_i - \phi_i)}{2\beta_i}\left(\left(1 + \dfrac{\left[\left(\dfrac{\rho_i - \psi_i}{\pi_i - \phi_i}\right) - \lambda_i\right]^2}{\mu_i^2}\right)^{1/2} + 1\right)\right]^{1/2} & i = (n_2 + 1), \cdots m_2 \end{cases} \tag{4.7}$$

$$0 \leqq \xi_i \equiv \begin{cases} \left[\dfrac{\delta_i}{2\beta_i}\left(\left(1 + \dfrac{(\zeta_i - \lambda_i)^2}{\mu_i^2}\right)^{1/2} - 1\right)\right]^{1/2} & i = 1, 2, \cdots n_2 \\ \left[\dfrac{\kappa_i(\pi_i - \phi_i)}{2\beta_i}\left(\left(1 + \dfrac{\left[\left(\dfrac{\rho_i - \psi_i}{\pi_i - \phi_i}\right) - \lambda_i\right]^2}{\mu_i^2}\right)^{1/2} - 1\right)\right]^{1/2} & i = (n_2 + 1) \cdots m_2 \end{cases} \tag{4.8}$$

These parameters must satisfy

$$\left(\lambda_0 + \frac{1}{\beta_0} \sum_{i=n_2+1}^{m2} \kappa_i\right) - \sum_{i=1}^{m2} \frac{\beta_i}{\beta_0} (\nu_i + \xi_i) \equiv \epsilon_1 \geqq 0 \tag{4.9}$$

and

$$(\lambda_i - \mu_i) - \nu_i \equiv \epsilon_2^{(i)} \geqq 0; \qquad i = 1, 2, \cdots, m_2 \tag{4.10}$$

In this case, it is not as readily apparent what the maximum value of Λ such that $Z^*(s) \equiv Z(s - \Lambda) \in Z_{\text{FMO}}$ is. $Z^*(s)$ may be expressed in the same form as (4.6), except that some parameters are changed. Denoting the parameters of $Z^*(s)$ by stars, it is evident that the following are invariant: β_0^*, γ_i^*, δ_i^*, μ_i^*, and κ_i^*, i.e. $\beta_0^* = \beta_0$ etc. Also it may be seen that $\lambda_0^* = \lambda_0 - \Lambda$, and similarly for λ_i^*, η_i^* and $(\tau_i\eta_i)^*$. Hence for $Z^* \in Z_{\text{FMO}}$ one requires

(i) Term 1: $\Lambda \leqq \lambda_0$, by inspection.

(ii) Term 2: $\Lambda \leqq \lambda_0$, also by inspection.

(iii) Term 3: $\Lambda \leqq (2 - \tau_i)\eta_i$ which can be seen from the requirement that $\tau_i^* \leqq 2$.

(iv) Terms 4 and 5: $\Lambda \leqq \min\{\epsilon_1, \epsilon_2^{(i)}\}$ which is evident since by inspection ν_i and ξ_i are invariant.

Hence the maximum value of Λ is the smallest of these maxima, or

$$\Lambda_{\text{FMO}} = \min\{\lambda_0, (2 - \tau_i)\eta_i, \epsilon_1, \epsilon_2^{(i)}\} \tag{4.11}$$

Since $s^2 + \pi_i s + \rho_i = (s + \lambda_i + j\mu_i)(s + \lambda_i - j\mu_i)$, it is important to note that (4.10) requires that any pole of Z_{FMO} lie in the sector S_1 (2.7).

(4) $f(\sigma) \in$ FMOS: the form of the multiplier Z_{FMOS} is exactly that of Z_{FMO} (4.6); the difference lies chiefly in relaxed parameter constraints (or, for the same parameters, a larger value of Λ). Again, all parameters are nonnegative. The first two terms (RL) are unchanged. In the third (RC) term, $1 < \tau_i < \infty$ is allowed. This now permits any RC impedance with poles at $s = -\eta_i$ to be represented by this expansion.

In the fourth and fifth terms, $\zeta_i < \pi_i$, and $\phi_i < \pi_i$, $\psi_i < \rho_i$ are still required, and the parameters ν_i and ξ_i must be defined as before (4.7), (4.8). However, the inequalities of (4.9) and (4.10) are relaxed, viz.

$$\left(\lambda_0 + \frac{1}{\beta_0}\sum_{i=n_2+1}^{m_2} \kappa_i\right) - \sum_{i=1}^{n_2} \frac{\beta_i}{\beta_0}(\alpha_i\nu_i + \xi_i) - \sum_{i=n_2+1}^{m_2} \frac{\beta_i}{\beta_0}\nu_i \equiv \epsilon_1 \geqq 0 \tag{4.12}$$

$$\begin{aligned} (\lambda_i - \tfrac{1}{2}\mu_i) - (1 - \alpha_i)\nu_i &\equiv \epsilon_2^{(i)} \geqq 0, \quad i = 1, 2, \cdots, n_2 \\ (\lambda_i - \tfrac{1}{2}\mu_i) - \nu_i &\equiv \epsilon_2^{(i)} \geqq 0, \quad i = (n_2 + 1), \cdots, m_2 \end{aligned} \tag{4.13}$$

where α_i are arbitrary real numbers in the range $0 \leqq \alpha_i \leqq (1 - \xi_i/\nu_i) \leqq 1$ which may be adjusted to maximize $\min\{\epsilon_1, \epsilon_2^{(i)}\}$.

Using the arguments of section (3), one can see that

$$\Lambda_{\mathrm{FMOS}} = \min\{\lambda_0, \eta_{n_1+1}, \epsilon_1, \epsilon_2^{(i)}\}. \tag{4.14}$$

It is noteworthy that the poles of the fourth and fifth terms now may lie in S_2 rather than S_1, which is an important relaxation of constraints.

V. COMMENTS

The theorem for $f(\sigma) \in \mathrm{F}$ is simply a generalization of Popov's result to extend its application to nonlinear time-varying systems; this well-known result was obtained for the linear case [$F_{\min} = 2$] by Brockett and Forys (1964), and in its final form by Narendra and Cho (1967).

For $f(\sigma) \in \mathrm{FM}$, the theorem is an extension of a result of Narendra and Neuman (5/1966) to cover nonlinear time-varying systems. This result was originally obtained by Narendra and Cho (1967) using the passive operator technique.

The development of the result for $f(\sigma) \in \mathrm{FMO}$ is as follows: Narendra and Neuman (5/1966) obtained a multiplier corresponding to the first three terms ($Z_{\mathrm{RL}} + Z_{\mathrm{RC}}$) for a time-invariant system. Both Thathachar, Srinath and Ramapriyan (1967) [using a Lyapunov function approach similar to that of this paper] and Narendra and Cho (1967) [using the passive operator technique] obtained the more complex RLC terms, again only for time-invariant situations. The upper bound on dk/dt is original with this study.

In the case that $f(\sigma) \in \mathrm{FMOS}$, the result is completely new. The importance of this new class of nonlinearity is two-fold:

(1) If $Z \in Z_{\mathrm{FMO}}$ exists such that $G(s)Z_{\mathrm{FMO}}(s) =$ s.p.r. and Λ_{FMO} is found, then if one can say that $f(\sigma)$ is a saturating function, this same $Z(s)$ will yield $\Lambda_{\mathrm{FMOS}} \geqq \Lambda_{\mathrm{FMO}}$, where the increase is generally significant.

(2) If only a multiplier with poles in S_2 (not S_1) may be used to satisfy $G(s)Z(s) =$ s.p.r., then only a saturating nonlinearity may be included in the system.

It is important to note that the expansions are not unique, so since Λ is determined by the parameters of Z, one must obtain the best expansion. This is clear from the following example: say

$$f(\sigma) \in \mathrm{FMO} \text{ and } Z(s) = \frac{s^2 + 4s + 5}{s + 2} = \frac{(s + 2 + j)(s + 2 - j)}{s + 2}.$$

The best expansion is $Z(s) = (s + 1) + (s + 3)/(s + 2)$ in which case by (4.11), $\Lambda = \min\{1, [2 - (3/2)]2\} = 1$. However, it can be seen that $Z(s) = (s + 1\frac{1}{2}) + \frac{1}{2}(s + 4)/(s + 2)$ also, in which case $\Lambda = 0$.

It may seem that the parameter constraints and the computation of Λ for the most general RLC multiplier (4.6) are quite complex, and in the general form, this is so. It is well to recall however that in a practical problem n is in general not large, so there are in reality few terms in $Z(s)$ (no more than n), and the computation involved is not too formidable.

In a more general case than that presented here, it may not be possible to say that $f(\sigma, t) = f(\sigma)k(t)$, in which case (4.3) of the theorem need only be modified to read

$$\max_{\sigma,t}\left\{\frac{\int_0^{\sigma_0}\frac{\partial f(z,t)}{\partial t}\,dz}{\sigma f(\sigma, t)}\right\} \leqq \Lambda_N . \tag{4.15}$$

In this paper, it is assumed that $\bar{F}$ and $\bar{K}$, while bounded, are arbitrarily large. If this is not the case, then it is relatively simple to extend the theorem by the standard transformation $G^*(s) = [G(s) + 1/\bar{F}\bar{K}]$; condition (iii) of the theorem becomes

$$\frac{dk}{dt} \leqq \Lambda_N\, F_{\min}\, k\left(1 - \frac{k}{\bar{K}}\right) \tag{4.15}$$

In conclusion, Table I is included as a synopsis of the results presented in this paper.

VI. BASIC LEMMAS

First consider the inequalities satisfied for each class of nonlinearities:

LEMMA 1. *For all* FM *nonlinearities,*

$$(\sigma_1 - \sigma_2)[f(\sigma_1) - f(\sigma_2)] \geqq 0 \quad \textit{all} \quad \sigma_1, \sigma_2$$

LEMMA 2. *For all* FMO *nonlinearities,*

$$\sigma_1 f(\sigma_1) + \sigma_2 f(\sigma_2) \pm [\sigma_1 f(\sigma_2) \pm \sigma_2 f(\sigma_1)] \geqq 0 \quad \textit{all} \quad \sigma_1, \sigma_2$$

LEMMA 3. *For all* FMOS *nonlinearities,*

$$\alpha_0\sigma_1 f(\sigma_1) + [1 - \alpha_0]\sigma_2 f(\sigma_2) \pm [\sigma_2 f(\sigma_1) - \sigma_1 f(\sigma_2)] \geqq 0$$

for all σ_1 *and* σ_2, *and* $0 \leqq \alpha_0 \leqq 1$.

Next we consider various lemmas concerning $G(s)$:

LEMMA 4. *If*

$$G(s)|_{s=-\eta} = 0, \qquad \textit{then the vector}^{2}\, c, \qquad c^T \equiv h^T(\eta I + A)^{-1}$$

has the properties

(i) $c^T(sI - A)^{-1}b = G(s)/(s + \eta)$
(ii) $c^Tb = 0$
(iii) $c^TAx = h^Tx - \eta c^Tx$

LEMMA 5. *If*

$$G(s)|_{s=-\lambda\pm j\mu} = 0, \qquad \textit{then the vector}^{2}\, d, \qquad d^T \equiv h^T(\rho I + \pi A + AA)^{-1}$$

has the properties

(i)

$$d^T(sI - A)^{-1}b = \frac{G(s)}{s^2 + \pi s + \rho} = \frac{G(s)}{(s + \lambda + j\mu)(s + \lambda - j\mu)}$$

(ii) $d^Tb = 0$
(iii) $d^TAx = e^Tx$ [see Lemma 6 below]

LEMMA 6. *Under the conditions of Lemma* 5

$$e^T \equiv d^TA$$

has the properties

(i) $e^T(sI - A)^{-1}b = \dfrac{sG(s)}{s^2 + \pi s + \rho}$

(ii) $e^Tb = 0$
(iii) $e^TAx = h^Tx - \rho\, d^Tx - \pi e^Tx$

LEMMA 7. [Lefschetz form of the Kalman–Yakubovich lemma]. *Given the stable matrix A, a symmetric matrix $D > 0$, vectors $b \neq 0$, and k, and scalars $\tau \geqq 0$,*[3] *$\epsilon > 0$, then a necessary and sufficient condition for the*

[2] Zeros of the numerator of $G(s)$ must not be eigenvalues of A in order for $(\eta I + A)^{-1}$ and $(\rho I + \pi A + AA)^{-1}$ to exist. This is ensured by the observability of the system.

[3] The reviewer has pointed out a recent correction to this lemma (Lemma 7) by Lefschetz, Meyer and Wonham (1967) viz.

"If $\tau = 0$ then $k^TAb \neq 0$"

The significance of this is as follows: if $\tau = 0$ then the Kalman relation requires that $H(j\omega) \equiv k^T(j\omega I - A)^{-1}b$ be strictly positive real. This transfer function is of *existence of a solution as a matrix P (necessarily >0) and a vector q of the system*

(a) $A^TP + PA = -qq^T - \epsilon D$

(b) $Pb - k = (\tau)^{1/2}q$

is that ϵ be small enough and that the Kalman relation

(c) $\tau + 2\,\mathrm{Re}\,[k^T(j\omega I - A)^{-1}b] > 0$

be satisfied for all ω.

The proofs of lemmas one through six are given in Narendra and Taylor (1967), and Lemma 7 is proved in Lefschetz (1965).

VII. OUTLINE OF THE THEOREM PROOF

In the proof of the theorem there are five basic steps:

(i) Choose the signals $\sigma_i = r_i^Tx$ to be used in the integrals of V corresponding to each multiplier term.

(ii) Insert the correct form of r_i^Tb and r_i^TAx in $\dot{V}$ [Eq. (3.5)] according to Lemmas 4 through 6.

(iii) Lemmas 1 through 3 define positive semidefinite forms for each class of nonlinearities. These forms are added to and subtracted from $\dot{V}$.

(iv) Lefschetz' lemma (Lemma 7) is applied to the first three terms of $\dot{V}$ in order to render them negative definite. This requires that the Kalman relation be satisfied, and from this the frequency domain criterion is derived.

(v) Group all remaining terms of $\dot{V}$ into negative semidefinite forms. Any restrictions required to ensure this become stability conditions in the theorem.

These steps will be illustrated in the proof of the theorem for FM nonlinearities that follows. ($f(\sigma) \in$ F is actually a special case of this proof; the nonlinearity class is broader than FM because no FM positive semidefinite forms are needed in $\dot{V}$ to yield the Popov multiplier.)

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the same form as (2.4) with each h_i replaced by k_i. As ω becomes arbitrarily large,

$$H(j\omega) \approx \frac{k_n j\omega + k_{n-1}}{j\omega(j\omega + a_n)}$$

and

$$\mathrm{Re}\; H(j\omega) = \frac{a_n k_n - k_{n-1}}{\omega^2 + a_n^2} = \frac{-k^TAb}{\omega^2 + a_n^2},$$

so if $k^TAb < 0$, $H(j\omega) =$ s.p.r. is ensured for large ω.

(i) $\sigma_i = r_i^T x = \dfrac{\gamma_i}{\beta_i} c_i^T x = \dfrac{\gamma_i}{\beta_i} h^T(\eta_i I + A)^{-1} x$

(ii) $r_i^T b = 0, r_i^T A x = \dfrac{1}{\beta_i}[\gamma_i \sigma_0 - \beta_i \eta_i \sigma_i]$

(iii) add and subtract $\gamma_i k(t)(\sigma_0 - \sigma_i)[f(\sigma_0) - f(\sigma_i)]$

At this point one has

$$\begin{aligned}\dot{V} = {} & \frac{1}{2} x^T(PA + A^T P)x - k(t) f(\sigma_0) x^T \Big[Pb - \beta_0(\lambda_0 h + A^T h) \\ & - \sum_{i=1}^{m1} \gamma_i(h - r_i) \Big] - \beta_0 h^T b[k(t) f(\sigma_0)]^2 \\ & - \sum_{i=1}^{m1} \gamma_i k(t)(\sigma_0 - \sigma_i)[f(\sigma_0) - f(\sigma_i)] \\ & - \beta_0 \int_0^{\sigma_0} f(z)\, dz \left[\lambda_0 k(t) F(\sigma_0) - \frac{dk}{dt} \right] \\ & - \sum_{i=1}^{m1} \beta_i \int_0^{\sigma_i} f(z)\, dz \left[\left(\eta_i - \frac{\gamma_i}{\beta_i} \right) k(t) F(\sigma_i) - \frac{dk}{dt} \right]\end{aligned} \tag{6.1}$$

(iv) The first three terms of the above are negative definite as long as for all real ω,

$$\beta_0 h^T b + \operatorname{Re}\{[\beta_0(\lambda_0 h + A^T h) + \sum_{i=1}^{m1} \gamma_i(h - r_i)]^T (j\omega I - A)^{-1} b\} > 0. \tag{6.2}$$

In order to recast this requirement into the form $Z(s)G(s) = \text{s.p.r.}$ note that

$$h^T b + h^T A (j\omega I - A)^{-1} b = h^T[(j\omega I - A) + A](j\omega I - A)^{-1} b = j\omega G(j\omega)$$

by inspection, and

$$(h - r_i)^T (j\omega I - A)^{-1} b = \frac{j\omega + \left(\eta_i - \dfrac{\gamma_i}{\beta_i} \right)}{j\omega + \eta_i} G(j\omega)$$

by Lemma 4. Since each β_i is an arbitrary positive constant (the γ_i determine the magnitude of the multiplier terms it may be chosen so that $\beta_i = \gamma_i/(\eta_i - \lambda_0)$, or

$$\left(\eta_i - \frac{\gamma_i}{\beta_i} \right) = \lambda_0\,; \tag{6.3}$$

this is always possible since λ_0 is by definition smaller than any pole of Z_{RL} and hence $\lambda_0 < \eta_i$ for each i. Using these relations, (6.2) becomes

$$\mathrm{Re}\left\{\left[\beta_0(j\omega + \lambda_0) + \sum_{i=1}^{m1} \gamma_i \frac{j\omega + \lambda_0}{j\omega + \eta_i}\right] G(j\omega)\right\} > 0,$$

so defining Z_{FM} as in T2, this is tantamount to requiring that $G(s)Z_{FM} = \text{s.p.r.}$

(v) By substituting Eq. (6.3) in $\dot{V}$, the remaining terms are seen to be negative semidefinite as long as

$$-\infty \leqq \frac{dk}{dt} \leqq \lambda_0\, k(t) F_{\min}$$

which completes the proof of the theorem for $f(\sigma) \in \mathrm{FM}$.

In the proof of the theorem for the other classes, precisely the same procedure is followed. The algebra involved, while considerably more tedious, is equally straightforward. The crux of each proof lies in the choice of the signals used in the integral upper limits and the positive semidefinite forms (psf's) added to and subtracted from $\dot{V}$. For the sake of completeness, this information is included in tabular form below. The vectors c, d, and e are as defined in Lemmas 4 through 6, and the scalars ν_i and ξ_i are as defined in Eqs. (4.7) and (4.8). Subscripting is omitted for simplicity.

$f(\sigma) \in \mathrm{FMO}$; see (4.6) for multiplier expansion:

(i) Terms 1 and 2 (Z_{RL}): signals and psf's as above.

(ii) Term 3 (Z_{RC}):

$$\sigma = r^T x = \frac{\gamma}{\beta} c^T x$$

$$\text{psf} = \sigma_0 f(\sigma_0) + \sigma f(\sigma) + \sigma f(\sigma_0) - \sigma_0 f(\sigma)$$

(iii) Term 4 $\left(\delta \dfrac{s + \zeta}{s^2 + \pi s + \rho}\right)$:

$$\sigma_1 = (\lambda\nu + \mu\xi)\, d^T x + \nu e^T x$$

$$\sigma_2 = (\lambda\xi - \mu\nu)\, d^T x + \xi e^T x$$

$$\text{psf}^{(1)} = \sigma_0 f(\sigma_0) + \sigma_1 f(\sigma_1) + \sigma_1 f(\sigma_0) - \sigma_0 f(\sigma_1)$$

$$\text{psf}^{(2)} = (\sigma_0 - \sigma_2)[f(\sigma_0) - f(\sigma_2)]$$

$$\text{psf}^{(3)} = \sigma_1 f(\sigma_1) + \sigma_2 f(\sigma_2) + \sigma_1 f(\sigma_2) - \sigma_2 f(\sigma_1)$$

(iv) Term 5 $\left(\kappa \dfrac{s^2 + \phi s + \psi}{s^2 + \pi s + \rho}\right)$:

as term 4 except

$$\mathrm{psf}^{(1)} = (\sigma_0 - \sigma_1)[f(\sigma_0) - f(\sigma_1)]$$

$$\mathrm{psf}^{(2)} = \sigma_0 f(\sigma_0) + \sigma_2 f(\sigma_2) + \sigma_2 f(\sigma_0) - \sigma_0 f(\sigma_2)$$

$f(\sigma) \in$ FMOS; see (4.6) for multiplier expansion:

(i) Terms 1 and 2: as above

(ii) Term 3: as above except

$$\mathrm{psf} = \sigma_0 f(\sigma_0) + \sigma f(\sigma_0) - \sigma_0 f(\sigma)$$

(iii) Term 4: as above except

$$\mathrm{psf}^{(1)} = \alpha\sigma_0 f(\sigma_0) + (1 - \alpha)\sigma_1 f(\sigma_1) + \sigma_1 f(\sigma_0) - \sigma_0 f(\sigma_1)$$

$$\mathrm{psf}^{(3)} = \tfrac{1}{2}[\sigma_1 f(\sigma_1) + \sigma_2 f(\sigma_2)] + \sigma_1 f(\sigma_2) - \sigma_2 f(\sigma_1)$$

(iv) Term 5: as above except

$$\mathrm{psf}^{(2)} = \sigma_2 f(\sigma_2) + \sigma_2 f(\sigma_0) - \sigma_0 f(\sigma_2)$$

$$\mathrm{psf}^{(3)} = \tfrac{1}{2}[\sigma_1 f(\sigma_1) + \sigma_2 f(\sigma_2)] + \sigma_1 f(\sigma_2) - \sigma_2 f(\sigma_1)$$

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