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A stability of the generalized sine functional equations ☆

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Abstract

In this paper we will investigate the stability of the generalized (pexiderized) sine functional equation

$$g(x)h(y) = f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2.$$

Keywords: Stability; Superstability; Functional equation; Sine functional equation

1. Introduction

In 1940, the stability problem raised by S.M. Ulam [11] was solved in the case of the additive mapping by Hyers [7] in the next year.

J. Baker, J. Lawrence and F. Zorzitto in [4] introduced that if f satisfies the stability inequality $|E_1(f) - E_2(f)| \leqslant \varepsilon$, then either f is bounded or $E_1(f) = E_2(f)$. This is now frequently referred to as *Superstability*. Baker [3] showed the superstability of the cosine functional equation $f(x+y) + f(x-y) = 2f(x)f(y)$ which is called the d'Alembert functional equation. The stability of the generalized cosine functional equation has been researched in many papers [1,2,6,8–10].

The superstability of the sine functional equation

$$f(x)f(y) = f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2, \tag{S}$$

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bounded by constant is investigated by P.W. Cholewa [5], and is improved in R. Badora and R. Ger [2].

In this paper, we introduce the pexiderized sine functional equation

$$g(x)h(y)=f\left(\frac{x+y}{2}\right)^2-f\left(\frac{x-y}{2}\right)^2 \tag{S_{gh}}$$

of the sine functional equation (S) and its special cases as follow:

$$g(x)f(y)=f\left(\frac{x+y}{2}\right)^2-f\left(\frac{x-y}{2}\right)^2, \tag{S_{gf}}$$

$$f(x)h(y)=f\left(\frac{x+y}{2}\right)^2-f\left(\frac{x-y}{2}\right)^2, \tag{S_{fh}}$$

$$g(x)g(y)=f\left(\frac{x+y}{2}\right)^2-f\left(\frac{x-y}{2}\right)^2, \tag{S_{gg}}$$

in which f, g, h are unknown functions to be determined.

Given the mappings $f, g, h : G \to \mathbb{C}$, we define a difference operator $DS_{gh} : G \to \mathbb{C}$ as

$$DS_{gh}(x,y) := g(x)h(y)-f\left(\frac{x+y}{2}\right)^2+f\left(\frac{x-y}{2}\right)^2,$$

which is called the approximate remainder of the functional equation (S_{gh}) and acts as a perturbation of the equation.

The aim of this paper is to investigate the stability problem for the pexiderized sine functional equation (S_{gh}) under the condition $|DS_{gh}(x,y)| \leqslant \varepsilon$.

Applying $h = f$, $g = f$, or $h = g = f$ in the obtained results, we also obtain justly the stability for the sine functional equation (S), the generalized sine functional equations (S_{gf}), (S_{fh}), and (S_{gg}) as corollaries.

In this paper, let $(G, +)$ be a uniquely 2-divisible Abelian group, $\mathbb{C}$ the field of complex numbers, $\mathbb{R}$ the field of real numbers, and $\mathbb{N}$ the field of natural numbers. We may assume that f and g are nonzero functions and ε is a nonnegative real constant. If all results of this article be given by the Kannappan's condition $f(x+y+z) = f(x+z+y)$ in [6], we will get same results on semigroup $(G, +)$.

2. Stability of the equation (S_{gh})

We will investigate the superstability of the pexiderized sine functional equation (S_{gh}) of the sine functional equation (S).

Theorem 1. *$f, g, h : G \to \mathbb{C}$ satisfy the inequality*

$$\left|g(x)h(y)-f\left(\frac{x+y}{2}\right)^2+f\left(\frac{x-y}{2}\right)^2\right| \leqslant \varepsilon \tag{1}$$

for all $x, y \in G$. Then either g is bounded or h satisfies (S).

Proof. Let g be unbounded. Then we can choose a sequence $\{x_n\}$ in G such that

$$0 \neq \left|g(2x_n)\right| \to \infty, \quad \text{as } n \to \infty. \tag{2}$$

Inequality (1) may equivalently be written as

$$\left|g(2x)h(2y) - f(x+y)^2 + f(x-y)^2\right| \leqslant \varepsilon \quad \forall x, y \in G. \tag{3}$$

Taking $x = x_n$ in (3) we obtain

$$\left|h(2y) - \frac{f(x_n+y)^2 - f(x_n-y)^2}{g(2x_n)}\right| \leqslant \frac{\varepsilon}{|g(2x_n)|},$$

that is, using (2)

$$h(2y) = \lim_{n\to\infty} \frac{f(x_n+y)^2 - f(x_n-y)^2}{g(2x_n)} \quad \forall x \in G. \tag{4}$$

Using (1) we have

$$\begin{aligned}
2\varepsilon &\geqslant \left|g(2x_n+x)h(y) - f\left(x_n + \frac{x+y}{2}\right)^2 + f\left(x_n + \frac{x-y}{2}\right)^2\right| \\
&\quad + \left|g(2x_n-x)h(y) - f\left(x_n + \frac{-x+y}{2}\right)^2 + f\left(x_n - \frac{x+y}{2}\right)^2\right| \\
&= \left|\big(g(2x_n+x) + g(2x_n-x)\big)h(y) - \left(f\left(x_n + \frac{x+y}{2}\right)^2 - f\left(x_n - \frac{x+y}{2}\right)^2\right)\right. \\
&\quad \left. + \left(f\left(x_n + \frac{x-y}{2}\right)^2 - f\left(x_n - \frac{x-y}{2}\right)^2\right)\right|
\end{aligned}$$

for all $x, y \in G$ and every $n \in \mathbb{N}$. Consequently, we get

$$\begin{aligned}
\frac{2\varepsilon}{|g(2x_n)|} \geqslant &\left|\frac{g(2x_n+x) + g(2x_n-x)}{g(2x_n)}h(y) - \frac{f(x_n + \frac{x+y}{2})^2 - f(x_n - \frac{x+y}{2})^2}{g(2x_n)}\right. \\
&\left. + \frac{f(x_n + \frac{x-y}{2})^2 - f(x_n - \frac{x-y}{2})^2}{g(2x_n)}\right|
\end{aligned}$$

for all $x, y \in G$ and every $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ with the use of (2) and (4), we conclude that, for every $x \in G$, there exists the limit

$$k(x) := \lim_{n\to\infty} \frac{g(2x_n+x) + g(2x_n-x)}{g(2x_n)},$$

where the obtained function $k : G \to \mathbb{C}$ satisfies the equation

$$h(x+y) - h(x-y) = k(x)h(y) \quad \forall x, y \in G. \tag{5}$$

From the definition of k, we get the equality $k(0) = 2$, which jointly with (5) implies that h has to be odd. Keeping this in mind, by means of (5), we infer the equality

$$\begin{aligned}
h(x+y)^2 - h(x-y)^2 &= \big[h(x+y) + h(x-y)\big]\big[h(x+y) - h(x-y)\big] \\
&= \big[h(x+y) + h(x-y)\big]k(x)h(y) \\
&= \big[h(2x+y) + h(2x-y)\big]h(y) \\
&= \big[h(y+2x) - h(y-2x)\big]h(y) \\
&= k(y)h(2x)h(y).
\end{aligned}$$

The oddness of h forces it to vanish at 0. Putting $x = y$ in (5) we conclude with the above result that

$$h(2y) = k(y)h(y) \quad \forall y \in G.$$

This, in return, leads to the equation

$$h(x + y)^2 - h(x - y)^2 = h(2x)h(2y) \tag{6}$$

valid for all $x, y \in G$ which, in the light of the unique 2-divisibility of G, states nothing else but (S). □

Theorem 2. *Suppose that $f, g, h : G \to \mathbb{C}$ satisfy the stability inequality*

$$\left| g(x)h(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x-y}{2}\right)^2 \right| \leqslant \varepsilon,$$

which satisfies one of the cases $g(0) = 0$, $f(x)^2 = f(-x)^2$. Then either h is bounded or g satisfies (S)*.*

Proof. Let h be unbounded. Then we can choose a sequence $\{y_n\}$ in G such that $h(2y_n)| \to \infty$ as $n \to \infty$. An obvious slight change in the proof steps applied in the start of Theorem 1 gives us

$$g(2x) = \lim_{n\to\infty} \frac{f(x + y_n)^2 - f(x - y_n)^2}{h(2y_n)} \quad \forall x \in G, \tag{7}$$

and, with an applying of (7), allows one to state the existence of a limit function

$$p(y) := \lim_{n\to\infty} \frac{h(y + 2y_n) + h(-y + 2y_n)}{h(2y_n)},$$

where $p : G \to \mathbb{C}$ obtained in that way has to satisfy the equation

$$g(x)p(y) = g(x + y) + g(x - y) \quad \forall x, y \in G.$$

From the definition of p, it yields an even function. Clearly, so is also the function $\tilde{p} := \frac{1}{2}p$. Moreover, $\tilde{p}(0) = \frac{1}{2}p(0) = 1$ and

$$g(x + y) + g(x - y) = 2g(x)\tilde{p}(y) \quad \forall x, y \in G. \tag{8}$$

First, let us consider the case $g(0) = 0$, then it forces by (8) that g is odd. Putting $y = x$ in (8), we get a duplication formula

$$g(2x) = 2g(x)\tilde{p}(x).$$

From (8), the oddness and the duplication of g, we obtain the equation

$$\begin{aligned} g(x + y)^2 - g(x - y)^2 &= 2g(x)\tilde{p}(y)\big[g(x + y) - g(x - y)\big] \\ &= g(x)\big[g(x + 2y) - g(x - 2y)\big] \\ &= 2g(x)g(2y)\tilde{p}(x) \\ &= g(2x)g(2y), \end{aligned}$$

that holds true for all $x, y \in G$, which, in the light of the unique 2-divisibility of G, states nothing else but (S).

In the next case $f(x)^2 = f(-x)^2$, it is enough to show that $g(0) = 0$. Suppose that this is not the case. Then in what follows, without loss of generality, we may assume that $g(0) = 1$.

Putting $x = 0$ in (3) which is equivalent to stability inequality (1), and from the above assumption, a given condition and the 2-divisibility of group G, we obtain the inequality

$$\left|h(y)\right| \leqslant \varepsilon \quad \forall y \in G.$$

This inequality means that h is globally bounded—a contradiction. Thus the claimed $g(0) = 0$ holds, so the proof of theorem is completed. □

Replacing h by f in Theorems 1 and 2, respectively, we obtain the superstability for the generalized functional equation (S_{gf}) of (S) as corollaries.

Corollary 1. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left|g(x)f(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x-y}{2}\right)^2\right| \leqslant \varepsilon \tag{9}$$

for all $x, y \in G$. Then either g is bounded or f and g satisfy (S).

Proof. The case f is trivial from the Theorem 1.

Next, by showing $g = f$, we will prove that g also satisfies (S).

If f is bounded, choose $y_0 \in G$ such that $f(2y_0) \neq 0$, then by (9), we have

$$\left|g(2x)\right| - \left|\frac{f(x+y_0)^2 - f(x-y_0)^2}{f(2y_0)}\right| \leqslant \left|\frac{f(x+y_0)^2 - f(x-y_0)^2}{f(2y_0)} - g(2x)\right| \leqslant \frac{\varepsilon}{|f(2y_0)|},$$

which shows that g is also bounded on G.

Since the unbounded assumption of g implies that f also is unbounded, we can choose a sequence $\{y_n\}$ such that $0 \neq |f(2y_n)| \to \infty$ as $n \to \infty$.

A slight change applied after (2) gives us

$$g(2x) = \lim_{n\to\infty} \frac{f(x+y_n)^2 - f(x-y_n)^2}{f(2y_n)} \quad \forall x \in G. \tag{10}$$

Since we have shown in (i) that f satisfies (S) whenever g is unbounded, Eq. (10) is represented as

$$g(2x) = f(2x) \quad \forall x \in G.$$

By the 2-divisibility of group G, we obtain $g = f$. Therefore g also satisfies (S). □

Corollary 2. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left|g(x)f(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x-y}{2}\right)^2\right| \leqslant \varepsilon \quad \forall x, y \in G,$$

which satisfies one of the cases $g(0) = 0$, $f(x)^2 = f(-x)^2$. Then either f is bounded or g satisfies (S).

Secondly, substitute g by f in Theorems 1 and 2, respectively, then we obtain the superstability for the generalized functional equation (S_{fh}) of (S) as corollaries.

Corollary 3. *Suppose that $f, h : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f(x)h(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x-y}{2}\right)^2 \right| \leqslant \varepsilon \quad \forall x, y \in G.$$

Then either f is bounded or h satisfies (S).

Corollary 4. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f(x)h(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x-y}{2}\right)^2 \right| \leqslant \varepsilon \quad \forall x, y \in G.$$

Then:

(i) *under the one of the cases $f(0) = 0$ and $f(x)^2 = f(-x)^2$, either h is bounded or f satisfies* (S);
(ii) *either h is bounded or h satisfies* (S).

Proof. The case (i) is trivial from Theorem 2.

(ii) As the proof of Corollary 1, we can see easily that h is bounded whenever f is bounded. Namely, the unboundedness of h implies that of f. Hence it is completed by Corollary 3 that h satisfies (S). □

Thirdly, replacing h by g in Theorem 1 or Theorem 2, we obtain the superstability for the generalized functional equation (S_{gg}) of (S) as corollaries.

Corollary 5. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| g(x)g(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x-y}{2}\right)^2 \right| \leqslant \varepsilon \quad \forall x, y \in G.$$

Then either g is bounded or g satisfies (S).

Lastly, consider the case $g = f$ in Corollary 5, namely, put $g = h = f$ in Theorems 1 and 2. Then we obtain the superstability of the sine functional equation (S) which is found in Cholewa [5].

Corollary 6. [5, Theorem] *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f(x)f(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x-y}{2}\right)^2 \right| \leqslant \varepsilon$$

for all $x, y \in G$. Then either f is bounded or f satisfies (S).

3. Application to the Banach algebra

Let us consider the functions into the Banach algebra, then we can obtain the same results as in Section 2 for each functional equations (S_{gh}), (S_{gf}), (S_{fh}), (S_{gg}), and (S). To simplify, we will combine two theorems into one.

Theorem 3. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g, h : G \to E$ satisfy the inequality*

$$\left\| g(x)h(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x-y}{2}\right)^2 \right\| \leqslant \varepsilon \quad \forall x, y \in G. \tag{11}$$

For an arbitrary linear multiplicative functional $x^* \in E^*$,

(i) *if the superposition $x^* \circ g$ fails to be bounded, then h satisfies* (S),
(ii) *if the superposition $x^* \circ h$ under the case $g(0) = 0$ or $f(x)^2 = f(-x)^2$ fails to be bounded, then g satisfies* (S).

Proof. The proofs of each case are very similar, so it suffices to show the proof of (i). Assume (i) and fix arbitrarily a linear multiplicative functional $x^* \in E^*$. As well known we have $\|x^*\| = 1$ hence, for every $x, y \in G$, we have

$$\begin{aligned}
\varepsilon &\geqslant \left\| g(x)h(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x-y}{2}\right)^2 \right\| \\
&= \sup_{\|y^*\|=1} \left| y^*\left(g(x)h(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x-y}{2}\right)^2 \right) \right| \\
&\geqslant \left| x^*\big(g(x)\big) \cdot x^*\big(h(y)\big) - x^*\left(f\left(\frac{x+y}{2}\right)^2 \right) + x^*\left(f\left(\frac{x-y}{2}\right)^2 \right) \right|,
\end{aligned}$$

which states that the superpositions $x^* \circ g$ and $x^* \circ h$ yield a solution of stability inequality (1) of Theorem 1. Since, by assumption, the superposition $x^* \circ g$ is unbounded an appeal to Theorem 1 shows that the superposition $x^* \circ h$ solves Eq. (S). In other words, bearing the linear multiplicativity of x^* in mind, for all $x, y \in G$, the sine difference $S(x, y)$ for the function h falls into the kernel of x^*. Therefore, in view of the unrestricted choice of x^*, we infer that

$$S(x, y) \in \bigcap \left\{ \ker x^* \colon\ x^* \text{ is a multiplicative member of } E^* \right\}$$

for all $x, y \in G$. Since the algebra E has been assumed to be semisimple, the last term of the above formula coincides with the singleton $\{0\}$, i.e.

$$h(x)h(y) - h\left(\frac{x+y}{2}\right)^2 + h\left(\frac{x-y}{2}\right)^2 = 0 \quad \text{for all } x, y \in G,$$

as claimed. The case (ii) runs the same procedure. □

As in Section 2, let us consider the each case $h = f$, $g = f$, $h = g$, $g = h = f$ in the inequality (11) of Theorem 3, respectively, then we can obtain the same results as in Section 2 for each functional equation.

Corollary 7. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : G \to E$ satisfy the inequality*

$$\left\| g(x)f(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x-y}{2}\right)^2 \right\| \leqslant \varepsilon \quad \forall x, y \in G.$$

For an arbitrary linear multiplicative functional $x^* \in E^*$,

(i) *if the superposition $x^* \circ g$ fails to be bounded, then f satisfies* (S),
(ii) *if the superposition $x^* \circ f$ under the case $g(0) = 0$ or $f(x)^2 = f(-x)^2$ fails to be bounded, then g satisfies* (S).

Corollary 8. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, h : G \to E$ satisfy the inequality*

$$\left\| f(x)h(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x-y}{2}\right)^2 \right\| \leqslant \varepsilon \quad \forall x, y \in G.$$

For an arbitrary linear multiplicative functional $x^* \in E^*$,

(i) *if the superposition $x^* \circ f$ fails to be bounded, then h satisfies* (S),
(ii) *if the superposition $x^* \circ h$ under the case $f(0) = 0$ or $f(x)^2 = f(-x)^2$ fails to be bounded, then f and h satisfy* (S).

Proof. The case (i) and f of (ii) are trivial from Theorem 3.
For the case h, it follows from (ii) of Corollary 4 that h satisfies (S). □

Corollary 9. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : G \to E$ satisfy the inequality*

$$\left\| g(x)g(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x-y}{2}\right)^2 \right\| \leqslant \varepsilon \quad \forall x, y \in G.$$

Then, for an arbitrary linear multiplicative functional $x^* \in E^*$, either the superposition $x^* \circ g$ is bounded or g satisfies (S).

Corollary 10. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f : G \to E$ satisfies the inequality*

$$\left\| f(x)f(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x-y}{2}\right)^2 \right\| \leqslant \varepsilon \quad \forall x, y \in G.$$

Then, for an arbitrary linear multiplicative functional $x^* \in E^*$, either the superposition $x^* \circ f$ is bounded or f satisfies (S).

Remark. If the operation of a group G is multiplication, then the investigated equations (S_{gh}), (S_{gf}), (S_{fh}), (S_{gg}), and (S) are represented by

$$g(2x)h(2y) = f(xy)^2 - f\left(xy^{-1}\right)^2,$$
$$g(2x)f(2y) = f(xy)^2 - f\left(xy^{-1}\right)^2,$$

$$f(2x)h(2y) = f(xy)^2 - f\big(xy^{-1}\big)^2,$$
$$g(2x)g(2y) = f(xy)^2 - f\big(xy^{-1}\big)^2,$$
$$f(2x)f(2y) = f(xy)^2 - f\big(xy^{-1}\big)^2.$$

Hence we can obtain the superstability for the above functional equations on a group $(G, \cdot)$ with multiplication.

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