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# Amenable Orders on Orthodox Semigroups*

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## INTRODUCTION

In his investigation into compatible orders definable on an inverse semigroup $S$, McAlister [4] introduced the notion of a *cone* as a full subsemigroup $Q$ of the subsemigroup

$$R(E) = \{x \in S; (\forall e \in E(S))\ exe = xe\}$$

with the properties $Q \cap Q^{-1} = E(S)$ and $xQx^{-1} \subseteq Q$ for all $x \in S$. He then established a bijection between the set of cones in $S$ and the set of compatible orders $\leq$ on $S$ that extend the natural order $\preceq$ on the idempotents and are *left amenable* in the sense that

$$x \leq y \Rightarrow x^{-1}x \leq y^{-1}y.$$

The one-sided nature of these orders suggests that they have some significance in a more general setting, and the objective of the present paper is to highlight the rôle that they play in right inverse (or $\mathscr{L}$-unipotent) semigroups. To achieve this, we require to extend the notion of a cone. This we do in such a semigroup $S$ by considering a multiplicative inverse transversal $S^0$ of $S$ and effectively taking inverses in $S^0$. We show that every cone relative to a multiplicative inverse transversal defines a left-amenable order on $S$, and that every such order is obtained from a cone in this way. In this situation we can define a closure $\vartheta\colon Q \mapsto \hat{Q}$ on

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the set of cones that has the property that $\vartheta$-equivalent cones define the same left-amenable order. The locally maximal cones (those such that $Q = \hat{Q}$) are shown to be independent of the inverse transversal chosen, as are the orders that they define. We then prove that the set of locally maximal cones forms an $\cap$-semilattice LMC with a smallest element, namely $E$, and establish an order-preserving bijection from the set LMC to the set of left-amenable orders definable on $S$. Finally, we show the usefulness of the results by applying them, together with their duals, to a symmetric situation. Specifically, using a deep constructive result of McAlister and McFadden [6], we prove that if $S$ if an orthodox semigroup that has a multiplicative inverse transversal then $S$ can be amenably ordered. We also give a description of all the amenable orders definable on $S$.

## 1. Inverse Transversals of Right Inverse Semigroups

An *inverse transversal* of a regular semigroup $S$ is an inverse subsemigroup $T$ with the property that $|T \cap V(x)| = 1$ for every $x \in S$, where $V(x)$ denotes the set of inverses of $x \in S$. In what follows we shall write the unique element of $T \cap V(x)$ as $x^0$, and $T$ as $S^0 = \{x^0;\ x \in S\}$. Then in $S^0$ we have $(x^0)^{-1} = x^{00}$. If $E(S^0)$ is the semilattice of idempotents of $S^0$ then the inverse transversal $S^0$ is said to be *multiplicative* if $x^0xyy^0 \in E(S^0)$ for all $x, y \in S$; and *weakly multiplicative* if $(x^0xyy^0)^0 \in E(S^0)$ for all $x, y \in S$. The complicated structure of regular semigroups with inverse transversals of these types has been determined by Saito [7]. In particular, he has established the following important facts that we shall use without mention:

(1) $S$ is orthodox if and only if $(xy)^0 = y^0x^0$ for all $x, y \in S$, in which case $e^0 \in E$ for every $e \in E$;

(2) if $S$ is orthodox then all inverse transversals are weakly multiplicative.

A *right inverse semigroup* is an orthodox semigroup in which every $\mathscr{L}$-class contains a single idempotent (another name for this is an $\mathscr{L}$-*unipotent* semigroup). Equivalently, a right inverse semigroup is a regular semigroup $S$ in which the set $E$ of idempotents forms a right regular band, in the sense that

$$(\forall e, f \in E) \qquad efe = fe.$$

The following examples illustrate situations where a right inverse semigroup (a) has no inverse transversal; (b) has a weakly multiplicative inverse transversal that is not multiplicative; (c) has a multiplicative inverse transversal. We shall see later that in case (c) all inverse transversals are necessarily multiplicative and precisely how this situation is controlled by the band of idempotents.

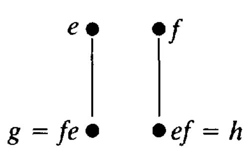
EXAMPLE 1. In order to give an example of a right inverse semigroup that has no inverse transversal, we recall that the natural order on the set $E$ of idempotents is given by

$$e \preceq f \Leftrightarrow e = ef = fe.$$

It is clear from the identity $efe = fe$ that in a right inverse semigroup we therefore have the relation $fe \preceq e$, so consider the semigroup

$$S = \langle e, f \mid e^2 = e,\ f^2 = f,\ efe = fe,\ fef = ef \rangle.$$

This can be depicted as follows, the order in the Hasse diagram being the natural order:




| | $e$ | $g$ | $f$ | $h$ |
|---|---|---|---|---|
| $e$ | $e$ | $g$ | $h$ | $h$ |
| $g$ | $g$ | $g$ | $h$ | $h$ |
| $f$ | $g$ | $g$ | $f$ | $h$ |
| $h$ | $g$ | $g$ | $h$ | $h$ |

It is readily seen that $S$ is right inverse with

$$V(e) = \{e\}, \qquad V(f) = \{f\}, \qquad V(g) = \{g, h\} = V(h).$$

Now any inverse transversal must contain $e$ and $f$, so it must also contain $fe = g$ and $ef = h$, a contradiction; so $S$ has no inverse transversals.

EXAMPLE 2. For an example of a right inverse semigroup having an inverse transversal that is weakly multiplicative without being multiplicative, consider the set

$$A = \left\{ \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix};\ x, y \in \mathbb{R},\ x \neq 0 \right\}$$

and let $S = A \cup \{I_2\}$, where $I_2$ is the identity matrix. Then it is readily seen that $S$ is a regular semigroup with

$$V(I_2) = \{I_2\}, \qquad V\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} = \left\{ \begin{bmatrix} x^{-1} & z \\ 0 & 0 \end{bmatrix};\ z \in \mathbb{R} \right\}.$$

The set of idempotents of $S$ is

$$E = \{I_2\} \cup \left\{\begin{bmatrix} 1 & y \\ 0 & 0 \end{bmatrix}; y \in \mathbb{R}\right\}.$$

A simple calculation reveals that $S$ is right inverse. Now let

$$A^0 = \left\{\begin{bmatrix} x & x \\ 0 & 0 \end{bmatrix}; x \neq 0\right\}$$

and consider the subset $S^0 = \{I_2\} \cup A^0$. Further elementary calculations show that $S^0$ is an inverse semigroup with

$$S^0 \cap V(I_2) = \{I_2\}, \qquad S^0 \cap V\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} = \left\{\begin{bmatrix} x^{-1} & x^{-1} \\ 0 & 0 \end{bmatrix}\right\}.$$

Thus $S^0$ is an inverse transversal of $S$ and we can write

$$I_2^0 = I_2, \qquad \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}^0 = \begin{bmatrix} x^{-1} & x^{-1} \\ 0 & 0 \end{bmatrix}.$$

Then, for $X = \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}$ with $x \neq y$, we have

$$X^0 X I_2 I_2^0 = X^0 X = \begin{bmatrix} 1 & x^{-1}y \\ 0 & 0 \end{bmatrix} \notin E(S^0).$$

Thus the inverse transversal $S^0$, which is weakly multiplicative since $S$ is orthodox, is not multiplicative.

Example 3. For an example of a right inverse semigroup having a multiplicative inverse transversal, consider the cartesian product semigroup $S = T \times R \times G$, where $T$ is a semilattice, $R$ is a right zero semigroup, and $G$ is a group. The multiplication is given by

$$(x, a, g)(y, b, h) = (xy, b, gh).$$

It is easily seen that $S$ is regular with $V(x, a, g) = \{(x, b, g^{-1}); b \in R\}$. Moreover, $E = \{(x, a, 1_G); x \in T, a \in R\}$ which is a right regular band, so $S$ is right inverse. For every $(x, a, g) \in S$ define

$$(x, a, g)^0 = (x, \alpha, g^{-1}),$$

where $\alpha$ is a fixed element of $R$. Then $S^0$ is an inverse transversal of $S$. Since $(x, a, g)^0(x, a, g)(y, b, h)(y, b, h)^0 = (x, a, 1_G)(y, \alpha, 1_G) = (xy, \alpha, 1_G) \in E(S^0)$ it follows that $S^0$ is multiplicative.

Of particular importance relative to a given inverse transversal $S^0$ are the sets

$$I(S^0) = \{e \in E; e = ee^0\}, \qquad \Lambda(S^0) = \{f \in E; f = f^0 f\}$$

which we shall write simply as $I$ and $\Lambda$.

THEOREM 1. *If $S$ is a right inverse semigroup with inverse transversal $S^0$ then*

$$I = I^0 = \{e \in E; e = e^0\} = E(S^0), \qquad \Lambda = E.$$

*Proof.* If $e \in I$ then, since $e^0 \in E$, we have $e = ee^0 = e^0 ee^0 = e^0$ so that $I = \{e \in E; e = e^0\} = I^0 \subseteq E(S^0)$. But if $e \in E(S^0)$ then

$$e \in V(e) \cap S^0 = \{e^0\}$$

and, consequently, $e = e^0$, so that $E(S^0) \subseteq I$.

That $\Lambda = E$ follows from the observation that if $e \in E$ then we have $e = ee^0 e = e^0 e$ and so $e \in \Lambda$. ▌

If $S$ is a right inverse semigroup then the natural order $\preceq$ on the band $E$ of idempotents is right compatible with multiplication; for if $e \preceq f$ then from $ef = e = fe$ we obtain, for every $g \in E$,

$$egfg = efg = eg = feg = fgeg$$

and, therefore, $eg \preceq fg$.

THEOREM 2. *Let $S$ be a right inverse semigroup that admits an inverse transversal $S^0$. Then the following statements are equivalent*:

(1) *$S^0$ is multiplicative*;

(2) *$E$ is right normal* $[(\forall e, f, g \in E)\ efg = feg]$;

(3) *$\preceq$ is left compatible*;

(4) *every inverse transversal is multiplicative.*

*Proof.* (1) $\Rightarrow$ (2) If $S^0$ is multiplicative then by Saito [7] and Theorem 1 it follows that $E = \Lambda$ is a right normal band.

(2) $\Rightarrow$ (3) If $e \preceq f$ then $e = ef = fe$ and so, for all $g \in E$, it follows by (2) that

$$ge = gef = egf = gegf, \qquad ge = gfe = fge = gfge$$

and therefore $ge \preceq gf$.

(3) $\Rightarrow$ (4) Let $S^0$ be an inverse transversal of $S$. We have to show that $x^0 xyy^0 \in E(S^0)$ for all $x, y \in S$. Since, by Theorem 1, we have $E(S^0) = I$

and $x^0xyy^0 \in \Lambda I$, it suffices to show that $\Lambda I \subseteq I$. Suppose then that $e \in \Lambda$ and $f \in I$, so that $e = e^0e$ and $f = f^0$. Since $ef \preceq f$ it follows by (3) that $ef = e^0ef \preceq e^0f$. Consequently,

$$ef = efe^0f = ee^0f = e^0ee^0f = e^0f.$$

Since $e^0 \in I$ and $f \in I$ we therefore have $ef \in I$ and hence $\Lambda I \subseteq I$.

(4) $\Rightarrow$ (1) This is clear. ∎

## 2. Cones Associated with Inverse Transversals

Let $S$ be a right inverse semigroup. Consider the subset

$$R(E) = \{x \in S; (\forall e \in E)\ exe = xe\}.$$

We have $R(E) \neq \varnothing$ since, $S$ being right inverse, $E \subseteq R(E)$. Moreover, $R(E)$ is a subsemigroup of $S$; for if $x, y \in R(E)$ then

$$xye = xeye = exeye = exye.$$

If $S^0$ is an inverse transversal of $S$ then by an *$S^0$-cone* we shall mean a nonempty subset $Q$ of $S$ such that

(C1) $Q$ is a subsemigroup of $R(E)$;

(C2) $Q \cap Q^0 = E(S^0)$;

(C3) $(\forall x \in S)\ xQx^0 \subseteq Q$.

This notion of a cone generalises that of McAlister [4] for an inverse semigroup (and reduces to his on taking $S$ to be inverse and $S^0 = S$).

Theorem 3. *If $S$ is right inverse then $E$ is an $S^0$-cone for every inverse transversal $S^0$.*

*Proof.* Since $R(E)$ is a full subsemigroup, (C1) holds with $Q = E$. As for (C2), since $S$ is orthodox we have $e^0 \in E$ for every $e \in E$, so $E^0 \subseteq E(S^0) \subseteq E$. But if $x \in E(S^0)$ then $x = x^{00}$ and $x^0 \in E$, so $x \in E^0$ and hence $E(S^0) \subseteq E^0$. Thus $E^0 = E(S^0) \subseteq E$ and (C2) follows. As for (C3), this is an immediate consequence of the fact that in an orthodox semigroup every conjugate $xex'$ of an idempotent is idempotent. ∎

Theorem 4. *If $S$ is right inverse then $E(S^0)$ is the smallest $S^0$-cone of $S$.*

*Proof.* For every cone $Q$ we have $I = E(S^0) \subseteq Q$, so it suffices to show that $I$ is a cone. For this purpose, we observe that $I \subseteq E \subseteq R(E)$ so that (C1) holds; and that $I = I^0$ so that $I \cap I^0 = I = E(S^0)$ and (C2) holds. As for (C3), we note that for all $e \in E$ we have

$$\begin{aligned} xex^0 &= xx^0x^{00}x^0xex^0xx^0 \\ &= xx^0x^{00}ex^0xx^0 && \text{since } e \in E \subseteq R(E) \\ &= x^{00}ex^0xx^0 && \text{since } x^{00}ex^0 \in E \subseteq R(E) \\ &= x^{00}ex^0. \end{aligned}$$

Consequently, if $e \in I$ then $xex^0 = x^{00}e^0x^0 = (xex^0)^0$ and so $xex^0 \in I$. Thus $xIx^0 \subseteq I$ and (C3) holds. ▮

COROLLARY 1. $(\forall x \in S)\ x = x^{00}x^0x,\ x^{00} = xx^0x^{00}$.

*Proof.* Since $xx^0 = xx^0x^{00}x^0 = xx^0(xx^0)^0$ we have $xx^0 \in I$ whence $xx^0 = (xx^0)^0 = x^{00}x^0$, from which the result follows immediately. ▮

COROLLARY 2. *If $Q$ is an $S^0$-cone then $x \in Q \Rightarrow x^{00} \in Q$.*

*Proof.* If $x \in Q$ then by (C3) we have $x^0x^{00} \cdot x \cdot (x^0x^{00})^0 \in Q$. But

$$\begin{aligned} x^0x^{00} \cdot x \cdot (x^0x^{00})^0 &= x^0x^{00}xx^0x^{00} \\ &= xx^0x^{00} && \text{since } x \in Q \subseteq R(E) \\ &= x^{00} && \text{by Corollary 1.} \end{aligned}$$ ▮

THEOREM 5. *If $S$ is right inverse and $Q$ is an $S^0$-cone of $S$ then so is $\hat{Q} = \langle Q \cup E \rangle$, the subsemigroup generated by $Q \cup E$.*

*Proof.* Since $Q, E \subseteq R(E)$ it follows that $\hat{Q} \subseteq R(E)$ and so (C1) holds. As for (C2), it is clear that

$$\hat{Q} \cap \hat{Q}^0 = \langle Q \cup E \rangle \cap \langle Q \cup E \rangle^0 \supseteq E(S^0).$$

To establish the reverse inclusion, let $x \in \hat{Q} \cap \hat{Q}^0$. Then we have

$$x = q_1e_1 \dots q_ne_n = f_m^0p_m^0 \dots f_1^0p_1^0,$$

where $e_i, f_j \in E$ and $q_i, p_j \in Q$. Since each $f_j^0 \in E(S^0) = Q \cap Q^0$ we have

$$x = f_m^0p_m^0 \dots f_1^0p_1^0 \in Q^0$$

and therefore $x = q^0$ for some $q \in Q$. Then $x = x^{00}$ and by Corollary 2 of Theorem 4 we have $x^0 = q^{00} \in Q$. But from $x = q_1e_1 \dots q_ne_n$ we have, as before,

$$x^0 = e_n^0q_n^0 \dots e_1^0q_1^0 \in Q^0.$$

Hence $x^0 \in Q \cap Q^0 = E(S^0)$ and so $x = x^{00} \in E(S^0)$. Thus we see that $\hat{Q} \cap \hat{Q}^0 \subseteq E(S^0)$ and (C2) follows.

As for (C3), if $y = q_1 e_1 \dots q_n e_n \in \hat{Q}$ then

$$\begin{aligned} xyx^0 &= xq_1 e_1 \dots q_n e_n x^0 \\ &= xq_1 x^0 xe_1 x^0 x \dots x^0 xq_n x^0 xe_n x^0 \qquad \text{since } q_i, e_i \in R(E) \\ &\in \langle Q \cup E \rangle = \hat{Q} \qquad \text{since } xq_i x^0 \in Q,\ xe_i x^0 \in E. \quad \blacksquare \end{aligned}$$

EXAMPLE 4. Let $k > 1$ be a fixed positive integer and for every $x \in \mathbb{Z}$ let $x_k$ be the biggest multiple of $k$ that is less than or equal to $x$. Let $S = \mathbb{Z} \times \mathbb{Z}$, together with the multiplication

$$(x, a)(y, b) = (x \wedge y, a_k + b).$$

Then $S$ is a regular semigroup; we have

$$V(x, a) = \{(x, b); b_k = -a_k\}.$$

The set of idempotents of $S$ is

$$E = \{(x, a); a_k = 0\}$$

which is readily seen to be a right normal band, so $S$ is right inverse. For every $(x, a) \in S$ define

$$(x, a)^0 = (x, -a_k).$$

Then it is easy to see that $S^0 = \{(x, a)^0; (x, a) \in S\}$ is an inverse transversal. It follows by Theorem 2 that $S^0$ is multiplicative.

A simple calculation reveals that here we have $R(E) = S$. The following subsets are then easily seen to be examples of $S^0$-cones:

(1) $Q_1 = \{(x, a); a \geq 0\}$;

(2) $Q_2 = \{(x, a_k); a_k \geq 0\}$;

(3) $Q_3 = \{(x, a); x \leq 0, a \geq 0\} \cup \{(x, 0); x \in \mathbb{Z}\}$.

Recalling Theorem 5 above, we can see that

$$\widehat{Q_1} = Q_1 = \widehat{Q_2}, \qquad \widehat{Q_3} = Q_3 \cup E.$$

## 3. Orders Defined by Cones

For a given $S^0$-cone $Q$ consider now the relation $\leq_Q$ defined on $S$ by

$$x \leq_Q y \Leftrightarrow x^0 x \preceq y^0 y, \quad yx^0 \in Q.$$

Theorem 6. *Let $S$ be a right inverse semigroup with an inverse transversal $S^0$. For every $S^0$-cone $Q$ the relation $\leq_Q$ is an order that is compatible on the right with multiplication and coincides with $\preceq$ on $E$. A necessary and sufficient condition for $\leq_Q$ to be compatible on the left with multiplication is that the inverse transversal $S^0$ be multiplicative.*

*Proof.* Since $xx^0 \in I$ and $I \subseteq Q$ by Theorem 4, it is clear that $\leq_Q$ is reflexive. Suppose now that $x \leq_Q y$ and $y \leq_Q x$. Then we have $x^0 x = y^0 y$ (whence $x = xy^0 y$) and $yx^0, xy^0 \in Q$. Consequently, by Corollary 1 of Theorem 4,

$$yx^0 = y(xy^0y)^0 = yy^0y^{00}x^0 = y^{00}x^0 = (xy^0)^0.$$

It now follows that $yx^0 \in Q \cap Q^0 = E(S^0)$. Then $x^0 = x^0xx^0 = y^0yx^0$ gives $x^0 \preceq y^0$ in the inverse semigroup $S^0$. Similarly, we have $y^0 \preceq x^0$, whence $x^0 = y^0$. Consequently,

$$x = x^{00}x^0x = y^{00}y^0y = y$$

and therefore $\leq_Q$ is anti-symmetric.

To show that $\leq_Q$ is transitive, suppose that $x \leq_Q y$ and $y \leq_Q z$. Then $x^0x \preceq y^0y \preceq z^0z$; $yx^0, zy^0 \in Q$ give

$$zx^0 = zx^0xx^0 = zy^0yx^0xx^0 = zy^0 \cdot yx^0 \in Q.$$

Hence $x \leq_Q z$, and so $\leq_Q$ is an order on $S$.

If now $x \leq_Q y$ then for every $z \in S$ we have

$$(xz)^0xz = z^0x^0xz = z^0y^0yx^0xy^0yz \preceq z^0y^0yz = (yz)^0yz$$

and

$$\begin{aligned} yz(xz)^0 &= yzz^0x^0 = yzz^0x^0xx^0 \\ &= yx^0xzz^0x^0xx^0 \qquad \text{since } zz^0 \in R(E) \\ &= yx^0 \cdot xzz^0x^0, \end{aligned}$$

so $yz(xz)^0 \in Q$ since $yx^0 \in Q$, and $zz^0 \in I \subseteq Q$ gives $xzz^0x^0 \in Q$. Hence $xz \leq_Q yz$ and so $\leq_Q$ is compatible on the right.

To see that $\leq_Q$ coincides with $\preceq$ on $E$, recall from Theorem 1 that $e^0e = e$ for every $e \in E$ and observe that if $e \preceq f$ then

$$fe^0 = f(ef)^0 = ff^0e^0 = f^{00}f^0ff^0e^0 = f^{00}f^0e^0 = f^0e^0 \in E(S^0) \subseteq Q.$$

Consequently, $e \leq_Q f$ if and only if $e \preceq f$.

As for compatibility on the left, suppose that $x \leq_Q y$. Then for all $z \in S$ we have

$$zy(zx)^0 = zyx^0z^0 \in zQz^0 \subseteq Q.$$

Now if the inverse transversal $S^0$ is multiplicative then $E$ is right normal by Theorem 2, so we have

$$\begin{aligned} x^0z^0zx \cdot y^0z^0zy &= x^0z^0zx \cdot y^0z^0zy \cdot y^0y \\ &= y^0z^0zy \cdot x^0z^0zx \cdot y^0y \\ &= y^0z^0zy \cdot x^0z^0zx \qquad \text{since } x^0x \preceq y^0y, \end{aligned}$$

and, since $yx^0 \in Q \subseteq R(E)$, this last product can be written as $y^0yx^0z^0zx$ which reduces to $x^0z^0zx$ since $x^0x \preceq y^0y$. We thus have

$$(zx)^0zx = x^0z^0zx \preceq y^0z^0zy = (zy)^0zy$$

so $zx \leq_Q zy$ and $\leq_Q$ is compatible on the left.

Conversely, if $\leq_Q$ is compatible on the left then so is its restriction $\preceq$ to $E$. It follows by Theorem 2 that $S^0$ is multiplicative. ∎

DEFINITION. If $S$ is a regular semigroup then by a *natural order* on $S$ we shall mean an order $\leq$ that extends the natural order $\preceq$ on the idempotents, in the sense that $e \preceq f \Rightarrow e \leq f$. When such an order $\leq$ coincides with $\preceq$ on the idempotents we shall call it a *strict natural order*.

On an inverse semigroup $S$ every compatible natural order $\leq$ is a strict natural order. In fact, if $e \leq f$ then by [2, Theorem 1.1] we have $e = efe$, whence, since the idempotents of $S$ commute, $e = ef = fe$ and so $e \preceq f$.

DEFINITION. If $S$ is a regular semigroup with an inverse transversal $S^0$ then by a *left amenable order* on $S$ we shall mean a compatible strict natural order $\leq$ with the property that

$$x \leq y \Rightarrow x^0x \preceq y^0y.$$

It follows by Theorem 6 that every $S^0$-cone $Q$ gives rise to a left amenable order $\leq_Q$ on $S$. We shall now show that every such order arises from a cone in this way.

THEOREM 7. *Let $S$ be a right inverse semigroup with an inverse transversal $S^0$. If $S$ admits a compatible natural order $\leq$ with respect to which $\mathscr{L}$ is regular, in the sense that*

$$x \leq y \Rightarrow x^0x \leq y^0y,$$

*then the subset*

$$C = \{x \in S;\ x^0x \leq x\}$$

*is an $S^0$-cone and $\leq$ extends $\leq_C$. If $\leq$ is left amenable then $\leq$ coincides with $\preceq_C$ and all inverse transversals on $S$ are multiplicative.*

*Proof.* To prove that $C$ is an $S^0$-cone, we proceed as follows:

(C1) Given $x \in C$ we have $(x, x^0x) \in \mathscr{L}$ and so, for every $e \in E$, we have $(xe, x^0xe) \in \mathscr{L}$ since $\mathscr{L}$ is a right congruence. As every $\mathscr{L}$-class contains a single idempotent, it follows that

$$(xe)^0xe = x^0xe. \qquad (*)$$

But $x^0xe \leq xe$ and so $x^0xe \leq x^0xe \cdot xe$ and therefore, since $\mathscr{L}$ is regular, we have

$$\begin{aligned} x^0xe &\leq (x^0xe \cdot xe)^0 x^0xe \cdot xe \\ &= e^0x^0e^0x^0x^{00}x^0xexe \\ &= (xe)^0e^0x^0xexe \\ &= (xe)^0e^0x^0xe^0exe \qquad \text{since } e^0e = e \\ &= (xe)^0x^0xe^0exe \qquad \text{since } x^0x \in E \subseteq R(E) \\ &= (xe)^0x^0xexe \qquad \text{since } e^0e = e \\ &\leq (xe)^0xe \qquad \text{since } \leq \text{ extends } \preceq . \end{aligned}$$

This, together with $(*)$ gives $(xe)^0xe = (xe)^0 \cdot x^0xe \cdot xe$, whence we obtain

$$xe = xe(xe)^0 \cdot x^0xe \cdot xe = x^0xe \cdot xe, \qquad (**)$$

the last equality resulting from the identity $fef = ef$. Pre-multiplying $(**)$ by $e$ and using this identity again, we obtain $exe = xe$. This then shows that $C \subseteq R(E)$. To show that $C$ is a subsemigroup, take $e = x^0x$ in $(**)$.

We obtain $x = x^0x^2$. Consequently, if $x, y \in C$ then

$$\begin{aligned}(xy)^0xy &= y^0x^0xy = y^0x^0xy^0y^2\\ &= y^0y^0yx^0xy^0y^2 \qquad \text{since } x^0x \in R(E)\\ &\le y^0yxy^0y^2 \qquad \text{since } x, y \in C\\ &= xy^0y^2 \qquad \text{since } x \in C \subseteq R(E)\\ &= xy.\end{aligned}$$

Thus $C$ is also a subsemigroup. Hence (C1) holds.

(C2) If $x \in C \cap C^0$ then we have $x^0x \le x = y^0$ with $y^0y \le y$. The first of these gives $y^{00}y^0 \le y^0$, whence $y^0 \le y^0y^0$; and the second gives $y^0 \le yy^0$, whence $y^0y^0 \le y^0$. Hence $x = y^0 \in E(S^0)$ and so $C \cap C^0 \subseteq E(S^0)$. The reverse inclusion follows from the fact that $C$ contains $E$. Hence we have (C2).

(C3) If $x \in C$ then for every $y \in S$ we have

$$\begin{aligned}(yxy^0)^0yxy^0 &= y^{00}x^0y^0yxy^0\\ &= y^{00}x^0xy^0 \qquad \text{since } x \in C \subseteq R(E)\\ &= yy^0y^{00}x^0xy^0y^{00}y^0\\ &= yx^0xy^0 \qquad \text{since } x^0x \in R(E)\\ &\le yxy^0,\end{aligned}$$

so $yxy^0 \in C$ and, therefore, $yCy^0 \subseteq C$.

Thus we see that $C$ is an $S^0$-cone. Consider now the order $\le_C$. We have

$$\begin{aligned}x \le_C y &\Rightarrow x^0x \preceq y^0y, \qquad (yx^0)^0yx^0 \le yx^0\\ &\Rightarrow x^{00}x^0 = x^{00}y^0yx^0 \le yx^0\\ &\Rightarrow x = x^{00}x^0x \le yx^0x \le yy^0y = y,\end{aligned}$$

so $\le$ extends $\le_C$.

If now $\le$ is left amenable then $x \le y$ gives $x^0x \preceq y^0y$, whence

$$(yx^0)^0yx^0 = x^{00}y^0yx^0 = x^{00}x^0 = xx^0x^{00}x^0 = xx^0 \le yx^0,$$

so that $x \le_C y$. Thus $\le$ coincides with $\le_C$. Since the restriction $\preceq$ of $\le$ to $E$ is compatible, it follows by Theorem 2 that all inverse transversals are then multiplicative. ∎

In view of Theorem 7 it is natural to consider, for a given $S^0$-cone $Q$, the set

$$C_Q = \left\{x \in S;\ x^0 x \leq_Q x\right\}.$$

Since $\leq_Q$ is left amenable, $C_Q$ is also an $S^0$-cone.

THEOREM 8. *Let $S$ be a right inverse semigroup and let $S^0$ be a multiplicative inverse transversal of $S$. If $Q$ is an $S^0$-cone of $S$ then*

$$C_Q = \{x \in S;\ x^{00} \in Q\} = \langle Q \cup E \rangle.$$

*Proof.* We have

$$x^0 x \leq_Q x \Leftrightarrow (x^0 x)^0 x^0 x \preceq x^0 x,\ x(x^0 x)^0 \in Q.$$

The first condition reduces to $x^0 x \preceq x^0 x$, and the second reduces to $x^{00} = xx^0 x^{00} \in Q$. Thus we see that $C_Q = \{x \in S;\ x^{00} \in Q\}$. It follows from this and Corollary 2 of Theorem 4 that $Q \subseteq C_Q$; clearly we have $E \subseteq C_Q$. Since $C$ is a subsemigroup, it follows that $\langle Q \cup E\rangle \subseteq C_Q$. To obtain the reverse inclusion, suppose that $x \in C_Q$. Then from the above we have $x^{00} \in Q$, whence

$$x = x^{00} \cdot x^0 x \in QE \subseteq \langle Q \cup E \rangle.$$

Thus we have $C_Q = \langle Q \cup E \rangle$. ∎

In what follows, we shall write $C_Q = \langle Q \cup E \rangle = \hat{Q}$.

THEOREM 9. *Let $S$ be a right inverse semigroup with a multiplicative inverse transversal $S^0$. If $X$ is the set of $S^0$-cones then $Q \mapsto \hat{Q}$ is a closure on $X$. Moreover,*

$$Q = \hat{Q} \Leftrightarrow E \subseteq Q,$$

*and $\leq_Q$ coincides with $\leq_{\hat{Q}}$.*

*Proof.* If $Q, R$ are $S^0$-cones with $Q \subseteq R$ then by Theorem 8 we have

$$x \in \hat{Q} \Rightarrow x^{00} \in Q \Rightarrow x^{00} \in R \Rightarrow x \in \hat{R},$$

so the mapping $Q \mapsto \hat{Q}$ is isotone on $X$. That it is idempotent is immediate from Theorem 8 and the identity $x^{000} = x^0$. Since, as we have seen above, $Q \subseteq C_Q = \hat{Q}$ it follows that $Q \mapsto \hat{Q}$ is a closure on $X$.

Since $\hat{Q} = \langle Q \cup E \rangle$ it follows immediately that $Q = \hat{Q}$ if and only if $E \subseteq Q$. Finally, that $\leq_Q$ coincides with $\leq_{\hat{Q}}$ follows from Theorem 7 on taking $\leq$ to be $\leq_Q$. ∎

COROLLARY. *If $Q_1, Q_2 \in X$ then the following statements are equivalent*:

(1) $(\forall x, y \in S)\ x \leq_{Q_1} y \Rightarrow x \leq_{Q_2} y$;

(2) $\widehat{Q_1} \subseteq \widehat{Q_2}$.

*Proof.* (1) $\Rightarrow$ (2) This follows from the fact that

$$\widehat{Q_i} = \{x \in S;\ x^0 x \leq_{Q_i} x\}.$$

(2) $\Rightarrow$ (1) If (2) holds then

$$x \leq_{\widehat{Q_1}} y \Rightarrow x^0 x \preceq y^0 y, \quad yx^0 \in \widehat{Q_1} \subseteq \widehat{Q_2} \Rightarrow x \leq_{\widehat{Q_2}} y$$

and (1) follows from the final statement in Theorem 9. ∎

## 4. LOCALLY MAXIMAL CONES

We shall say that an $S^0$-cone $Q$ is *locally maximal* if $Q = \hat{Q}$; equivalently, by Theorem 9, if $Q$ is a full subsemigroup.

THEOREM 10. *Let $S$ be a right inverse semigroup with a multiplicative inverse transversal $S^0$. If $Q$ is a locally maximal $S^0$-cone then $Q$ is a locally maximal $S^*$-cone for every multiplicative inverse transversal $S^*$ of $S$.*

*Proof.* With the obvious notation, it suffices to prove that

(1) $Q \cap Q^* = E(S^*)$;

(2) $(\forall x \in S)\ xQx^* \subseteq Q$.

To establish (1), let $x \in Q \cap Q^*$. Then $x \in Q$ and $x = y^*$ with $y \in Q$. Since every $\mathscr{L}$-class contains precisely one idempotent, we have $z^0 z = z^* z$ for every $z \in S$. Consequently $y^0 = y^0 y y^0 = y^* y y^0 \in Q \cdot I(S^0) \subseteq Q$ by Theorems 1 and 4. Thus $y^0 \in Q \cap Q^0 = E(S^0)$ and therefore $y^* = y^* y y^* = y^0 y y^* \in E$. It follows that $x \in E(S^*)$ and, hence, that $Q \cap Q^* \subseteq E(S^*)$. For the reverse inclusion, suppose that $e \in E(S^*)$. Since, by Theorems 1 and 4 (applied to $S^*$), $E(S^*) = I(S^*)$ and therefore $e = e^*$, it follows from the fact that $Q$ is locally maximal that $e \in Q \cap Q^*$, whence $E(S^*) \subseteq Q \cap Q^*$.

As for (2), for every $y \in Q$ we have, since $E \subseteq Q$,

$$xyx^* = xyx^*xx^* = xyx^0xx^* \in QE \subseteq Q. \quad ∎$$

COROLLARY. *The left amenable order* $\leq_Q$ *is independent of the multiplicative inverse transversal.*

*Proof.* Suppose that $S^0, S^*$ are multiplicative inverse transversals of $S$. Let $Q$ be a cone relative to both $S^0$ and $S^*$, and let $\leq_Q^0$, $\leq_Q^*$ be the corresponding left amenable orders. If $x \leq_Q^0 y$ then we have $x^*x = x^0x \preceq y^0y = y^*y$ and $yx^0 \in Q$. Consequently,

$$\begin{aligned} yx^* &= yx^*xx^* = yx^0xx^* \\ &= xx^*yx^0xx^* \qquad \text{since } yx^0 \in Q \subseteq R(E) \\ &= xx^*yx^0(xx^*)^* \qquad \text{since } xx^* \in I(S^*) \\ &\in Q \qquad \text{since } zQz^* \subseteq Q. \end{aligned}$$

It follows that $x \leq_Q^* y$. Similarly, we can show that

$$x \leq_Q^* y \Rightarrow x \leq_Q^0 y$$

and therefore the orders $\leq_Q^0$, $\leq_Q^*$ coincide. ∎

We can now combine the above results to obtain the following result that relates locally maximal cones with left amenable orders.

THEOREM 11. *Let $S$ be a right inverse semigroup with a multiplicative inverse transversal. Then, independently of the inverse transversal chosen, the set* LMC *of locally maximal cones is an* $\cap$*-semilattice with smallest element* $E$ *and there is an order-preserving bijection from* LMC *to the set* LAO *of left amenable orders definable on* $S$. *Consequently,* LAO *is a meet semilattice with smallest element the order* $\leq_E$ *given by*

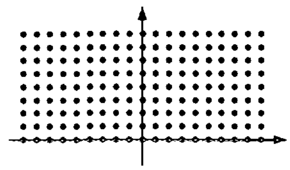
$$x \leq_E y \Leftrightarrow x^0x \preceq y^0y, \ yx^0 \in E.$$

*Proof.* That the intersection of two locally maximal cones is also a locally maximal cone results from the inclusions

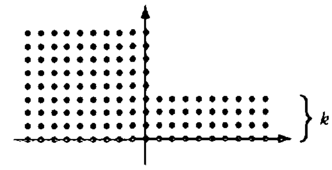
$$E(S^0) \subseteq Q_1 \cap Q_2 \cap (Q_1 \cap Q_2)^0 \subseteq Q_1 \cap Q_2 \cap Q_1^0 \cap Q_2^0 = E(S^0).$$

Thus LMC is an $\cap$-semilattice. By Theorem 10, this is independent of the inverse transversal chosen. By Theorems 3 and 9, the smallest element of LMC is $E$. The inclusion-preserving bijection from LMC to LAO results from Theorems 6, 7 and Theorem 9 and its Corollary. It is immediate that LAO is then a meet semilattice with smallest element $\leq_E$. ∎

EXAMPLE 5. By way of illustration, consider the locally maximal cones $Q_1$ and $\widehat{Q_3}$ as described in Example 4. These can be depicted as follows:



$Q_1$

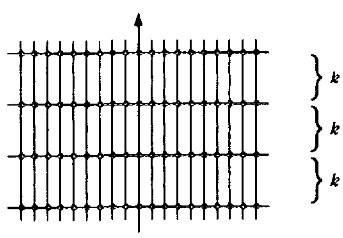



$\widehat{Q_3}$

Relative to $Q_1$ we have

$$(x, a) \leq_{Q_1} (y, b) \Leftrightarrow (x, a - a_k) \preceq (y, b - b_k), \qquad (y \wedge x, b_k - a_k) \in Q_1$$
$$\Leftrightarrow x \leq y, \qquad a - a_k = b - b_k, \qquad a_k \leq b_k.$$

The left amenable order $\leq_{Q_1}$ can therefore be depicted by $k$ disjoint Hasse diagrams of the form

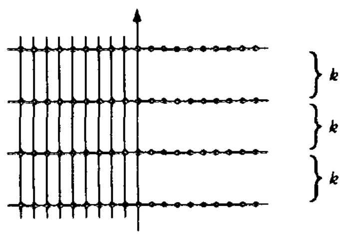



Relative to $\widehat{Q_3}$ we have

$$(x, a) \leq_{\widehat{Q_3}} (y, b) \Leftrightarrow x \leq y,\ a - a_k = b - b_k,\ (y \wedge x, b_k - a_k) \in \widehat{Q_3}$$
$$\Leftrightarrow \begin{cases} x \leq y \leq 0,\ a \leq b,\ a - a_k = b - b_k; \\ x \leq 0 \leq y,\ a \leq b,\ a - a_k = b - b_k; \\ 0 \leq x \leq y,\ a = b. \end{cases}$$

The left amenable order $\leq_{Q_3}$ can therefore be depicted by $k$ disjoint

Hasse diagrams of the form




As for the smallest locally maximal cone $E = \{(x,a);\ a_k = 0\}$, we have

$$(x,a) \leq_E (y,b) \Leftrightarrow x \leq y, \quad a - a_k = b - b_k, \quad b_k = a_k$$
$$\Leftrightarrow x \leq y, \quad a = b,$$

so the Hasse diagram in this case consists of the lines $y = \alpha$, where $\alpha \in \mathbb{Z}$.

## 5. Amenable Orders on Orthodox Semigroups

We now turn to a "symmetric" situation and consider the following notion.

Definition. If $S$ is a regular semigroup with an inverse transversal $S^0$ then by an *amenable order* on $S$ we shall mean a compatible strict natural order $\leq$ such that

$$x \leq y \Rightarrow x^0 x \preceq y^0 y, \qquad xx^0 \preceq yy^0.$$

In what follows we shall require the above results and their duals, together with the following known results.

Theorem 12 [6, Corollary 3.3]. *If $S$ is a regular semigroup with a multiplicative inverse transversal then $S$ can be embedded as an ideal in a naturally ordered regular semigroup with a biggest idempotent.*

Theorem 13 [5, Proposition 1.4]. *A regular semigroup can be naturally ordered if and only if it is locally inverse.*

Theorem 14 [2, Theorem 1.1]. *If $S$ is a naturally ordered semigroup and if $e, f$ are idempotents such that $e \leq f$ then $e = efe$.*

THEOREM 15 [5, PROPOSITION 1.9]. *Let $S$ be an ordered regular semigroup with a biggest idempotent $\alpha$. Then $S$ is naturally ordered and orthodox if and only if $\alpha$ is a middle unit.*

We can now establish the following.

THEOREM 16. *Let $S$ be an orthodox semigroup with a multiplicative inverse transversal. Then $S$ can be amenably ordered.*

*Proof.* By Theorem 12 there is a naturally ordered regular semigroup $(A, \leq)$ that has a biggest idempotent $\alpha$ and contains $S$ as an ideal. In fact, by its construction in [6], $A = S \cup \{\alpha\}$, where $\alpha \notin S$. Since $A$ is naturally ordered under $\leq$ it is locally inverse by Theorem 13. Since by hypothesis $S$ is orthodox, so is $A$; for if $e \in E(S)$ then $e \leq \alpha$ gives $e = e\alpha e$ by Theorem 14, and it follows from this and the fact that $S$ is an ideal of $A$ that $\alpha e \in E(S)$ and $e\alpha \in E(S)$. It now follows by Theorem 15 that $\alpha$ is a middle unit of $A$. Consequently, $A^0 = \alpha A \alpha$ is a multiplicative inverse transversal of $A$. Here we have $x^0 = \alpha x' \alpha$ for every $x' \in V(x)$ and is the biggest inverse of $x$ under $\leq$, and $x^{00} = \alpha x \alpha$. Thus $\alpha^0 = \alpha$ and if $x \in S$ then, since $S$ is an ideal, $x^0 \in S$. It follows that $\alpha S \alpha = \alpha A \alpha \setminus \{\alpha\}$ is a multiplicative inverse transversal of $S$. Then $\alpha S$ (resp. $S\alpha$) is a right inverse (resp. left inverse) subsemigroup of $S$ with multiplicative inverse transversal $\alpha S \alpha$.

Let $Q$ be a locally maximal cone of $\alpha S$ and let $R$ be a locally maximal cone of $S\alpha$. Consider the relation $\sqsubseteq_{Q,R}$ defined on $S$ by

$$x \sqsubseteq_{Q,R} y \Leftrightarrow \alpha x \leq_Q \alpha y, \qquad x\alpha \leq_R y\alpha.$$

Clearly, $\sqsubseteq_{Q,R}$ is reflexive and transitive. To see that it is also anti-symmetric, let $x \sqsubseteq_{Q,R} y$ and $y \sqsubseteq_{Q,R} x$. Then $\alpha x = \alpha y$ and $x\alpha = y\alpha$, so $x^{00} = \alpha x \alpha = \alpha y \alpha = y^{00}$ and therefore $x^0 = y^0$. It follows that $x^0 x = x^0 \alpha x = y^0 \alpha y = y^0 y$ and $x x^0 = x \alpha x^0 = y \alpha y^0 = y y^0$. Consequently,

$$x = x x^0 x^{00} x^0 x = y y^0 y^{00} y^0 y = y.$$

Thus $\sqsubseteq_{Q,R}$ is an order. Since $\leq_Q$ and $\leq_R$ are compatible orders and $\alpha$ is a middle unit, it follows that $\sqsubseteq_{Q,R}$ is also compatible.

Now for $e, f \in E = E(S)$ we have

$$e \sqsubseteq_{Q,R} f \Leftrightarrow \alpha e \leq_Q \alpha f, \qquad e\alpha \leq_R f\alpha.$$

Since

$$\alpha f(\alpha e)^0 = \alpha f e^0 = \alpha f \alpha e^0 = f^{00} e^0 \in (\alpha E)^0 \subseteq Q$$

and, similarly, $(e\alpha)^0 f\alpha \in R$, we see that

$$e \sqsubseteq_{Q,R} f \Leftrightarrow e^0 e \preceq f^0 f, \qquad ee^0 \preceq ff^0.$$

But $e^0 e \preceq f^0 f$ gives $ef^0 f = e$, whence $ef = e$; similarly, $ee^0 \preceq ff^0$ gives $fe = e$. It follows that we have

$$e \sqsubseteq_{Q,R} f \Rightarrow e \preceq f.$$

Conversely, since $\alpha$ is a middle unit,

$$\begin{aligned} e \preceq f &\Rightarrow \alpha e \preceq \alpha f, \qquad e\alpha \preceq f\alpha \\ &\Rightarrow \alpha e \leq_Q \alpha f, \qquad e\alpha \leq_R f\alpha \qquad \text{by Theorem 6} \\ &\Rightarrow e \sqsubseteq_{Q,R} f. \end{aligned}$$

Thus $\sqsubseteq_{Q,R}$ coincides with $\preceq$ on $E$ and so $\sqsubseteq_{Q,R}$ is an amenable order on $S$. ∎

COROLLARY 1. *Every amenable order on $S$ is of the form $\sqsubseteq_{Q,R}$ for some locally maximal cone $Q$ of $\alpha S$ and some locally maximal cone $R$ of $S\alpha$.*

*Proof.* Let $\leq$ be a given amenable order on $S$. Then, by Theorem 7, the restriction of $\leq$ to the right inverse semigroup $\alpha S$ is given by $\leq_C$ where $C$ is the locally maximal cone

$$C = \{x \in \alpha S;\ x^0 x \leq x\}.$$

Correspondingly, the restriction to $S\alpha$ of $\leq$ is given by $\leq_D$, where $D$ is the locally maximal cone

$$D = \{x \in S\alpha;\ xx^0 \leq x\}.$$

Consider the amenable order $\sqsubseteq_{C,D}$ on $S$. We have

$$x \sqsubseteq_{C,D} y \Leftrightarrow x^0 x \preceq y^0 y, \quad \alpha y(\alpha x)^0 \in C, \quad xx^0 \preceq yy^0, \quad (x\alpha)^0 y\alpha \in D.$$

Now, under the hypothesis that $x^0 x \preceq y^0 y$, we have that

$$\alpha y(\alpha x)^0 = \alpha yx^0 \in C \Leftrightarrow \alpha xx^0 = x^{00} x^0 = x^{00} y^0 yx^0 \leq \alpha yx^0.$$

Moreover, $\alpha xx^0 \leq \alpha yx^0$ gives

$$\begin{aligned} x^{00} &= \alpha xx^0 x\alpha \leq \alpha yx^0 x\alpha \\ &\leq \alpha yy^0 y\alpha \qquad \text{since } x^0 x \leq y^0 y \\ &= y^{00}, \end{aligned}$$

and, conversely, $\alpha x \alpha = x^{00} \leq y^{00} = \alpha y \alpha$ gives $\alpha x x^0 \leq \alpha y x^0$. Using a similar observation for $D$, we therefore see that $\sqsubseteq_{C,D}$ can be described by

$$x \sqsubseteq_{C,D} y \Leftrightarrow x^0x \preceq y^0y, \qquad xx^0 \preceq yy^0, \qquad x^{00} \leq y^{00}. \qquad (*)$$

Since $\leq$ is a natural order, it follows that

$$\begin{aligned} x \sqsubseteq_{C,D} y &\Rightarrow x^0x \leq y^0y, \qquad xx^0 \leq yy^0, \qquad x^{00} \leq y^{00} \\ &\Rightarrow x = xx^0x^{00}x^0x \leq yy^0y^{00}y^0y = y, \end{aligned}$$

and so $\sqsubseteq_{C,D}$ is a refinement of $\leq$. Now $x\alpha = xx^0x\alpha = xx^0\alpha x \alpha = xx^0x^{00} = xx^0x(x^0x)^0$ and so, since $\leq$ is amenable, we have that $x \leq y$ implies $x\alpha \leq y\alpha$. Similarly, $x \leq y$ implies $\alpha x \leq \alpha y$. Hence

$$x \leq y \Rightarrow x^{00} = \alpha x \alpha \leq \alpha y \alpha = y^{00}.$$

It follows from $(*)$ that $\leq$ is a refinement of $\sqsubseteq_{C,D}$. Thus $\leq$ coincides with $\sqsubseteq_{C,D}$. ∎

COROLLARY 2. *The amenable orders definable on S form a meet semilattice with smallest element* $\sqsubseteq_{\alpha E, E\alpha}$.

*Proof.* It suffices to observe that

$$\sqsubseteq_{Q,R} \wedge \sqsubseteq_{Q',R'} = \sqsubseteq_{Q \cap Q', R \cap R'}$$

and use Theorem 11. ∎

COROLLARY 3. *The smallest amenable order* $\sqsubseteq_{\alpha E, E\alpha}$ *can be described by*

$$x \sqsubseteq_{\alpha E, E\alpha} y \Leftrightarrow x^0x \preceq y^0y, \qquad xx^0 \preceq yy^0, \qquad yx^0 \in E,$$

*and coincides with the Nambooripad order* $\leq_N$ *given by*

$$x \leq_N y \Leftrightarrow (\exists e, f \in E)\ x = ey = yf.$$

*Proof.* As for the description of $\sqsubseteq_{\alpha E, E\alpha}$ it suffices to note that if $\alpha x \in \alpha E \subseteq E$ then $x \in E$, for $\alpha x \alpha x = \alpha x$ gives (on pre-multiplication by $xx^0$) the equality $xx = x$; and that $yx^0 \in E$, together with $x^0x \preceq y^0y$, gives

$$x^0y = x^0xx^0y = y^0yx^0xx^0y = y^0 \cdot yx^0yx^0 \cdot xx^0y = x^0yx^0y,$$

so that $x^0y \in E$.

Now since $S$ is locally inverse it is well known that the Nambooripad order is compatible. It is then a strict natural order. Since

$$\begin{aligned} x \leq_N y &\Rightarrow (\exists e, f \in E)\ x = ey = yf \\ &\Rightarrow x^0 = y^0 e^0 = f^0 y^0 \\ &\Rightarrow x^0 x = y^0 e^0 ey, \qquad xx^0 = yff^0 y^0 \\ &\Rightarrow x^0 x \preceq y^0 y, \qquad xx^0 \preceq yy^0, \end{aligned}$$

we see that $\leq_N$ is amenable with respect to every multiplicative inverse transversal. Also, since $S$ is orthodox,

$$x \leq_N y \Rightarrow x^0 = y^0 e^0 \Rightarrow yx^0 = yy^0 e^0 \in E$$

and so $x \leq_N y$ implies that $x \sqsubseteq_{\alpha E, E\alpha} y$. Since $\sqsubseteq_{\alpha E, E\alpha}$ is the smallest amenable order, it follows that $\leq_N$ coincides with $\sqsubseteq_{\alpha E, E\alpha}$. ∎

EXAMPLE 6. With the notation as in Example 4, endow $\mathbb{Z} \times \mathbb{R} \times \mathbb{Z}$ with the multiplication

$$(p, x, a)(q, y, b) = (p + q_k, x \wedge y, a_k + b).$$

Under the cartesian order, $\mathbb{Z} \times \mathbb{R} \times \mathbb{Z}$ becomes a naturally ordered orthodox semigroup with

$$V(p, x, a) = \{(q, x, b);\ q_k = -p_k, b_k = -a_k\}.$$

Let $S$ be the subsemigroup $\mathbb{Z} \times [0, 1) \times \mathbb{Z}$ and let $A = S \cup \{(k-1, 1, k-1)\}$. Then $S$ and $A$ are naturally ordered orthodox semigroups. Clearly,

$$E(A) = \{(p, x, a);\ p_k = 0 = a_k\},$$

so $A$ has a biggest idempotent $\alpha = (k-1, 1, k-1) \notin S$. It follows that $\alpha$ is a middle unit and that $S^0 = \alpha S \alpha$ is a multiplicative inverse transversal of $S$. Simple calculations now give

$$(p, x, a) \leq_N (q, y, b) \Leftrightarrow p = q,\ x \leq y,\ a = b.$$

Finally, we observe that the above can be applied to the particular situation of a naturally ordered orthodox Dubreil–Jacotin semigroup $S$, called an *orthodox epigroup* in [3] to which we refer for notation and properties. In such a semigroup $x^0 = (\xi : x)x(\xi : x)$ is the biggest inverse of $x$ and the bimaximum element $\xi$ is such that $S^0 = \xi S \xi$ is a multiplicative inverse transversal of $S$. Then, by Theorem 16, $S$ can be amenably ordered.

THEOREM 17. *If $S$ is an orthodox epigroup then the smallest amenable order definable on $S$ is given by*

$$x \leq_N y \Leftrightarrow (\xi : x)x \preceq (\xi : y)y, \qquad x(\xi : x) \preceq y(\xi : y), \qquad \xi : x = \xi : y.$$

*Proof.* By Corollary 3 of Theorem 16 we have

$$x \leq_N y \Leftrightarrow x \sqsubseteq_{\xi E(S), E(S)\xi} y \Leftrightarrow x^0 x \preceq y^0 y, \qquad xx^0 \preceq yy^0, \qquad yx^0 \in E.$$

Now $yx^0 \in E$ gives $y(\xi : x)x(\xi : x) \in E$, whence $\xi = \xi : y(\xi : x)$, since, $S$ being orthodox, $\xi : e = \xi$ for every $e \in E$. Consequently, using [1, Theorem 25.5], we have

$$\xi : x = \xi .\!\!\cdot x = [\xi : y(\xi : x)] .\!\!\cdot x = \xi : y(\xi : x)x = \xi : y.$$

Conversely, $\xi : x = \xi : y$ gives

$$\xi : yx^0 = (\xi : y) .\!\!\cdot x^0 = (\xi : x) .\!\!\cdot x^0 = \xi : xx^0 = \xi,$$

whence $yx^0 \in E$. ∎

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