# Some BPS configurations of the BLG theory

Shankhadeep Chakrabortty [a], Sudipto Paul Chowdhury [b],*, Koushik Ray [b]

[a] *Institute of Physics, Bhubaneswar, India*
[b] *Department of Theoretical Physics, Indian Association for the Cultivation of Science, Calcutta-700032, India*

**A R T I C L E   I N F O**

**A B S T R A C T**

We obtain BPS configurations of the BLG theory and its variant including mass terms for scalars and fermions in addition to a background field with different world-volume and $R$-symmetries. Three cases are considered, with world-volume symmetries $SO(1, 1)$ and $SO(2)$ and preserving different amounts of supersymmetry. In the former case we obtain a singular configuration preserving $N = (3, 3)$ supersymmetry and an one-quarter BPS configuration corresponding to intersecting M2-M5-M5-branes. In the latter instance the BPS equations are reduced to those in the self-dual Chern–Simons theory with two complex scalars. In want of an exact solution, we find a topological vortex solution numerically in this case. Other solutions are given by combinations of domain walls.

© 2011 Elsevier B.V. Open access under CC BY license.

## 1. Introduction

The Bagger–Lambert–Gustavsson (BLG) theory [1–5] is an $N = 8$, supersymmetric Chern–Simons type gauge theory based on a ternary gauge algebra coupled to matter in $(2 + 1)$-dimensions with $SO(8)$ $R$-symmetry. The theory is deemed to have $Osp(8|4)$ superconformal symmetry based on strong evidences [6] and is thus a candidate for a world-volume theory of M2-branes. It contains eight scalar fields interpreted as eight directions transverse to the world-volume of M2-branes in M-theory and eight corresponding fermions in addition to a gauge field. Imposing complete antisymmetry of the structure constant of the ternary gauge algebra along with the closure of the supersymmetry algebra constrains the gauge group to be $SO(4)$. The theory has a sixteen-dimensional moduli space lending itself to the interpretation as a theory of two M2-branes [7–9]. Various aspects and variants of the BLG theory have been considered [10–21].

In this Letter we shall be concerned with BPS configurations in the BLG theory and a particular modification of it. This entails the inclusion of mass terms for the scalars and the fermion and a flux term [20]. BPS states of the BLG theory have been classified [22] according to world-volume symmetries, namely, $SO(1, 2)$, $SO(1, 1)$ and $SO(2)$. A BPS configuration of the modified theory with $SO(1, 2)$ world-volume symmetry and $SO(4)$ $R$-symmetry has been studied earlier [9]. In this Letter we study BPS configurations in three cases with $SO(1, 1)$ and $SO(2)$ world-volume symmetries. We consider BPS configurations in the BLG theory preserv-

ing $N = (3, 3)$ supersymmetry with $SO(1, 1)$ world-volume symmetry and $SO(3) \times SO(3) \times SO(2)$ $R$-symmetry and obtain a solution to the BPS equations. The solution has scalars diverging at a finite distance of a world-volume coordinate. We then consider the deformed variant with $SO(1, 1)$ world-volume symmetry and $SO(4) \times SO(4)$ $R$-symmetry preserving $N = (4, 0)$ supersymmetry. Finally we deal with an $N = 4$ BPS configuration with $SO(2)$ world-volume symmetry and $SU(2) \times SO(4)$ $R$-symmetry. In the former case we obtain a configuration which may be interpreted as a system of intersecting M2-M5-M5-branes, following the popular interpretation of the BLG theory. The other solutions turn out to be combination of domain walls interpolating between pairs of classical vacua. In the latter case the problem, upon choosing an appropriate ansatz, is mapped to the self-dual $U(1)^2$ Chern–Simons theory with two complex scalar fields. This problem has been studied earlier [27–29]. In want of an analytic solution to the BPS equations we present a numerical one corresponding to a single vortex. In considering these examples we find that casting the BPS equations in terms of gauge-invariant variables used earlier [9] furnishes a useful guideline for the choice of ansätze for scalars in the BPS equations.

The Letter is organized as follows. In the following section we recall some aspects of the modified BLG theory. In Section 3 we obtain solution to the BPS configurations with $SO(1, 1)$ symmetry in the world-volume and $SO(3) \times SO(3) \times SO(2)$ $R$-symmetry. In Section 4 we discuss the $SO(1, 1)$-invariant BPS configuration with $SO(4) \times SO(4)$ $R$-symmetry and its domain-wall solution. In Section 5 we proceed to discuss the $SO(2)$-invariant BPS configuration having $SU(2) \times SO(4)$ $R$-symmetry, map it to the Chern–Simons theory with two complex scalars and present a numerical solution for an Abelian topological vortex, before concluding in Section 6.

* Corresponding author.
*E-mail addresses:* sankha@iopb.res.in (S. Chakrabortty), tpspc@iacs.res.in
(S.P. Chowdhury), koushik@iacs.res.in (K. Ray).

## 2. BLG theory

Let us begin with a brief description of the BLG theory and its deformation by a background four-form field. The modified BLG theory is an $N = 8$ supersymmetric theory in $(2+1)$-dimensions, given by the Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{BLG}} + \mathcal{L}_{\text{mass}} + \mathcal{L}_{\text{flux}}, \tag{1}$$

where the first term

$$
\begin{aligned}
\mathcal{L}_{\text{BLG}} = {} & -\frac{1}{2}\operatorname{Tr}\big(D_\mu X^I\big)\big(D^\mu X^I\big) + \operatorname{Tr}\frac{i}{2}\overline{\Psi}\gamma^\mu D_\mu \Psi \\
& + \frac{i}{4}\operatorname{Tr}\overline{\Psi}\Gamma_{IJ}\langle X^I, X^J, \Psi\rangle - \frac{1}{12}\operatorname{Tr}\langle X^I, X^J, X^K\rangle^2 \\
& + \frac{1}{2}\epsilon^{\mu\nu\lambda}\bigg(f^{abcd}A_{\mu ab}\partial_\nu A_{\lambda cd} \\
& + \frac{2}{3}f^{cda}{}_g f^{efgb} A_{\mu ab}A_{\nu cd}A_{\lambda ef}\bigg)
\end{aligned} \tag{2}
$$

is the original BLG Lagrangian. Here $\mu = 0, 1, 2$ designates the world-volume directions, $I = 1, \dots, 8$ indexes the flavors and $a = 1, 2, 3, 4$ the gauge algebra. $X_a^I$, $\Psi_a$ and $A_{\mu ab}$ are the scalars, the Majorana–Weyl spinor and the gauge field, respectively. The three- and eight-dimensional gamma matrices are denoted $\gamma$ and $\Gamma$, respectively. The ternary bracket of the gauge algebra is denoted as $\langle , , \rangle$, while its structure constants are denoted by $f^{abcd}$. Repeated indices are summed over in the above expression and in the following unless stated otherwise. Denoting the generators of the ternary algebra as $\tau_a$, the metric tensor raising and lowering gauge indices is written as

$$h_{ab} = \operatorname{Tr}\tau_a\tau_b. \tag{3}$$

We use the generators to write the fields valued in the ternary algebra as

$$X^I = h^{ab}X_a^I\tau_b, \tag{4}$$

$$\Psi = h^{ab}\Psi_a\tau_b. \tag{5}$$

Here $D_\mu$ denotes the covariant derivative,

$$D_\mu X_a^I = \partial_\mu X_a^I - \tilde{A}_\mu{}^b_a X_b^I. \tag{6}$$

In the presence of a four-form field $G_{IJKL}$ the BLG Lagrangian is augmented by a mass term

$$\mathcal{L}_{\text{mass}} = -\frac{1}{2}m^2\delta^{IJ}\operatorname{Tr}\big(X^I X^J\big) + c\operatorname{Tr}\big(\overline{\Psi}\Gamma^{IJKL}\Psi\big)\tilde{G}_{IJKL} \tag{7}$$

and a flux term

$$\mathcal{L}_{\text{flux}} = -c\tilde{G}_{IJKL}\operatorname{Tr}\big(X^I\langle X^J, X^K, X^L\rangle\big). \tag{8}$$

The four-form field satisfies a self-duality condition

$$\tilde{G}_{IJKL} = G_{IJKL}, \tag{9}$$

where the dual of the four-form field $G$ is defined as

$$\tilde{G}_{IJKL} = \frac{1}{4!}\epsilon_{IJKLPQRS}G^{PQRS}. \tag{10}$$

The mass $m$ is determined by the four-form field as $m^2 = \frac{c^2}{768}G^2$, with $G^2 = G^{IJKL}G_{IJKL}$ and $c$ is a parameter which is found to be equal to 2 [20]. Thus the BLG theory is recovered in the limit of vanishing $c$.

The action corresponding to the Lagrangian (2) is under the supersymmetry transformations [20]

$$\delta X^I = i\overline{\theta}\Gamma^I\Psi, \tag{11}$$

$$
\begin{aligned}
\delta\Psi = {} & \gamma^\mu \Gamma^I D_\mu X^I\theta - \frac{1}{6}\Gamma^{IJK}\langle X^I, X^J, X^K\rangle\theta \\
& + \frac{c}{8}\Gamma^{IJKL}\Gamma^M\tilde{G}_{IJKL}X^M\theta,
\end{aligned} \tag{12}
$$

$$\delta A_\mu(\phi) = i\overline{\theta}\gamma_\mu\Gamma^I\langle\Psi, X^I, \phi\rangle, \tag{13}$$

where $\phi$ in the transformation of the gauge field represents either a $X^I$ or $\Psi$ and $\theta$ denotes the parameter of supersymmetry variation, satisfying

$$\Gamma^9 = \Gamma^{1\dots 8}\theta = \theta,$$
$$\gamma^{txy}\theta = \theta. \tag{14}$$

The supersymmetry transformations close on-shell up to translation and local gauge transformations if the structure constant of the ternary algebra is the rank-four antisymmetric tensor, that is,

$$f^{abcd} = \epsilon^{abcd}, \tag{15}$$

so that the gauge group is $SO(4)$. The scalars and the fermion transform as vectors of the gauge group $SO(4)$. We thus choose, for example,

$$X^I = \begin{pmatrix} X_1^I \\ X_2^I \\ X_3^I \\ X_4^I \end{pmatrix}, \tag{16}$$

for all $I$. For future convenience we have set the level of the Chern–Simons action to be unity, and the metric is taken to be Euclidean,

$$h_{ab} = \delta_{ab}. \tag{17}$$

The ternary bracket then reads

$$\langle X^I, X^J, X^K\rangle = \epsilon^{abcd}X_a^I X_b^J X_c^K\tau_d. \tag{18}$$

We shall be concerned with the BPS configurations of the theory with Lagrangian (1). The BPS equation is obtained by setting the supersymmetry variation of the fermion to zero, that is

$$\delta\Psi = 0. \tag{19}$$

Depending on the subgroup of the $R$-symmetry as well as the world-volume symmetry to be maintained, the supersymmetry parameter $\theta$ is restricted by means of a projector, $\Omega$. Thus, the BPS equations are given by

$$
\begin{aligned}
\bigg[ & D_\mu X^I\gamma^\mu\Gamma^I - \frac{1}{6}\Gamma^{IJK}\langle X^I, X^J, X^K\rangle \\
& + \frac{c}{8}\Gamma^{IJKL}\Gamma^M\tilde{G}_{IJKL}X^M\bigg]\Omega\theta = 0.
\end{aligned} \tag{20}
$$

Let us note that only the anti-self-dual combination of the four-form field appears in the last term on the left hand side, linear in $X$. The $R$-symmetry in this formulation is realized explicitly in terms of the four-form field as

$$R^I_J = \overline{\theta_2}\Gamma^{IKLM}\theta_1\tilde{G}_{KLMJ}, \tag{21}$$

where $\theta_1$ and $\theta_2$ are two parameters of supersymmetry variation. The conserved charged under the global $SO(8)$ symmetry of the BLG theory is given by the $R$-charge, namely

$$R^{IJ} = \int d^2x\left(X^{Ia}D_0 X_a^J - X^{Ja}D_0 X_a^I + \frac{i}{2}\overline{\psi}^a\gamma^0\Gamma^{IJ}\psi_a\right) \tag{22}$$

where $R^{IJ}$ is antisymmetric in $I$ and $J$. We now proceed to study certain BPS configurations of the theory discussed above.

Furthermore, the BPS configurations have to satisfy the Gauss constraint, namely

$$F_{\mu\nu b}{}^{a} + \epsilon_{\mu\nu\lambda}\epsilon^{cda}_{b}X_c^I D^\lambda X_d^I = 0, \tag{23}$$

where $F$ denotes the field strength corresponding to the gauge field $\tilde{A}$.

## 3. BPS configuration with $SO(1,1) \times SO(3) \times SO(3) \times SO(2)$ symmetry

In this section we present a solution to the BPS equations preserving $N = (3,3)$ supersymmetry in the BLG theory without the mass and the four-form terms, corresponding to the Lagrangian (2). The world-volume has $SO(1,1)$ symmetry and the $R$-symmetry is $SO(3) \times SO(3) \times SO(2)$. The equations are [22]

$$D_t X^I = 0, \qquad D_x X^I = 0, \tag{24}$$

with

$$\mathcal{J}_p^{IJ} D_y X_J + \frac{1}{2}\mathcal{J}_p^{JK}\langle X^I, X_J, X_K\rangle = 0,$$

$$\mathcal{J}_{p+3}^{IJ} D_y X_J - \frac{1}{2}\mathcal{J}_{p+3}^{JK}\langle X^I, X_J, X_K\rangle = 0, \tag{25}$$

where $p = 1,2,3$, $I, J = 1,2,\ldots,8$ and the complex structures $\mathcal{J}$ are defined by

$$\frac{1}{2}\mathcal{J}_1^{IJ}\Gamma_{IJ} = \Gamma^{12} + \Gamma^{34} + \Gamma^{56} + \Gamma^{78},$$

$$\frac{1}{2}\mathcal{J}_4^{IJ}\Gamma_{IJ} = \Gamma^{12} + \Gamma^{43} + \Gamma^{56} + \Gamma^{87},$$

$$\frac{1}{2}\mathcal{J}_2^{IJ}\Gamma_{IJ} = \Gamma^{14} + \Gamma^{23} + \Gamma^{58} + \Gamma^{67},$$

$$\frac{1}{2}\mathcal{J}_5^{IJ}\Gamma_{IJ} = \Gamma^{17} + \Gamma^{28} + \Gamma^{53} + \Gamma^{64},$$

$$\frac{1}{2}\mathcal{J}_3^{IJ}\Gamma_{IJ} = \Gamma^{13} + \Gamma^{42} + \Gamma^{57} + \Gamma^{86},$$

$$\frac{1}{2}\mathcal{J}_6^{IJ}\Gamma_{IJ} = \Gamma^{18} + \Gamma^{72} + \Gamma^{54} + \Gamma^{36}. \tag{26}$$

Thus from (25) we have six expressions for each $D_y X^I$, for $I = 1,2,\ldots,8$. Comparing the various expressions for the same $D_y X^I$ we obtain a set of relations among the ternary brackets, namely,

$$\langle X^2, X^5, X^6\rangle = 0, \qquad \langle X^3, X^5, X^7\rangle = \langle X^4, X^5, X^8\rangle,$$
$$\langle X^3, X^6, X^8\rangle = -\langle X^4, X^6, X^7\rangle,$$
$$\langle X^1, X^5, X^6\rangle = 0, \qquad \langle X^4, X^6, X^8\rangle = \langle X^3, X^6, X^7\rangle,$$
$$\langle X^4, X^5, X^7\rangle = -\langle X^3, X^5, X^8\rangle,$$
$$\langle X^4, X^7, X^8\rangle = 0, \qquad \langle X^2, X^6, X^7\rangle = \langle X^1, X^5, X^7\rangle,$$
$$\langle X^2, X^5, X^8\rangle = -\langle X^1, X^6, X^8\rangle,$$
$$\langle X^3, X^7, X^8\rangle = 0, \qquad \langle X^2, X^6, X^8\rangle = \langle X^1, X^5, X^8\rangle,$$
$$\langle X^2, X^5, X^7\rangle = -\langle X^1, X^6, X^7\rangle,$$
$$\langle X^1, X^2, X^6\rangle = 0, \qquad \langle X^1, X^4, X^8\rangle = \langle X^1, X^3, X^7\rangle,$$
$$\langle X^2, X^4, X^7\rangle = -\langle X^2, X^3, X^8\rangle,$$
$$\langle X^1, X^2, X^6\rangle = 0, \qquad \langle X^2, X^4, X^8\rangle = \langle X^2, X^3, X^7\rangle,$$
$$\langle X^1, X^4, X^7\rangle = -\langle X^1, X^3, X^8\rangle,$$
$$\langle X^3, X^4, X^8\rangle = 0, \qquad \langle X^2, X^3, X^6\rangle = \langle X^1, X^3, X^5\rangle,$$
$$\langle X^2, X^4, X^5\rangle = -\langle X^1, X^4, X^6\rangle,$$

$$\langle X^3, X^4, X^7\rangle = 0, \qquad \langle X^2, X^4, X^6\rangle = \langle X^1, X^4, X^5\rangle,$$
$$\langle X^2, X^3, X^5\rangle = -\langle X^1, X^3, X^6\rangle, \tag{27}$$
$$\langle X^2, X^3, X^4\rangle = \langle X^2, X^7, X^8\rangle = \langle X^4, X^5, X^8\rangle + \langle X^4, X^6, X^7\rangle,$$
$$\langle X^1, X^3, X^4\rangle = \langle X^1, X^7, X^8\rangle = -\langle X^3, X^5, X^8\rangle - \langle X^3, X^6, X^7\rangle,$$
$$\langle X^1, X^2, X^4\rangle = \langle X^4, X^5, X^6\rangle = \langle X^1, X^6, X^8\rangle - \langle X^1, X^5, X^7\rangle,$$
$$\langle X^1, X^2, X^3\rangle = \langle X^3, X^5, X^6\rangle = \langle X^1, X^5, X^8\rangle + \langle X^1, X^6, X^7\rangle,$$
$$\langle X^3, X^4, X^6\rangle = \langle X^6, X^7, X^8\rangle = \langle X^1, X^3, X^7\rangle + \langle X^2, X^3, X^8\rangle,$$
$$\langle X^5, X^7, X^8\rangle = \langle X^3, X^4, X^4\rangle = \langle X^1, X^3, X^8\rangle - \langle X^2, X^3, X^7\rangle,$$
$$\langle X^1, X^2, X^8\rangle = \langle X^5, X^6, X^8\rangle = -\langle X^1, X^4, X^6\rangle - \langle X^1, X^3, X^5\rangle,$$
$$\langle X^1, X^2, X^7\rangle = \langle X^5, X^6, X^7\rangle = \langle X^1, X^4, X^5\rangle - \langle X^1, X^3, X^6\rangle, \tag{28}$$

together with a set of Basu–Harvey equations with respect to the world-volume coordinate $y$,

$$D_y X^1 = 2\langle X^2, X^3, X^4\rangle, \qquad D_y X^2 = -2\langle X^1, X^3, X^4\rangle,$$
$$D_y X^3 = 2\langle X^1, X^2, X^4\rangle, \qquad D_y X^4 = -2\langle X^1, X^2, X^3\rangle,$$
$$D_y X^5 = 2\langle X^3, X^4, X^6\rangle, \qquad D_y X^6 = -2\langle X^5, X^7, X^8\rangle,$$
$$D_y X^7 = 2\langle X^1, X^2, X^8\rangle, \qquad D_y X^8 = -2\langle X^1, X^2, X^7\rangle. \tag{29}$$

It will be useful to first write the BPS equations in terms of gauge-invariant variables [9]. This furnishes a guideline for the choice of ansätze for the $X$'s. Let us introduce the gauge-invariant fields

$$Y^{IJ} = \sum_{a=1}^{4} X^{Ia} X_a^J, \tag{30}$$

where indices are raised or lowered with the Euclidean bilinear (17). The gauge-invariants satisfy

$$\partial_\mu Y^{IJ} = X^{Ia} D_\mu X_a^J + X^{Ja} D_\mu X_a^I \tag{31}$$

due to the antisymmetry of the gauge field. Using this and (27) and (28) we obtain from (29) a set of first-order equations for the gauge-invariants

$$\partial_y Y^{11} = -4F_{1234}, \qquad \partial_y Y^{15} = -2F_{4567} - 2F_{1238},$$
$$\partial_y Y^{16} = -2F_{4568} + 2F_{1237},$$
$$\partial_y Y^{22} = -4F_{1234}, \qquad \partial_y Y^{25} = -2F_{1345} - 2F_{2346},$$
$$\partial_y Y^{26} = -2F_{1346} + 2F_{2345},$$
$$\partial_y Y^{33} = -4F_{1234}, \qquad \partial_y Y^{37} = 2F_{1247} + 2F_{1238},$$
$$\partial_y Y^{38} = 2F_{4568} - 2F_{5678},$$
$$\partial_y Y^{44} = -4F_{1234}, \qquad \partial_y Y^{47} = 2F_{1248}, \qquad \partial_y Y^{48} = 2F_{1247},$$
$$\partial_y Y^{55} = -4F_{3456}, \qquad \partial_y Y^{66} = -4F_{3456},$$
$$\partial_y Y^{77} = -4F_{1278}, \qquad \partial_y Y^{88} = -4F_{1278}, \tag{32}$$

and $\partial_y Y^{IJ} = 0$ for all other $I$, $J$. So far our analysis has been completely general. Now Let us assume that the constant $Y$'s, not appearing in (32), are zero. Also, from the above set of first-order equations we note that it is convenient to choose

$$X^i = f S^i, \tag{33}$$

where $i = 1,2,3,4$ and $S^i$ are the four mutually orthogonal canonical basis vectors of $\mathbf{R}^4$. Then from (32) it is apparent that $X^5$ and $X^6$ are linear combinations of $S^1$ and $S^2$, while $X^7$ and $X^8$ are linear combinations of $S^3$ and $S^4$. Using the relations (27) and (28) we can fix the coefficients of these linear combinations and obtain
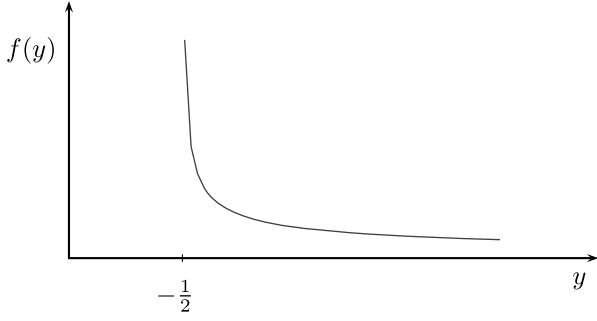
**Fig. 1.** Plot of $f(y)$ for the $N = (3, 3)$ configuration.

$$X^i = f S^i, \quad i = 1, 2, 3, 4,$$
$$X^5 = f S^1 \cos\theta + f S^2 \sin\theta, \qquad X^6 = -f S^1 \sin\theta + f S^2 \cos\theta,$$
$$X^7 = f S^3 \cos\theta + f S^4 \sin\theta, \qquad X^8 = -f S^3 \sin\theta + f S^4 \cos\theta,$$
$$(34)$$

where $f$ and $\theta$ are to be determined from the differential equations.

From the equation for $Y^{11}$, say, we obtain an equation for $f$,

$$\partial_y f = -2 f^3, \tag{35}$$

which is solved to obtain

$$f^2 = \frac{1}{c + 4y}, \tag{36}$$

where $c = 2$ and we have chosen the integration constant to be vanishing. Now, comparing the expressions for $D_y X^5$, $D_y X^1$ and $D_y X^2$, upon using (34), we obtain

$$\partial_y \log \cos\theta = 0, \tag{37}$$

implying that $\theta$ is a constant. We have thus fixed the solution (34) completely. A sketch of the function $f(y)$ is shown in Fig. 1. For this solution all the scalars diverge at a finite distance in the $y$ direction, namely, $y = -1/2$. Interpretation of this solution in terms of M2- and M5-branes is not clear.

## 4. BPS configurations with $SO(1, 1) \times SO(4) \times SO(4)$ symmetry

In this section we study $SO(1, 1)$-invariant $N = (4, 0)$ BPS configurations having $SO(4) \times SO(4)$ $R$-symmetry. The BPS equations are derived for the modified BLG theory by applying the appropriate BPS projection operator on the supersymmetry variation of the fermion and equating it to zero by (20). The generic form of the $SO(1, 1)$ BPS projector is given in terms of $32 \times 32$ gamma matrices as

$$\Omega = \frac{1}{16}\big(1 + \alpha_0 \gamma^{tx}\big)\big(1 - \alpha_1\alpha_2 \Gamma^{1278} + \alpha_1\alpha_3 \Gamma^{1368} - \alpha_1 \Gamma^{2468}$$
$$- \alpha_3 \Gamma^{3478} - \alpha_2 \Gamma^{5678} + \alpha_1\alpha_2\alpha_3 \Gamma^{2358} + \alpha_2\alpha_3 \Gamma^{1458}\big)\mathcal{P}$$
$$(38)$$

where $\alpha_0$, $\alpha_1$, $\alpha_2$, $\alpha_3$ are sign factors assuming values $\pm 1$ and $\gamma$ and $\Gamma$ designates the gamma matrices defined on the world-volume and transverse to the world-volume, respectively. The chiral projection operator $\mathcal{P}$ is defined in terms of $\Gamma^9 = \Gamma^{12\cdots8}$ as

$$\mathcal{P} = \frac{1}{2}\big(1 + \Gamma^9\big). \tag{39}$$

Different choices of the sign factors in (38) give the BPS projection matrix which correspond to breaking $R$-symmetry in a certain manner [22–24]. The projector preserving $SO(4) \times SO(4)$ $R$-symmetry is obtained by summing the four $N = 1$ projectors (38)

with the choice of $\alpha$'s as $\{+, +, +, +\}$, $\{+, +, +, -\}$, $\{+, +, -, +\}$, $\{+, +, -, -\}$ and is given by

$$\Omega = \frac{1}{4}\big(1 + \gamma^{tx}\big)\big(1 + \Gamma^{5678}\big)\big(1 + \Gamma^9\big). \tag{40}$$

By Eq. (14) this operator corresponds to the projections

$$\gamma^{txy}\theta = \theta, \qquad \gamma^{tx}\Gamma^{1234}\theta = \theta, \qquad \gamma^{tx}\Gamma^{5678}\theta = \theta, \tag{41}$$

the last one being a dependent one. Applying the projection matrix (40) on (20), we obtain the BPS equations. These comprise of differential equations namely

$$(D_t - D_x)X^I = 0, \tag{42}$$

for all $I = 1, 2, \ldots, 8$ and a set of modified Basu–Harvey equations,

$$D_y X^i - \frac{1}{6}\epsilon^{ijkl}\langle X_j, X_k, X_l\rangle - \eta_1 X^i = 0, \tag{43}$$

$$D_y X^p - \frac{1}{6}\epsilon^{pqrs}\langle X_q, X_r, X_s\rangle - \eta_2 X^p = 0, \tag{44}$$

where we have split the flavor indices as $i, j, k, l = 1, 2, 3, 4$ and $p, q, r, s = 5, 6, 7, 8$. $D_t$, $D_x$ and $D_y$ designate the covariant derivatives with respect to the world-volume coordinates (6). Coefficients of the terms linear in $X$'s in Eq. (43) are determined in terms of the four-form field by $\eta_1 = 3cG_{1234}$ and $\eta_2 = 3cG_{5678}$ with $c = 2$, as discussed above. We assume $\eta_1$ and $\eta_2$ to be positive in the following.

Our objective here is to find a topological solution to the BPS equations (42), (43) and (23). First let us write down the BPS equations in terms of the gauge-invariant variables $Y$ as before. Multiplying both sides of (43) and (44) with an appropriate $X$ and taking linear combinations, using (31), we obtain equations for the gauge-invariants

$$\partial_y Y^{ij} - 2\eta_1 Y^{ij} = -2\delta^{ij} F_{1234},$$
$$\partial_y Y^{rs} - 2\eta_2 Y^{rs} = -2\delta^{rs} F_{5678},$$
$$\partial_y Y^{ip} - \frac{1}{3!}\big(\varepsilon^{ijkl}F_{jklp} - \varepsilon^{pqrs}F_{iqrs}\big) - (\eta_1 + \eta_2)Y^{ip} = 0 \tag{45}$$

for $i, j = 1, \ldots, 4$ and $r, s = 5, 6, 7, 8$, where we defined the gauge-invariant four-form $F_{IJKL} = X_a^I X_b^J X_c^K X_d^L \epsilon^{abcd}$.

Further, using (42) in the Gauss constraint (23) we get

$$F_{xy} = F_{ty}, \tag{46}$$

while using (43) and (44) to eliminate the covariant $y$-derivatives in (23) we obtain

$$F_{tx} = 0. \tag{47}$$

The modified Basu–Harvey equations have been solved earlier to obtain a domain wall and a fuzzy funnel solution [3,25]. Let us point out that the equations for the scalars are the same for half and quarter BPS configurations with $SO(4) \times SO(4)$ $R$-symmetry. The gauge fields satisfy different equations in these two cases, however. For example, while we have Eq. (42) for the quarter-BPS configuration, the half-BPS configurations satisfy $D_t X^I = D_x X^I = 0$ [22]. Hence, it is important to write down the gauge fields explicitly, even if in a special gauge. In order to obtain explicit expressions for gauge field configurations we shall choose simplifying ansätze. The scalar fields $X^i$, $i = 1, 2, 3, 4$, are chosen to be mutually orthogonal $SO(4)$ vectors, as are $X^p$, $p = 5, 6, 7, 8$. However, if we choose the basis vectors to be the constant vectors $S^i$ as in the last section, the gauge fields will remain undetermined. Hence for the present case we choose a different set of mutually orthogonal basis vectors and express the scalars as

$$X^1 = f \begin{pmatrix} \cos\Theta \\ \sin\Theta \\ 0 \\ 0 \end{pmatrix}, \qquad X^2 = \kappa f \begin{pmatrix} -\sin\Theta \\ \cos\Theta \\ 0 \\ 0 \end{pmatrix},$$

$$X^3 = f \begin{pmatrix} 0 \\ 0 \\ \cos\Phi \\ \sin\Phi \end{pmatrix}, \qquad X^4 = f \begin{pmatrix} 0 \\ 0 \\ -\sin\Phi \\ \cos\Phi \end{pmatrix},$$

$$X^5 = g \begin{pmatrix} \cos\Theta \\ \sin\Theta \\ 0 \\ 0 \end{pmatrix}, \qquad X^6 = \kappa' g \begin{pmatrix} -\sin\Theta \\ \cos\Theta \\ 0 \\ 0 \end{pmatrix},$$

$$X^7 = g \begin{pmatrix} 0 \\ 0 \\ \cos\Phi \\ \sin\Phi \end{pmatrix}, \qquad X^8 = g \begin{pmatrix} 0 \\ 0 \\ -\sin\Phi \\ \cos\Phi \end{pmatrix}, \tag{48}$$

where $\kappa, \kappa' = \pm 1$. We shall find solutions corresponding to both signs of $\kappa, \kappa'$. The gauge-invariant co-ordinates for this choice are $Y^{ii} = f^2$, $i = 1, 2, 3, 4$, $Y^{pp} = g^2$, $p = 5, 6, 7, 8$. The gauge field is chosen to be of the form

$$\tilde{A}_\mu = \begin{pmatrix} 0 & \tilde{A}_{\mu 2}^1 & 0 & 0 \\ -\tilde{A}_{\mu 2}^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{A}_{\mu 4}^3 \\ 0 & 0 & -\tilde{A}_{\mu 4}^3 & 0 \end{pmatrix}, \tag{49}$$

in accordance with the above choice for the scalars, hence breaking the gauge group $SO(4)$ to $SO(2) \times SO(2)$. In Eq. (48) the functions $\Theta$ and $\Phi$ correspond to the freedom of gauge choice for the residual $SO(2) \times SO(2)$ subgroup. For future convenience we shall leave these arbitrary. However, the stress-energy tensor does not depend on them, as we find later.

We shall first obtain restrictions on $f$, $g$, $\Theta$, $\Phi$ and then determine the components of the gauge field in terms of them. We restrict ourselves to stationary solutions. Then all the fields are independent of time. From (45) we obtain equations for $f$ and $g$, namely

$$\partial_y f = \eta_1 f - \kappa f^3, \qquad \partial_y g = \eta_2 g - \kappa' g^3. \tag{50}$$

We now relate the components of the gauge field to $f$, $g$, $\Theta$, $\Phi$. First, using (48) and the equation for $X^1$ from (43) we obtain, for the first component,

$$\left(\partial_y f - \eta_1 f + \kappa f^3\right)\cos\Theta - f\left(\tilde{A}_{y2}^1 - \partial_y\Theta\right)\sin\Theta = 0, \tag{51}$$

and similarly for the first component of the equation for $X^3$ from (43). Comparing with (50) we obtain

$$\tilde{A}_{y2}^1 = \partial_y\Theta, \qquad \tilde{A}_{y4}^3 = \partial_y\Phi. \tag{52}$$

This demonstrates the utility of the gauge-invariant equations and justifies keeping $\Theta$ and $\Phi$ arbitrary in (48). Setting the temporal derivative to zero in (42) we obtain

$$\left(\tilde{A}_{ta}^b - \tilde{A}_{xa}^b\right)X_b^I = -\partial_x X_a^I. \tag{53}$$

Putting $a = 1, 2$, resp. $b = 2, 1$ and using $\tilde{A}_{\mu b}^a = -\tilde{A}_{\mu a}^b$ and Eqs. (48), we obtain

$$\partial_x f = 0, \tag{54}$$

that is, $f$ is independent of $x$. Similarly, from (44) and (50) we obtain $\partial_x g = 0$. In other words, $f$ and $g$ are functions of $y$ only. This leads to

$$\tilde{A}_{t2}^1 - \tilde{A}_{x2}^1 = -\partial_x\Theta, \qquad \tilde{A}_{t4}^3 - \tilde{A}_{x4}^3 = -\partial_x\Phi. \tag{55}$$

Thus the gauge fields are determined in terms of $\Theta$ and $\Phi$. Using (52), (42) and (55), the Gauss constraint (23) yields

$$\partial_y \tilde{A}_{t2}^1 = 2\left(f(y)^2 + g(y)^2\right)\tilde{A}_{t4}^3,$$
$$\partial_y \tilde{A}_{t4}^3 = 2\left(f(y)^2 + g(y)^2\right)\tilde{A}_{t2}^1, \tag{56}$$

while (47) yields

$$\partial_x \tilde{A}_{t2}^1 = 0 = \partial_x \tilde{A}_{t4}^3, \tag{57}$$

so that $\tilde{A}_t$ is a function of $y$ alone. Now, eliminating the combination $f^2 + g^2$ between the two equations in (56) we obtain

$$\partial_y\left((\tilde{A}_{t2}^1)^2 - (\tilde{A}_{t4}^3)^2\right) = 0, \tag{58}$$

leading to the conclusion that the squares of the components of the gauge field $A_t$ may differ by a constant only. By linearly combining Eqs. (56) we can cast them as first order differential equations for the combinations $\tilde{A}_{t2}^1 \pm \tilde{A}_{t4}^3$ as

$$\partial_y\left(\tilde{A}_{t2}^1 \pm \tilde{A}_{t4}^3\right) = \pm 2\left(f^2 + g^2\right)\left(\tilde{A}_{t2}^1 \pm \tilde{A}_{t4}^3\right), \tag{59}$$

which are solved to obtain

$$\tilde{A}_{t2}^1 = A_0 e^{\int(f^2 + g^2)\,dy} + B_0 e^{-\int(f^2 + g^2)\,dy},$$
$$\tilde{A}_{t4}^3 = A_0 e^{\int(f^2 + g^2)\,dy} - B_0 e^{-\int(f^2 + g^2)\,dy}, \tag{60}$$

where $A_0$ and $B_0$ are constants. This solution satisfies (58). The solution to (50) is

$$f(y) = \pm\frac{\sqrt{\eta_1}\,a}{\sqrt{e^{-2\eta_1 y} + \kappa a^2}}, \qquad g(y) = \pm\frac{\sqrt{\eta_2}\,a'}{\sqrt{e^{-2\eta_2 y} + \kappa' a'^2}}, \tag{61}$$

where $a, a'$ are constants of integration.

Different configurations ensue from the choices of $\kappa, \kappa'$. By allowing the supersymmetry variation of $\mathcal{L}$ to vanish [20], the mass parameter in the scalar term in $\mathcal{L}_{\text{mass}}$ gets related to the four-form field $\tilde{G}_{IJKL}$ as

$$\Gamma^{IJKL}\tilde{G}_{IJKL}\Gamma^{MNOP}\tilde{G}_{MNOP} = \frac{32m^2}{c^2}\left(1 + \Gamma^{12345678}\right), \tag{62}$$

resulting in $m^2 = 9c^2 G^2$, where $G^2 = G_{1234}G^{1234} = G_{5678}G^{5678}$. Using this value of $m$ and the ansatz for the scalars (48), the scalar potential obtained from the sextic term in $\mathcal{L}_{\text{BLG}}$, the scalar term in $\mathcal{L}_{\text{mass}}$ along with $\mathcal{L}_{\text{flux}}$ is

$$\begin{aligned} V &= -\frac{1}{12}\,\text{Tr}\langle X^I, X^J, X^K\rangle^2 - \frac{1}{2}m^2\delta^{IJ}\,\text{Tr}(X^I X^J) \\ &\quad - c\tilde{G}_{IJKL}\,\text{Tr}(X^I\langle X^J, X^K, X^L\rangle) \\ &= -2f^2\left(f^2 - \eta_1\right)^2 - 2g^2\left(g^2 - \eta_2\right)^2, \end{aligned} \tag{63}$$

with $\eta_1, \eta_2 > 0$. The classical vacua are therefore,

$$\begin{aligned} &V_I: \quad f(y) = g(y) = 0; \\ &V_{II}: \quad f(y) = \pm\sqrt{\eta_1}, \qquad g(y) = \pm\sqrt{\eta_2}, \\ &V_{III}: \quad f(y) = 0, \qquad g(y) = \pm\sqrt{\eta_2}, \\ &V_{IV}: \quad f(y) = \pm\sqrt{\eta_1}, g(y) = 0. \end{aligned} \tag{64}$$

The solution (61) with $\kappa = \kappa' = 1$ interpolates between the vacua $V_I$ and $V_{II}$ as $y$ varies from $-\infty$ to $\infty$. Taking into account that the solution is independent of $x$, this is therefore a domain wall solution. By (60), the temporal component of the gauge field is

$$\tilde{A}_{t\,2}^{1} = A_0 \sqrt{\left(1 + a^2 e^{2\eta_1 y}\right)\left(1 + a'^2 e^{2\eta_2 y}\right)}$$
$$+ \frac{B_0}{\sqrt{(1 + a^2 e^{2\eta_1 y})(1 + a'^2 e^{2\eta_2 y})}},$$
$$\tilde{A}_{t\,4}^{3} = A_0 \sqrt{\left(1 + a^2 e^{2\eta_1 y}\right)\left(1 + a'^2 e^{2\eta_2 y}\right)}$$
$$- \frac{B_0}{\sqrt{(1 + a^2 e^{2\eta_1 y})(1 + a'^2 e^{2\eta_2 y})}}. \tag{65}$$

However, in order to keep $\tilde{A}_t$ finite in the whole domain of $y$, we have to set $A_0$ to zero. Thus, finally, the two components of $\tilde{A}_t$ are given by

$$\tilde{A}_{t\,2}^{1} = -\tilde{A}_{t\,4}^{3} = \frac{B_0}{\sqrt{(1 + a^2 e^{2\eta_1 y})(1 + a'^2 e^{2\eta_2 y})}}, \tag{66}$$

where we have taken the difference of their squares to be vanishing. Having thus obtained $\tilde{A}_t$, $\tilde{A}_x$ and $\tilde{A}_y$ are determined, up to gauge transformation, by (55) and (52), respectively. The energy–momentum tensor is obtained by varying the Lagrangian $\mathcal{L}$ with respect to the world-volume metric,

$$T_{\mu\nu} = \frac{1}{2} D_\mu X^{Ia} D_\nu X_a^I - \frac{1}{4} g_{\mu\nu}\left(D^\alpha X^{Ia} D_\alpha X_a^I - V\right), \tag{67}$$

where the potential $V$ is given by (63). Energy density of the configuration obtained above is given by $T_{tt}$. Plugging in the solutions for the scalars, (61), and the gauge fields, (52), (55) and (66) in the expression for $T_{tt}$, we obtain the energy density to be

$$T_{tt} = 2\eta_1 e^{2\eta_1 y}\left(\frac{1 + 3\eta_1^2(4 - a^2 e^{2\eta_1 y} - 1)^2}{(1 + a^2 e^{2\eta_1 y})^3}\right)$$
$$+ 2\eta_2 e^{2\eta_2 y}\left(\frac{1 + 3\eta_2^2(4 - a'^2 e^{2\eta_2 y} - 1)^2}{(1 + a'^2 e^{2\eta_2 y})^3}\right). \tag{68}$$

For $\kappa = \kappa' = -1$, the $f$ and $g$ and hence the gauge invariants $Y^{ii}$ and $Y^{rr}$ diverge at $y = -\frac{1}{\eta_1}\ln a$ and $y = -\frac{1}{\eta_2}\ln a'$, respectively. The corresponding solutions for the gauge fields $\tilde{A}_{t\,2}^{1}$ and $\tilde{A}_{t\,4}^{3}$ are given by

$$\tilde{A}_{t\,2}^{1} = C_0 \sqrt{\left(1 - a^2 e^{2\eta_1 y}\right)\left(1 - a'^2 e^{2\eta_2 y}\right)}$$
$$+ \frac{D_0}{\sqrt{(1 - a^2 e^{2\eta_1 y})(1 - a'^2 e^{2\eta_2 y})}},$$
$$\tilde{A}_{t\,4}^{3} = C_0 \sqrt{\left(1 - a^2 e^{2\eta_1 y}\right)\left(1 - a'^2 e^{2\eta_2 y}\right)}$$
$$- \frac{D_0}{\sqrt{(1 - a^2 e^{2\eta_1 y})(1 - a'^2 e^{2\eta_2 y})}}, \tag{69}$$

where $C_0$ and $D_0$ are constants. In accordance with (41) the solution thus describes an M2-brane ending on two M5-branes with world-volumes spanning directions 1, 2, 3, 4 and 5, 6, 7, 8, respectively and sharing the directions $x$ and $t$ with the M2-brane [26]. Thus we obtained a quarter-BPS configuration of the mass deformed BLG theory given by a bound state of M2-M5-M5-branes.

## 5. BPS configuration with $SO(2) \times SU(2) \times SO(4)$ symmetry

In this section we consider $N = 4$ BPS configurations in the modified BLG theory with world-volume symmetry $SO(2)$. We find that upon choosing a certain ansatz the equations for the scalars reduce to the scalar equations of a self-dual $U(1)^2$ Chern–Simons theory with two complex scalars. Topological configurations in the latter case have been investigated in earlier [27–29]. However, no analytic solution to the equations appears to be known. We shall consider special cases in which we obtain certain solutions.

As before, to write down the $SO(2)$-invariant BPS equations we project the supersymmetry variation of the fermion with the $SO(2)$ invariant BPS projector. In terms of the $32 \times 32$ gamma matrices the projector is

$$\Omega = \frac{1}{8}\left(1 + \alpha_1 \gamma^{xy} \Gamma^{12}\right)\left(1 + \alpha_2 \gamma^{xy} \Gamma^{12}\right)\left(1 + \alpha_3 \gamma^{xy} \Gamma^{12}\right)\mathcal{P} \tag{70}$$

where $\alpha_1$, $\alpha_2$ and $\alpha_3$ are sign factors $\pm 1$ and $\mathcal{P}$ denotes the chiral projection matrix as before (39). We shall consider the situation in which the $R$-symmetry is broken to $SU(2) \times SO(4)$. The projector (70) assumes the form

$$\Omega = \frac{1}{4}\left(1 + \gamma^{xy}\left(\Gamma^{12} + \Gamma^{34}\right) - \Gamma^{1234}\right)\mathcal{P} \tag{71}$$

corresponding to the choice of the $\alpha$'s as $\{+, +, +\}$, $\{+, +, -\}$. The projector corresponds to

$$\gamma^{xy}\Gamma^{12}\theta = \gamma^{xy}\Gamma^{34}\theta = \theta, \tag{72}$$

or, equivalently, $\Gamma^{1234}\theta = -\theta$. For simplicity, we set the four scalars $X^5$, $X^6$, $X^7$, $X^8$ to zero and write the BPS equations for the non-zero fields only. This reduction in the flavor degrees of freedom breaks the $R$-symmetry further to $SU(2)$. The BPS equations in terms of the non-vanishing scalars, $X^I$, $I = 1, 2, 3, 4$, are

$$D_x X^1 + D_y X^2 = 0, \qquad D_x X^2 - D_y X^1 = 0,$$
$$D_x X^3 + D_y X^4 = 0, \qquad D_x X^4 - D_y X^3 = 0, \tag{73}$$

along with

$$D_t X^1 + \langle X^1, X^3, X^4 \rangle + \eta_1 X^2 = 0,$$
$$D_t X^2 + \langle X^2, X^3, X^4 \rangle - \eta_1 X^1 = 0,$$
$$D_t X^3 + \langle X^3, X^1, X^2 \rangle + \eta_1 X^4 = 0,$$
$$D_t X^4 + \langle X^4, X^1, X^2 \rangle - \eta_1 X^3 = 0. \tag{74}$$

Using (31) these can be written in terms of the gauge-invariant variables $Y$. From (74) we obtain

$$\partial_t Y^{11} + 2\eta_1 Y^{12} = 0, \qquad \partial_t Y^{22} - 2\eta_1 Y^{12} = 0,$$
$$\partial_t Y^{33} + 2\eta_1 Y^{34} = 0, \qquad \partial_t Y^{44} - 2\eta_1 Y^{34} = 0 \tag{75}$$

and

$$\partial_t Y^{12} + \eta_1\left(Y^{22} - Y^{11}\right) = 0, \qquad \partial_t Y^{13} + \eta_1\left(Y^{23} + Y^{14}\right) = 0,$$
$$\partial_t Y^{14} + \eta_1\left(Y^{24} - Y^{13}\right) = 0, \qquad \partial_t Y^{23} + \eta_1\left(Y^{24} - Y^{13}\right) = 0,$$
$$\partial_t Y^{24} - \eta_1\left(Y^{14} + Y^{23}\right) = 0, \qquad \partial_t Y^{34} + \eta_1\left(Y^{44} - Y^{33}\right) = 0. \tag{76}$$

These gauge-invariant equations provide restrictions on the choice of ansatz for the scalar fields. For stationary configurations the time derivatives are set to zero and these equations yield relations between the $Y$'s. In particular, they imply that $Y^{12} = Y^{34} = 0$, meaning, $X^1$, $X^2$ are mutually orthogonal, as are $X^3$, $X^4$. They also require $Y^{11} = Y^{22}$ and $Y^{33} = Y^{44}$. An ansatz satisfying these relations compatible with the remaining $SU(2)$ $R$-symmetry is

$$X^1 = \begin{pmatrix} f_1 \\ f_2 \\ 0 \\ 0 \end{pmatrix}, \qquad X^2 = \begin{pmatrix} f_2 \\ -f_1 \\ 0 \\ 0 \end{pmatrix},$$
$$X^3 = \begin{pmatrix} 0 \\ 0 \\ g_1 \\ g_2 \end{pmatrix}, \qquad X^4 = \begin{pmatrix} 0 \\ 0 \\ g_2 \\ -g_1 \end{pmatrix}. \tag{77}$$

For this choice for this ansatz the gauge group breaks down to $U(1)^2$. We retain the ansatz (49) for the gauge field $\tilde{A}_\mu$. Let us introduce complex combinations,

$$z = x + iy, \qquad \phi = f_1 + if_2, \qquad \chi = g_1 + ig_2,$$
$$\tilde{A}_{zb}^a = \tilde{A}_{xb}^a - i\tilde{A}_{yb}^a. \tag{78}$$

We also define $A_z = \tilde{A}_{z2}^1$ and $B_z = \tilde{A}_{z4}^3$ for the spatial components of the gauge field and $A_t = \tilde{A}_{t2}^1$ and $B_t = \tilde{A}_{t4}^3$ for the temporal parts. The gauge-invariant variables are, then

$$Y^{11} = Y^{22} = |\phi|^2, \qquad Y^{33} = Y^{44} = |\chi|^2,$$
$$Y^{IJ} = 0 \quad \text{for } I \neq J. \tag{79}$$

In terms of these complex quantities the BPS equations (73) are written as

$$D_z\phi = \partial_z\phi + iA_z\phi = 0,$$
$$D_z\chi = \partial_z\chi + iB_z\chi = 0, \tag{80}$$

which can be solved to express the gauge fields to be expressed as in terms of the scalars as

$$A_z = i\partial_z \ln\phi, \qquad B_z = i\partial_z \ln\chi. \tag{81}$$

Similarly, assuming stationarity Eqs. (74) relate the temporal part of the gauge field to the complex scalars $\phi$ and $\chi$, namely,

$$A_t = \eta_1 - |\chi|^2, \qquad B_t = \eta_1 - |\phi|^2. \tag{82}$$

Using (81) and (82) we can now rewrite the action (2) and the Gauss constraint (23) in terms of the complex scalars $\phi$ and $\chi$. The Lagrangian reads

$$\mathcal{L} = -2\big(|\partial_z\phi|^2 - |\partial_z\chi|^2 + 2|\chi|^2(\eta_1 - |\phi|^2)^2$$
$$+ 2|\phi|^2(\eta_1 - |\chi|^2)^2\big), \tag{83}$$

while the Gauss constraint leads to two differential equations for the scalars, namely,

$$\nabla^2 \ln|\phi|^2 - 2|\chi|^2(|\phi|^2 - \eta_1) = 0,$$
$$\nabla^2 \ln|\chi|^2 - 2|\phi|^2(|\chi|^2 - \eta_1) = 0, \tag{84}$$

where $\nabla^2 = \partial_z\partial_{\bar{z}}$. Defining rescaled fields

$$\tilde{\phi} = \phi/\sqrt{\eta_1}, \qquad \tilde{\chi} = \chi/\sqrt{\eta_1}, \tag{85}$$

these two equations take the form

$$\nabla^2 \ln|\tilde{\phi}|^2 + \lambda(1 - |\tilde{\phi}|^2)|\tilde{\chi}|^2 = 0,$$
$$\nabla^2 \ln|\tilde{\chi}|^2 + \lambda(1 - |\tilde{\chi}|^2)|\tilde{\phi}|^2 = 0, \tag{86}$$

where $\lambda = 2\eta_1^2$. These coupled elliptic partial differential equations have been studied in the context of $U(1)^2$ self-dual Chern–Simons theory with two complex scalar fields [27,28]. In particular, existence of topological vortex solutions, characterized by the boundary conditions

$$|\tilde{\phi}| \to 1, \qquad |\tilde{\chi}| \to 1,$$

as $|z| \to \infty$, have been established [27]. For a single vortex at the origin, the solution is also proved to be unique [28]. However, no explicit analytic construction of vortex solutions seem to exist in literature. The conserved $R$-charges for the $SO(2) \times SU(2)$ BPS configuration are given by

$$R_{12} = \int d^2x\, |\phi|^2(\eta_1 - |\chi|^2),$$
$$R_{34} = \int d^2x\, |\chi|^2(\eta_1 - |\phi|^2), \tag{87}$$

while the total energy of the configuration is given by

$$\mathcal{E} = \frac{1}{4} \int d^2x \big(|\partial_z\phi|^2 + |\partial_z\chi|^2 + 4(\eta_1 - |\chi|^2)^2|\phi|^2$$
$$+ 4(\eta_1 - |\phi|^2)^2|\chi|^2\big). \tag{88}$$

### 5.1. A special case

Let us consider the special case of the BLG theory without the four-form field, corresponding to the Lagrangian $\mathcal{L}_{\text{BLG}}$. Putting $\eta_1 = 0$ in (84) we obtain the Gauss constraint equations for this case as

$$\nabla^2 \ln|\phi|^2 - 2|\chi|^2|\phi|^2 = 0,$$
$$\nabla^2 \ln|\chi|^2 - 2|\phi|^2|\chi|^2 = 0. \tag{89}$$

Subtracting these we obtain

$$\nabla^2 \ln \frac{|\phi|}{|\chi|} = 0. \tag{90}$$

Adding Eqs. (89), on the other hand, we obtain a Liouville-like equation for $\rho = |\phi\chi|^2$, namely

$$\nabla^2 \ln\rho = 4\rho, \tag{91}$$

which is solved by

$$\rho = \frac{1}{2}\left|\frac{d\xi/dz}{1 - |\xi(z)|^2}\right|^2, \tag{92}$$

where $\xi(z)$ is an analytic function of $z$. From (90) and (92) we conclude that both $|\phi|^2$ and $|\chi|^2$ are proportional to $|\frac{d\xi/dz}{1-|\xi(z)|^2}|$, modulo analytic functions. Thus, $|\phi|^2$, $|\chi|^2$, hence $Y^{II}$, $I = 1, 2, 3, 4$, are singular on the curve $\xi(z) = 1$. Given a $\xi$, this corresponds to two M2-brane spikes extended along 1–2 and 3–4 directions corresponding to the two $U(1)$ factors of the gauge group on the original M2-brane of the BLG theory lying on the $z$-plane [26].

### 5.2. Numerical solution

While general closed form solutions to (86) are not known, we can solve the equations numerically. Here we present a numerical solution for the simplest case of a single vortex. Uniqueness of the solution for this case has been established earlier [28]. Using the $SO(2)$ world-volume symmetry of the BPS configuration let us now write $z = re^{i\theta}$ and drop the $\theta$-dependence of all the functions. Writing $|\phi|^2 = e^{\rho(r)}$ and $|\chi|^2 = e^{\sigma(r)}$ Eqs. (86) become

$$\rho''(r) + \frac{1}{r}\rho'(r) + 2e^{\sigma(r)}(\eta_1 - e^{\rho(r)}) = 0,$$
$$\sigma''(r) + \frac{1}{r}\sigma'(r) + 2e^{\rho(r)}(\eta_1 - e^{\sigma(r)}) = 0, \tag{93}$$

where a prime denotes a derivative with respect to $r$. To solve these equations numerically to obtain vortex solutions we have to impose two boundary conditions on each of $\rho$ and $\sigma$. The first set of asymptotic boundary conditions are chosen as $\rho(r) \to 0$ and $\sigma(r) \to 0$ as $r \to \infty$. The second set of boundary conditions arise from the quantization of magnetic fluxes corresponding to the gauge fields $A_z$ and $B_z$, namely, $\int dz \wedge d\bar{z}\, F_{z\bar{z}} = 2\pi N_1$ and
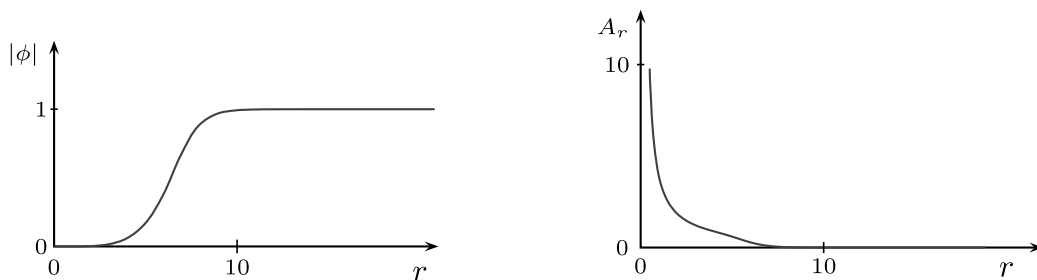
**Fig. 2.** Plot of $|\phi(r)|$ and $A_\theta$ with $\eta_1 = 1$.

$\int dz \wedge d\bar{z} \, F'_{z\bar{z}} = 2\pi N_2$, where $N_1$ and $N_2$ are integers representing vorticity. In the limit $r \to \infty$ the flux quantization conditions written in terms of $\rho(r)$ and $\sigma(r)$ require

$$\left|\rho'(r)\right|_{r\to\infty} = \frac{N_1}{r}, \qquad \left|\sigma'(r)\right|_{r\to\infty} = \frac{N_2}{r}. \tag{94}$$

Solutions for $\rho$ and $\sigma$ can now be obtained numerically. A plot of $|\phi|^2$ and the corresponding gauge field $A_\theta$ (in polar coordinates) is shown in Fig. 2 with unit vorticity for both vortices and $\eta_1 = 1$. The plots of $|\chi|$ and $B_r$ are similar.

## 6. Summary

To summarize, we have studied BPS configurations of the BLG theory with and without the mass and four-form deformations. We considered three cases of interest. In the first case the solution with world-volume symmetry $SO(1,1)$ preserving $N = (3,3)$ supersymmetry in the absence of any deformation has eight scalars which blow up at a finite value of the world-volume coordinate $y$. We then considered a quarter BPS configuration with $SO(4) \times SO(4)$ $R$-symmetry. In this case there are two types of solutions. One of them is a pair of domain walls each extending along four directions in agreement with the $R$-symmetry. The other solution features the M2-brane merging into two M5-branes at a finite distance in $y$, the latter intersecting along the $x$ direction. This has been interpreted as a system of intersecting M2-M5-M5-branes [30]. Finally we considered a configuration with $SO(2)$ symmetry in the world-volume and $SU(2) \times SO(4)$ $R$-symmetry. We chose to turn off the four scalar corresponding to the $SO(4)$. By choosing an appropriate ansatz for the scalars and the gauge field, the system maps into the self-dual $U(1)^2$ Chern–Simons theory with two complex scalars. Existence of vortex solutions to these equations has been established earlier. We presented a solution for the special case with no deformation, giving rise to a Liouville-like equation. We also presented a numerical solution for the single topological vortex, which is known to be unique. In dealing with the system of BPS equations we found that expressing them in terms of the gauge-invariant variables introduced earlier [9] appears to be of immense help in the choice of ansätze for the solutions in all cases. Other cases in the classification of BPS configurations of the BLG theory may also be considered in a similar fashion. However, the solutions for those are given either by constant scalars or combinations of domain walls or the singular solutions of Section 3.

## References

[1] J. Bagger, N. Lambert, Phys. Rev. D 75 (2007) 045020, arXiv:hep-th/0611108.
[2] J. Bagger, N. Lambert, Phys. Rev. D 77 (2008) 065008, arXiv:0711.0955 [hep-th].
[3] J. Bagger, N. Lambert, JHEP 0802 (2008) 105, arXiv:0712.3738 [hep-th].
[4] J. Bagger, N. Lambert, Phys. Rev. D 79 (2009) 025002, arXiv:0807.0163 [hep-th].
[5] A. Gustavsson, Algebraic structures on parallel M2-branes, arXiv:0709.1260 [hep-th].
[6] M. Bandres, A. Lipstein, J. Schwarz, JHEP 0805 (2008) 025, arXiv:0710.5073.
[7] N. Lambert, D. Tong, Phys. Rev. Lett. 101 (2008) 041602, arXiv:0804.1114 [hep-th].
[8] J. Distler, S. Mukhi, C. Papageorgakis, M. Van Raamsdonk, JHEP 0805 (2008) 038, arXiv:0804.1256 [hep-th].
[9] S. Chakrabortty, S. Paul Chowdhury, T. Köppe, K. Ray, Phys. Lett. B 695 (2011) 285, arXiv:1005.3707 [hep-th].
[10] S. Mukhi, C. Papageorgakis, JHEP 0805 (2008) 085, arXiv:0803.3218 [hep-th].
[11] B. Ezhuthachan, S. Mukhi, C. Papageorgakis, JHEP 0807 (2008) 041, arXiv:0806.1639 [hep-th].
[12] M. Alishahiha, S. Mukhi, JHEP 0810 (2008) 032, arXiv:0808.3067 [hep-th].
[13] M. Van Raamsdonk, JHEP 0805 (2008) 105, arXiv:0803.3803 [hep-th].
[14] P. De Medeiros, J.M. Figueroa-O'Farrill, E. Mendez-Escobar, JHEP 0807 (2008) 111, arXiv:0805.4363 [hep-th].
[15] F. Passerini, JHEP 0808 (2008) 062, arXiv:0806.0363 [hep-th].
[16] S. Cherkis, C. Saemann, Phys. Rev. D 78 (2008) 066019, arXiv:0807.0808 [hep-th].
[17] B.E.W. Nilsson, J. Palmkvist, Class. Quant. Grav. 26 (2009) 075007, arXiv:0807.5134.
[18] S. Paul Chowdhury, S. Mukhopadhyay, K. Ray, BLG theory with generalized Jordan triple systems, arXiv:0903.2951 [hep-th].
[19] C. Krishnan, C. Maccaferri, JHEP 0807 (2008) 005, arXiv:0805.3125 [hep-th].
[20] N. Lambert, P. Richmond, JHEP 0910 (2009) 084, arXiv:0908.2896 [hep-th].
[21] A. Gustavsson, JHEP 0911 (2009) 071, arXiv:0909.2518 [hep-th].
[22] I. Jeon, J. Kim, N. Kim, S.W. Kim, J.H. Park, JHEP 0807 (2008) 056, arXiv:0805.3236 [hep-th].
[23] D. Bak, K.M. Lee, J.H. Park, Phys. Rev. D 66 (2002) 025021, arXiv:hep-th/0204221.
[24] E.K. Loginov, Phys. Rev. D 78 (2008) 065010.
[25] C.S. Chu, G.S. Sehmbi, J. Phys. A 44 (2011) 135404.
[26] I. Jeon, J. Kim, B.-H. Lee, et al., Int. J. Mod. Phys. A 24 (2009) 5779, arXiv:0809.0856 [hep-th].
[27] C. Lin, A. Ponce, Y. Yang, J. Funct. Anal. 247 (2007) 289.
[28] J. Chern, Z. Chen, C. Lin, Commun. Math. Phys. 296 (2010) 323.
[29] J. Dziarmaga, Phys. Rev. D 49 (1994) 5469, arXiv:hep-th/9403127.
[30] D.S. Berman, M.J. Perry, E. Sezgin, D.C. Thompson, JHEP 1004 (2010) 025, arXiv:0912.3504 [hep-th].