MINIMUM MATRIX REPRESENTATION OF CLOSURE OPERATIONS

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Let $a$ be a column of the $m \times n$ matrix $M$ and $A$ a set of its columns. We say that $A$ implies $a$ iff $M$ contains no two rows equal in $A$ but different in $a$. It is easy to see that if $\mathcal{C}_M(A)$ denotes the columns implied by $A$, then $\mathcal{C}_M(A)$ is a closure operation. We say that $M$ represents this closure operation. $s(\mathcal{C})$ is the minimum number of the rows of the matrices representing a given closure operation. $s(\mathcal{C})$ is determined for some particular closure operations.

1. Introduction

A simple model of a data base [4] is a matrix. A row contains the data of one individual. A column contains the data of the same sort (e.g. date of birth). Let $X$ denote the set of columns. Choose a subset $A \subseteq X$ and suppose that the data of an individual are known in the columns belonging to $A$. The individual (or row) is not necessarily determined, there can be more individuals (rows) having these data in the columns belonging to $A$. However all these individuals (rows) might agree in a column $b \in A$. We say that $b$ belongs to the closure $\mathcal{C}(A)$ of $A$ if this happens for $b$ with any choice of data in the columns belonging to $A$. We will see that this function $\mathcal{C}$ mapping $2^X$ into $2^X$ satisfies (2)-(4). Such a function is called a closure operation. Conversely, if a closure operation $\mathcal{C}$ is given, one can find a matrix generating exactly this closure operation in the above defined way [1, 5, 6]. Let $s(\mathcal{C})$ denote the minimum cardinality of rows of such matrices.

The main aim of the paper is to investigate the function $s(\mathcal{C})$. There are three kinds of results. In Section 3 we more or less determine $s(\mathcal{C})$ for some very special closure operations: the closure of any set of cardinality $\geq k$ is $X$ while the closure of smaller sets $A$ is $A$. Section 4 determines $s(\mathcal{C}_1 \times \mathcal{C}_2)$ in terms of $s(\mathcal{C}_1)$ and $s(\mathcal{C}_2)$. Finally we quote a result producing an $\mathcal{C}$ with large $s(\mathcal{C})$. 
2. Definitions

Let $M$ be a matrix of $m$ rows and $n$ columns. The set of columns will be denoted by $X$. If $A \subseteq X$, $a \in X$ and $M$ contains no two rows equal in $A$ but different in $a$ then we say that $A$ implies $a$. The closure of $A$ is

$$(1) \quad \mathcal{L}_M(A) = \{a: a \in X, A \text{ implies } a\}.$$ 

It is easy to see that the following rules are valid for $\mathcal{L}_M = \mathcal{L}$:

$$(2) \quad A \subseteq \mathcal{L}(A),$$
$$(3) \quad A \subseteq B \Rightarrow \mathcal{L}(A) \subseteq \mathcal{L}(B),$$
$$(4) \quad \mathcal{L}(\mathcal{L}(A)) = \mathcal{L}(A).$$

A function $\mathcal{L}: \mathcal{P} \rightarrow \mathcal{P}$ is called a closure operation if it satisfies (2)-(4).

Conversely, if $\mathcal{L}$ is an arbitrary closure operation on an $n$-element groundset $X$, then there is an $m \times n$ matrix $M$ such that $\mathcal{L}_M = \mathcal{L}$ [1, 5, 6]. We say in this case that $M$ represents $\mathcal{L}$. The definition of our main target is the following:

$$(5) \quad s(\mathcal{L}) = \min \{m: M \text{ is an } m \times n \text{ matrix}, \mathcal{L}_M = \mathcal{L}\}.$$ 

A closure operation determines an important class of subsets, the class of keys. $K$ is a key in $\mathcal{L}$ if $\mathcal{L}(K) = X$. $\mathcal{K} = \mathcal{K}(\mathcal{L})$ denotes the family of minimal keys ($K$ is a minimal key if it is a key but no proper subset of $K$ is a key). It is obvious that $K_1$, $K_2 \in \mathcal{K}$, $K_1 \neq K_2$ imply $K_1 \not\subseteq K_2$. Families of subsets satisfying this condition are called Sperner-families. Hence $\mathcal{K}$ is a Sperner-family. We say that a matrix $M$ represents a given Sperner-family $\mathcal{K}$ if $\mathcal{K} = \mathcal{K}(\mathcal{L}_M)$ holds. The maximal non-keys are called antikeys. Their family is denoted by $\mathcal{K}^{-1} =\{A: \exists B \in \mathcal{K}, B \subseteq A, A \text{ is maximal for this property}\}$.

Lemma 1. $M$ represents the Sperner-family $\mathcal{K}$ iff for any $A \in \mathcal{K}^{-1}$ $M$ has two different rows having the same entries in the columns in $A$ and any two rows equal in a $K \in \mathcal{K}$ are equal everywhere.

Proof. If $M$ represents $\mathcal{K}$, then $\mathcal{K} = \mathcal{K}(\mathcal{L}_M)$ holds.

$K \in \mathcal{K}$ implies $\mathcal{L}_M(K) = X$, the second condition obviously follows. Similarly, $A \in \mathcal{K}^{-1}$ implies $\mathcal{L}_M(A) \neq X$ and hence we obtain the first condition.

Conversely, if both conditions are satisfied for $M$ and $\mathcal{K}$, then (i) $\mathcal{L}_M(A) \neq X$ holds for any $A \in \mathcal{K}^{-1}$ and (ii) $\mathcal{L}_M(K) = X$ holds for any $K \in \mathcal{K}$.

(ii) and (3) imply that $\mathcal{L}_M(C) = X$ if $C \supseteq K$ for some $K \in \mathcal{K}$. Suppose now that $C$ is not a superset of a member of $\mathcal{K}$. Then, by definition there is an $A \in \mathcal{K}^{-1}$ such that $C \subseteq A$. (i) and (3) imply $\mathcal{L}_M(C) \neq X$. That is, $\mathcal{L}_M(C) = X$ exactly for the supersets of members of $\mathcal{K}$. The proof is complete. $\square$
The following definition is an analogue of (5):

\( s(\mathcal{X}) = \min \{ m : \mathcal{M} \text{ is an } m \times n \text{ matrix representing } \mathcal{X} \} \)

where \( \mathcal{X} \) is a Sperner-family on an \( n \)-element set.

The \( k \)-uniform closure operation on an \( n \)-element groundset \( X \) is defined by

\[
\mathcal{Y}_k(A) = \begin{cases} X, & \text{if } |A| \geq k, \\ A, & \text{if } |A| < k. \end{cases}
\]

The family of all \( k \)-element subsets of \( X \) is denoted by \( \binom{X}{k} \). In general, there exist more than one closure operation with the same system \( \mathcal{X} \) of minimal keys \( \mathcal{X} = x(\mathcal{Y}) \). The next Lemma states that if \( \mathcal{X} \) is the family of all \( k \)-element subsets of \( X \), then \( \mathcal{Y} \) is uniquely determined by \( \mathcal{X} = x(\mathcal{Y}) \).

**Lemma 2.** Let any closure operation \( \mathcal{Y} \) be defined on an \( n \)-element set \( X \). Then

\[ x(\mathcal{Y}) = \binom{X}{k} \iff \mathcal{Y} = \mathcal{Y}_k. \]

**Proof.** \( x(\mathcal{Y}_k) = \binom{X}{k} \) is trivial. We have to verify the converse statement only. Suppose that \( a \in \mathcal{Y}(A) - A \) for some \( A \subseteq X \) such that \( |A| < k \). Then one can find a set \( B \) satisfying \( |B| = k \), \( B \supseteq A \cup \{ a \} \). (2) implies \( \mathcal{Y}(B - a) \supseteq B - a \); (3) implies \( \mathcal{Y}(B - a) \supseteq \mathcal{Y}(A) \cup a \). Hence \( \mathcal{Y}(B - a) \supseteq \mathcal{Y}(B) \) follows. We obtain \( \mathcal{Y}(B - a) = \mathcal{Y}(B) = X \) by (4) and (3). Consequently, there is a set \( B - a \) of cardinality \( < k \) with closure \( X \). This contradiction shows that \( a \in \mathcal{Y}(A) - A \) cannot exist if \( |A| < k \); \( \mathcal{Y}(A) = A \). \( \mathcal{Y}(A) = X \) for \( |A| \geq k \) easily follows from \( x(\mathcal{Y}) = \binom{X}{k} \) and (3).

3. Minimum representation of uniform closure operations

First we repeat some results of [9]. We prove these statements for sake of completeness.

**Lemma 3.** If an \( m \times n \) matrix \( M \) represents \( \mathcal{X} \), then

\[ \binom{m}{2} \geq |\mathcal{X}^{-1}|. \]

**Proof.** If \( A \in \mathcal{X}^{-1} \), then by Lemma 1 there are different rows \( i, j \) such that they are equal in \( A \). Take another member \( B \) of \( \mathcal{X}^{-1} \). Let the corresponding pair of rows be \( i' \) and \( j' \). If the unordered pairs \( \{i, j\}, \{i', j'\} \) are equal, then these two different rows are equal in \( A \cup B \). Consequently, \( \mathcal{Y}(A \cup B) \neq X \) and there is a \( C \subseteq A \cup B \) with \( C \in \mathcal{X}^{-1} \). By the definition of \( \mathcal{X}^{-1} \) this is possible only when \( C = A \) and \( C = B \), con-
J. Demetrovics et al.

Contradicting our supposition \( A \neq B \). Therefore to different members of \( \mathcal{X}^{-1} \) we have different pairs of rows satisfying the above condition. The number of pairs of rows of \( M \) must be \( \geq |\mathcal{X}^{-1}| \). The lemma is proved. \( \square \)

**Lemma 4.**

\[
\binom{s(\mathcal{X}^n)}{2} \geq \binom{n}{k-1}.
\]

**Proof.** Let \( M \) be an \( s(\mathcal{X}^n) \times n \) matrix representing \( \mathcal{X}^n \). By Lemma 2, \( M \) also represents \( \mathcal{X}(\mathcal{Y}^n) = (\mathcal{X}^n) \). It is easy to see that \( \mathcal{X}^{-1}(\mathcal{Y}^n) = (\mathcal{X}^{-1}) \). Then (8) follows by Lemma 3. \( \square \)

We will see that (8) gives a fairly good lower estimate on \( s(\mathcal{X}^n) \). It is sharp for \( k = 1, 2, n - 1 \). It seems to be sharp for \( k = 3 \) and \( n \geq 7 \). On the other hand it is sharp up to a constant depending on \( k \) for any fixed \( k \) when \( n \to \infty \). The case \( k = n \) needs another lemma. If \( M \) is an \( m \times n \) matrix let \( G(M) \) denote the graph whose vertices are the rows of \( M \), two vertices are connected with an edge iff the set \( A \) of columns where the two rows are equal is non-empty. The edge is labeled by \( A \).

**Lemma 5.** Let \( M \) be a matrix and let \( A_1, \ldots, A_r \) be the labels along a circuit of \( G(M) \). Then

\[
\left( \bigcap_{i=1}^{r} A_i \right) - A_j = \emptyset \quad (1 \leq j \leq r).
\]

**Proof.** Suppose that, on the contrary, (9) is non-empty, that is, there is a column, say the \( u \)th one, which is an element of all \( A_i \) but \( A_j \). Let the vertices of the circuit be \( k_1, \ldots, k_r \) in such a way that the edge \((k_i, k_{i+1})\) is labelled by \( A_i \) \((1 \leq i < r)\) and \((k_r, k_1)\) is labelled by \( A_r \). From \( u \in A_{j+1} \) it follows that the \( k_{j+1} \)th and \( k_{j+2} \)th entries of the \( u \)th column are equal. The same holds for the \( k_{j+2} \)th and \( k_{j+3} \)th entries, etc. Consequently, the \( k_{j+1} \)th, \( k_{j+2} \)th, \( k_{j+3} \)th, \( k_{j+4} \)th entries in the \( u \)th column are all equal. This leads to \( u \in A_j \) contradicting the assumption. The proof is complete. \( \square \)

**Theorem 1** \([9]\).

\[
\begin{align*}
\text{s}(\mathcal{X}^2) &= 2, & \text{s}(\mathcal{X}_2^n) &= \left\lceil (1 + \sqrt{1 + 8n})/2 \right\rceil, \\
\text{s}(\mathcal{Y}^n) &= n, & \text{s}(\mathcal{Y}_2^n) &= n + 1
\end{align*}
\]

where \( \lceil x \rceil \) denotes the smallest integer \( \geq x \).

**Proof.** By Lemma 2, \( s(\mathcal{Y}_2^n) = s(\mathcal{X}(\mathcal{Y}_2^n)) = s(\mathcal{X}_2^n) \). We use the last form in the proof.
For $k = 1$, (8) gives $s(\mathcal{Y}_1^n) \geq 2$. The construction
\[
\begin{array}{ccccccc}
0 & 0 & \ldots & 0 \\
1 & 1 & \ldots & 1
\end{array}
\]
proves the equality.

For $k = 2$, (8) gives
\[
\binom{s(\mathcal{Y}_2^n)}{2} \geq n.
\]
Suppose that $s(\mathcal{Y}_2^n)$ satisfies (10); we construct an $s(\mathcal{Y}_2^n) \times n$ matrix $M$ representing $(\mathcal{Y}_2)$. Any column of $M$ contains exactly two zeros. Different columns contain different pairs of zeros. The other entries of the $i$th row are $i (1 \leq i \leq s(\mathcal{Y}_2^n))$. It is easy to see, using Lemma 1, that $M$ represents $(\mathcal{Y}_2)$. The least integer satisfying (10) can be expressed in the form given in the theorem.

For $k = n - 1$, (8) gives $s(\mathcal{Y}_{n-1}^n) \geq n$. The construction
\[
\begin{array}{ccccccc}
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{array}
\]
gives the equality.

Finally, suppose that the $m \times n$ matrix $M$ realizes $(\mathcal{Y}_n^m) = \{X\}$. By Lemma 1 there is an edge in $G(M)$ labelled with $A$ for any $(n - 1)$-element subset of $X$. $G(M)$ has $n$ different edges of this kind. These edges cannot form a circuit because the $(n - 1)$-element subsets cannot satisfy (9), the lemma is applicable. $G(M)$ has at least $n + 1$ vertices: $s(\mathcal{Y}_n^n) \geq n + 1$. The following construction gives equality
\[
\begin{array}{ccccccc}
0 & 0 & \ldots & 0 \\
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{array}
\]
The proof is complete. \qed

Substituting $k = 3$ into (8) we obtain
\[
s(\mathcal{Y}_3^3) \geq n.
\]
$s(\mathcal{Y}_3^3) = 4 > 3$ is proved and $s(\mathcal{Y}_3^5) > 6$ can be verified by checking all the cases. We conjecture that the above inequality is sharp for all other cases:

**Conjecture 1.** $s(\mathcal{Y}_3^n) = n$ for $n \geq 7$.

We are able to reduce this conjecture in the case $n = 3r + 1$ for another conjecture concerning a certain kind of resolvable Steiner triple systems:
**Conjecture 2.** There is a system of 3-element subsets of an $n (= 3r + 1)$-element set $\{1, 2, \ldots, n\}$ satisfying the following conditions:

1. Any pair of elements is contained in exactly two 3-sets.
2. The family of 3-sets can be divided into $n$ subfamilies where the $i$th subfamily is a partition of $\{1, 2, \ldots, n\} - \{i\}$.
3. Exactly one pair of members of two different subfamilies meet in 2 elements.

We show the construction of an $n \times n$ matrix $M$ representing $\mathcal{F}_3^n (n = 3r + 1)$ if the family in Conjecture 2 exists. We write zeros in the main diagonal. The $i$th row $j$th entry will be 1 if $i$ is an element of the $i$th triple in the $j$th sub-family. It follows by condition (3) that for any two columns of $M$ there are two rows equal in these columns. The rows are, of course, different due to the zeros. The first condition of Lemma 1 is satisfied. Condition (1) implies that any two rows agree in exactly two entries. Hence there are no two rows equal in any given triple of columns. The second condition of Lemma 1 is also satisfied. $M$ represents $\mathcal{F}_3^n$, indeed.

Conjecture 2 follows for $n \equiv 1$ or $4 \pmod{12}$ from the following result of Hanani [11, 12]. There exists a Steiner system $\mathcal{S}(4, 2, n)$ for these $n$'s. (I.e. we have a 4-uniform subsystem $\mathcal{S}$ on $n$-element set $V$ such that for every two $u_1, u_2 \in V$ there exists exactly one member $S \in \mathcal{S}$ such that $\{u_1, u_2\} \subseteq S$.) Consider the 4-uniform set-system $\mathcal{F}$ over $\{1, 2, \ldots, n\}$ and replace every member $S \in \mathcal{F}$ with 4 3-element subsets. The obtained set-system $\mathcal{F}$ meets the condition of Conjecture 2, where the $i$th subfamily $\mathcal{F}_i = \{S - \{i\} : i \in S \in \mathcal{F}\}$.

We have proved the following.

**Theorem 2.**

1. $s(\mathcal{F}_3^n) \geq n$.
2. $s(\mathcal{F}_3^n) = n$ for $n = 12k + 1$ and $n = 12k + 4$.

**Corollary 1.** $n \geq s(\mathcal{F}_3^n) \geq n + 8$.

**Proof.** It follows from Theorem 2 and the inequality $s(\mathcal{F}_3^n) \leq s(\mathcal{F}_3^n+1)$. □

**Remark.** In Theorem 2 we could have proved the following stronger statement: For $n \equiv 1$ or $4 \pmod{12}$ we have a partition $G_1, G_2, \ldots G_n$ of the edge-set of complete directed graph $\overline{K}_n$ ($\overline{K}_n = \{u, v\} : 1 \leq u \neq v \leq n\}$, so it has $n(n - 1)$ oriented edges) such that $G_i$ consists of $(n - 1)/3$ pairwise vertex-disjoint oriented triangles on the points $\{1, 2, \ldots, n\} - \{i\}$ and $G_i \cup G_j (i \neq j)$ contains exactly one pair of oppositely oriented edges.

**Conjecture 2'.** The above-mentioned statement about the complete directed graph $\overline{K}_n$ holds for every $n \equiv 1 \pmod{3}$. 
This conjecture is much stronger than the usual statements about resolvable block-designs (cf. Hanani [12]).

Let us show the constructions for \( n = 4 \) and \( 7 \):

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 1 & 0 & 2 & 2 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 2 & 1 & 2 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 2 & 1 & 2 \\
1 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 2 & 1 \\
& 2 & 1 & 2 & 1 & 0 & 1 & 2 \\
& 2 & 1 & 1 & 2 & 2 & 0 & 1 \\
& 2 & 2 & 1 & 1 & 1 & 2 & 0 \\
\end{array}
\]

**Theorem 3** [10].

\[
\frac{\sqrt{2}}{k-1} \left( \frac{1}{k-1} \right)^{(k-1)/2} n^{(k-1)/2} < s \left( \gamma_k^n \right) < 2^{3k/2} n^{(k-1)/2} \ (2 \leq k \leq n).
\]

**Proof.** The left-hand side of (13) is an easy consequence of (8). We now give a construction proving the right-hand side.

Let \( p \) be a prime number. We show that there is a set \( D \) of cardinality \( 2 \left\lfloor \sqrt{p} \right\rfloor \) such that any integer satisfies

\[
i \equiv d_1 - d_2 \pmod{p}
\]

for some elements \( d_1, d_2 \) of \( D \). The set \( D \) is defined by

\[
D = \{0, 1, 2, \ldots, a-1, 2a, 3a, \ldots, (a-1)a\}
\]

where \( a = \left\lfloor \sqrt{p} \right\rfloor \). Suppose that \( i \) satisfies \( 0 \leq i < p \) and \( i = al + r \quad (0 \leq r < a) \). If \( 1 \leq i \leq a-2 \) and \( 0 < r < a \), then \( d_1 = i + 1 \) and \( d_2 = a - r \) satisfy (14). If \( i = al \) \((2 \leq l \leq a-1)\), then \( d_1 = al \) and \( d_2 = 0 \) are suitable. \( a = 3a - 2a \) and the rest can be expressed as a difference of zero and one of the numbers \( 1, 2, \ldots, a-1 \). (We used \( 3 \leq a - 1 \). Otherwise we have \( p \geq 9 \). These cases can be checked separately.)

As regards the cardinality of \( D \) we have

\[
|D| = 2a - 2 = 2 \left( \left\lfloor \sqrt{p} \right\rfloor - 1 \right) = 2 \left\lfloor \sqrt{p} \right\rfloor.
\]

Let \( \mathcal{P} \) be defined in the following way:

\[
\mathcal{P} = \{c_{k-1}x^{k-1} + c_{k-2}x^{k-2} + \ldots + c_1x + c_0 \mid c_k, \ldots, c_{k-1} \in D, c_{k-1} = 0 \text{ or } 1\}.
\]

Note that

\[
|\mathcal{P}| = 2^{k-1} \left\lfloor \sqrt{p} \right\rfloor^{k-1}.
\]

Let \( M \) be a \( |\mathcal{P}| \times p \) matrix. Its rows are associated with members of \( \mathcal{P} \). The \( j \)-th entry of the row associated with \( z(x) \in \mathcal{P} \) is \( z(j) \pmod{p} \) \((0 \leq j \leq p-1, 0 \leq z(j) \leq p-1)\). We prove now that \( M \) represents \( \gamma_k^n \). It is sufficient to prove, by Lemma 2, that \( M \)
represents \( (x^k) \) (where \( |X| = p \)). Here we may use Lemma 1; we have to verify its conditions with \( X = (x^k) \), \( X^{-1} = (x^{-1}) \) only.

Suppose that the rows associated with \( z_1(x) \) and \( z_2(x) \) have \( k \) equal entries:

\[ z_1(t_i) \equiv z_2(t_i) \pmod{p} \quad (0 \leq t_1 < \ldots < t_k < p). \]

Then the polynomial \( z_1(x) - z_2(x) \) of degree \( \leq k - 1 \) has \( k \) different roots. This contradiction proves that \( z_1 \) and \( z_2 \) are the same, the 'two' rows are only one.

Choose now the integers \( 0 \leq t_1 < \ldots < t_{k-1} < p \) arbitrarily. We have to find two different rows with equal entries in the \( t_1 \)st, \( t_2 \)nd, \ldots, \( t_{k-1} \)st places. Consider the polynomial

\[ w(x) = (x-t_1)(x-t_2)\ldots(x-t_{k-1}) = x^{k-1} + a_{k-2}x^{k-2} + \ldots + a_1x + a_0. \]

To \( a_i \) \((0 < i < k - 2)\) we can find two elements \( c_i \) and \( c_i' \) of \( D \) such that \( a_i \equiv c_i - c_i' \pmod{p} \). Then \( w(x) = z(x) - z'(x) \) holds where

\[ z(x) = x^{k-1} + c_{k-2}x^{k-2} + \ldots + c_1x + c_0 \quad \text{and} \]

\[ z'(x) = c_{k-2}'x^{k-2} + \ldots + c_1'x + c_0'. \]

\( z(x) \) and \( z'(x) \) are obviously different elements of \( \mathcal{Y} \). On the other hand, \( z(t_i) \equiv z'(t_i) \pmod{p} \) holds, indeed. Both conditions of Lemma 1 are verified. \( M \) represents \( \mathcal{Y}_k^p \), indeed. This proves

\[ s(\mathcal{Y}_k^p) \leq 2^k p^{(k-1)/2}. \]

For arbitrary \( n \) we choose a prime number \( p \) satisfying \( n \leq p \leq 2n \). It exists by Chebyshev's theorem. Then we construct a matrix representing \( \mathcal{Y}_k^p \) and omit \( p - n \) columns. The matrix represents \( \mathcal{Y}_k^n \). Hence

\[ s(\mathcal{Y}_k^n) \leq 2^k p^{(k-1)/2} \leq 2^k (2n)^{(k-1)/2} \leq 2^{k/2} n^{(k-1)/2}. \]

The theorem is proved.

The method of Theorem 3 gives a good estimate only for small \( k \). For instance, for \( k = n/2 \) a much better estimate is known. It is proved in [6] that

\[ s(x) \leq |x^{-1}| + 1 \]

holds for any Sperner-family. The following matrix proves it. Let the 0th row consist of zeros while the \( i \)th \((1 \leq i \leq |x^{-1}|)\) row contains zeros and \( i \)'s: zeros in the column corresponding to the elements of the \( i \)th member of \( \mathcal{Y}^{-1} \). By (15),

\[ s(\mathcal{Y}_{n/2}^{n/2}) = s\left( \binom{n}{n/2} \right) \leq \binom{n}{n/2} + 1 = 2^n + o(n) \]

follows. Our feeling is that the truth is closer to the lower estimate given by (8):

**Conjecture 3.** \( \log_2 s(\mathcal{Y}_{n/2}^{n/2}) = n/2 + o(n) \).
For comparison let us quote another related result [7, 8] stating

\[ s(\mathcal{U}) \leq (1 + o(1)) \binom{n}{n/2} \]

for any closure operation \( \mathcal{U} \) on \( n \) elements.

4. Direct products

Let \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \) be closure operations on the disjoint ground sets \( X_1 \) and \( X_2 \), resp. The direct product \( \mathcal{U}_1 \times \mathcal{U}_2 \) is defined by

\[ (\mathcal{U}_1 \times \mathcal{U}_2)(A) = \mathcal{U}_1(A \cap X_1) \cup \mathcal{U}_2(A \cap X_2), \quad A \subseteq X_1 \cup X_2. \]

We prove the following, perhaps surprising

**Theorem 4.** \( s(\mathcal{U}_1 \times \mathcal{U}_2) = s(\mathcal{U}_1) + s(\mathcal{U}_2) - 1 \).

**Proof.** (1) Let us first prove the inequality

\[ s(\mathcal{U}_1 \times \mathcal{U}_2) \leq s(\mathcal{U}_1) + s(\mathcal{U}_2) - 1 \]

with a construction. Let the \( s(\mathcal{U}_1) \times n_1 \) matrix \( M_1 \) and the \( s(\mathcal{U}_2) \times n_2 \) matrix \( M_2 \) represent \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \), resp. We denote by \( \alpha \) the last row of \( M_1 \) and by \( \beta \) the first row of \( M_2 \). The matrix \( M \) is constructed in Fig. 1.

\[
\begin{array}{c|c|c|c|c|c}
 & X_1 & & X_2 & & \\
\hline
M_1 & \beta & \vdots & \beta & & \\
\hline
\alpha & \vdots & \beta & \vdots & & \\
\hline
\alpha & \vdots & \beta & \vdots & & \\
\hline
\end{array}
\]

Fig. 1.

\( M \) is an \( (s(\mathcal{U}_1) + s(\mathcal{U}_2) - 1) \times (n_1 + n_2) \) matrix. We have to show that it represents \( \mathcal{U}_1 \times \mathcal{U}_2 \), that is,

\[ a \in \mathcal{U}_M(A) \iff a \in \mathcal{U}_1(A \cap X_1) \cup \mathcal{U}_2(A \cap X_2). \]

We may assume \( a \in X_1 \) because of the symmetry. Then the above condition can be divided into two implications:

\[ a \in \mathcal{U}_1(A \cap X_1) \iff \text{any two rows of } M \text{ equal in } A \text{ are equal in } a. \]

\[ a \in \mathcal{U}_1(A \cap X_1) \iff M \text{ has two rows equal in } A \text{ but different in } a. \]
To prove (17) suppose that \( a \in \mathcal{U}_1(A \cap X_1) \) choose two rows of \( M \) with equal entries in \( A \). If both of them start with \( a \), they are equal in \( a \). If one of them does not start with \( a \) then the first parts of these rows are two different rows of \( M_1 \). By the definition of \( M_1 \), if they are equal in \( A \cap X_1 \) then they are equal in \( a \).

To prove (18) suppose that \( a \notin \mathcal{U}_1(A \cap X_1) \). \( M_1 \) contains two rows equal in \( A \cap X_1 \) but different in \( a \). The extensions of these rows in \( M \) satisfy the right hand side of (18).

\( M \) represents \( \mathcal{U}_1 \times \mathcal{U}_2 \), consequently (16) is proved.

(2) With the help of two lemmas we prove now the inequality

\[
(19) \quad s(\mathcal{Y}_1 \times \mathcal{Y}_2) \geq s(\mathcal{Y}_1) + s(\mathcal{Y}_2) - 1.
\]

Let \( M \) be a matrix representing \( \mathcal{U}_1 \times \mathcal{U}_2 \) and suppose that the first \( n_1 \) columns correspond to the groundset \( X_1 \) of \( \mathcal{Y}_1 \) and the remaining \( n_2 \) columns correspond to the groundset \( X_2 \) of \( \mathcal{Y}_2 \). We want to prove that the number of rows of \( M \) is at least \( s(\mathcal{Y}_1) + s(\mathcal{Y}_2) - 1 \). The submatrix determined by the first \( n_1 \) columns in \( M \) is denoted by \( M_1 \). The rest is denoted by \( M_2 \).

Lemma 6. \( \mathcal{Y}_{M_1} = \mathcal{Y}_1 \).

**Proof.** Suppose that \( A \subseteq X_2 \), \( a \in X_2 \) and \( a \in \mathcal{Y}_2(A) \). Hence \( a \in (\mathcal{Y}_1 \times \mathcal{Y}_2)(A) \) follows.

If two rows of \( M \) are equal in \( A \) then, by the definition of \( M \), they are also equal in \( a \). Of course, this remains true if we consider the submatrix \( M_2 \) only. That is, we have proved \( a \in \mathcal{Y}_{M_2}(A) \). Conversely, suppose now \( A \subseteq X_2 \), \( a \in X_2 \) and \( a \notin \mathcal{Y}_2(A) \). Since \( a \notin (\mathcal{Y}_1 \times \mathcal{Y}_2)(A) \) follows, \( M \) has two rows equal in \( A \) and different in \( a \). These two rows in \( M_2 \) prove \( a \notin \mathcal{Y}_{M_2}(A) \). The proof is complete. \( \square \)

\( \mathcal{Y}_{M_1} = \mathcal{Y}_1 \) follows analogously. However, we need a somewhat stronger statement for \( M_1 \):

Lemma 7. Let \( N \) be a matrix. Suppose that the set of rows of \( N \) can be partitioned into \( k \) classes such that whenever \( a \notin \mathcal{Y}_N(A) \) holds, then there are two rows in one class which are equal in \( A \) and different in \( a \). Then

\[
(20) \quad (\text{number of rows of } N) \geq s(\mathcal{Y}_N) + k - 1.
\]

**Proof.** We use induction on \( k \). For \( k = 1 \), (20) follows by the definitions of \( s(\mathcal{Y}_N) \). Suppose now that \( N \) is partitioned into \( k \geq 2 \) classes satisfying the conditions of the lemma and that the statement is proved for smaller values.

\( \mathcal{Y}_N \) depends only on the relationship of the entries in \( N \): which ones are equal, which ones are different. Therefore we may suppose that \( N \) contains only positive integers.

If \( N \) has one or more columns with the same entry everywhere, then delete these columns and denote the new matrix by \( N_1 \). The partition of rows of \( N \) is a 'good'
Minimum matrix representation of closure operations

(21) \( s(\mathcal{Y}_N) = s(\mathcal{Y}_N) \)

obviously holds. Moreover \( \mathcal{Y}_N(\emptyset) = \emptyset \).

Let \( p_1, \ldots, p_k \) be different prime numbers greater than any entry of \( N_1 \). Multiply all entries in the \( i \)th class of rows by \( p_i (1 \leq i \leq k) \). The new matrix is denoted by \( N_2 \).

It is easy to see that

(22) \( \mathcal{Y}_N = \mathcal{Y}_N \)

(since \( \mathcal{Y}_N(\emptyset) = \emptyset \) and \( N_2 \) contains no equal entries in different classes of rows.

Let \( y = (y_1, \ldots, y_u) \) and \( \delta = (\delta_1, \ldots, \delta_u) \) be one of the rows of the first and second classes in \( N_2 \), resp. We now delete \( y \) and change any \( y_i \) for \( \delta_i \) in the \( i \)th column (for all \( i, 1 \leq i \leq u \)). The new matrix is denoted by \( N_3 \). The number of its rows is equal to the number of rows of \( N_2 \) minus 1. Let us prove that

(23) \( \mathcal{Y}_N = \mathcal{Y}_N \)

Suppose first that \( a \in \mathcal{Y}_N(A) \) and choose two rows, \( \mu_3 \) and \( \nu_3 \) of \( N_3 \) equal in \( A \). The corresponding rows in \( N_2 \) are denoted by \( \mu_2 \) and \( \nu_2 \), resp. If \( \mu_2 \) and \( \nu_2 \) are equal, we are still in the same class but not in the first class in \( N_2 \), then \( \mu_2 = \mu_3, \nu_2 = \nu_3 \). Therefore \( \mu_2 \) and \( \nu_2 \) are equal in \( A \); \( a \in \mathcal{Y}_N(A) \) implies that they are equal in \( a \). The same holds for \( \mu_3 \) and \( \nu_3 \). \( a \in \mathcal{Y}_N(A) \) is proved. If \( \mu_2 \) and \( \nu_2 \) are both in the first class, then \( \mu_2 \) and \( \mu_3 \) differ only in the sense that \( y_i \) is changed for \( \delta_i \) everywhere. The same holds for \( \nu_2 \) and \( \nu_3 \). It follows that \( \mu_2 \) and \( \nu_2 \) are equal in \( A \), consequently in \( a \). We obtained that \( \mu_3 \) and \( \nu_3 \) are also equal in \( a : a \in \mathcal{Y}_N(A) \). The last case is when \( \mu_2 \) and \( \nu_2 \) are in different classes. The supposition that \( \mu_3 \) and \( \nu_3 \) are equal in \( A \) implies either \( A = \emptyset \) or that \( \mu_2 \) and \( \nu_2 \) are in the first and second classes, resp. \( A = \emptyset \) is excluded by \( \mathcal{Y}_N(\emptyset) = \emptyset \). We may conclude that \( \mu_2 \) is in the first class, \( \nu_2 \) in the second one and they are different from \( y \) and \( \delta \). \( \nu_3 = \nu_1 \) is obvious. Since \( \mu_3 \) and \( \nu_3 \) are equal in \( A \) they both must contain \( \delta_i \) in the \( i \)th place if \( i \in A \). Then \( \nu_3 = \nu_1 \) and \( \delta \) are equal in \( A \), consequently they are also equal in \( a \). Their common entry here is \( \delta_u \). If \( \mu_3 \) contains \( \delta_i \) in the \( i \)th place when \( i \in A \), then \( \mu_3 \) contains \( y_i \) there. Consequently, \( \mu_2 \) and \( \nu_2 \) are equal in \( A \) and hence they are equal in \( a \). Their common entry here is \( y_u \). We obtained that \( \mu_3 \) contains \( \delta_u \) in this column. Hence \( \mu_3 \) and \( \nu_3 \) are equal in \( a : a \in \mathcal{Y}_N(A) \).

Suppose now that \( a \in \mathcal{Y}_N(A) \). \( N_2 \) contains two rows equal in \( A \) and different in \( a \). If \( A \neq \emptyset \), then the two rows are in the same class, consequently the corresponding rows in \( N_2 \) are also equal in \( A \) and different in \( a \). \( a \in \mathcal{Y}_N(A) \) follows. If \( A = \emptyset \), \( a \in \mathcal{Y}_N(\emptyset) \) would mean that there is a column with equal entries. This is impossible for \( k \geq 3 \). It is possible for \( k = 2 \) only when \( N_2 \) contains merely \( y_u \) and \( \delta_u \) in the column corresponding to \( u \). However in this case we are not able to find two rows in one class satisfying the conditions of the lemma for \( a \in \mathcal{Y}_N(\emptyset) \). This contradiction proves \( a \in \mathcal{Y}_N(A) \) and (23).

Moreover the conditions of the lemma are satisfied with at least \( k - 1 \) classes for
\[ N_j \]. Therefore we may use the induction hypothesis:
\[
\text{(number of rows of } N_j) \geq s(Y_{n_j}) + k - 2.
\]
Hence we obtain
\[
\text{(number of rows of } N) \geq s(Y_{n_i}) + k - 1
\]
by (21), (22) and (23). The lemma is proved. \( \Box \)

Let us turn back to the proof of the theorem, that is, more exactly, of (19). Form a partition of the rows of \( M_1 \) putting two rows in one class if their extension in \( M_2 \) is equal. Our aim is to apply Lemma 7 for \( M_1 \). We know that \( Y_{M_1} = Y_i \) by Lemma 6. Choose \( a \) and \( A \) so that \( a \notin Y_i(A) = Y_i(A) \). Then \( a \notin Y_i(U \cup X_2) = (Y_i \cup Y_2)(U \cup X_2) \) holds and \( M \) contains two rows equal in \( U \cup X_2 \) but different in \( a \). That is, there are two rows of \( M_1 \) equal in \( A \), different in \( a \) and being in the same class of the partition. We may apply Lemma 7 for \( M_1 \):
\[
(24) \quad (\text{number of rows of } M_1) \geq s(Y_i) + (\text{number of different rows of } M_2) - 1.
\]
Using Lemma 6, again, we obtain
\[
(25) \quad (\text{number of different rows of } M_2) \geq s(Y_2).
\]
(24) and (25) result in
\[
(\text{number of rows of } M) \geq s(Y_i) + s(Y_2) - 1.
\]
This proves (19) and the theorem. \( \Box \)

The analogous question for Sperner-families as minimal keys is not really answered. If \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are Sperner-families on the disjoint sets \( X_1 \) and \( X_2 \), resp., then \( \mathcal{F}_1 \times \mathcal{F}_2 \) is defined as the family \( \{ A \cup B : A \in \mathcal{F}_1 \} \) of subsets of \( X_1 \cup X_2 \). The proof of (16) works also here:

**Theorem 5.** \( s(\mathcal{F}_1 \times \mathcal{F}_2) \leq s(\mathcal{F}_1) + s(\mathcal{F}_2) - 1. \)

We found equality in many particular cases but it is not true in general, as the following example shows: let \( X_1 = \{1,2,3,4,5\} \), \( X_2 = \{6,7,8,9,10\} \), \( \mathcal{F}_1 = \{\{1,2\}, \{3,4\}, \{1,5\}, \{2,5\}, \{3,5\}, \{4,5\}\} \), \( \mathcal{F}_2 = \{\{6,7\}, \{8,9\}, \{6,10\}, \{7,10\}, \{8,10\}, \{9,10\}\} \). We show first \( s(\mathcal{F}_1) = s(\mathcal{F}_2) = 5 \). It is easy to see that
\[
\mathcal{F}_1^{-1} = \{\{5\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}.
\]
Lemma 3 implies \( s(\mathcal{F}_1) \geq 4 \). Suppose that \( s(\mathcal{F}_1) = 4 \) and the matrix \( M \) realizes it. \( G(M) \) (see Lemma 5) has 4 vertices and its 5 edges are labelled with the members of \( \mathcal{F}_1^{-1} \). Distinguishing several cases one can see that Lemma 5 implies that the sixth edge is labelled with \( \{1,2\}, \{1,2,5\}, \{3,4\} \) or \( \{3,4,5\} \). This contradicts the supposition that the key-set of \( M \) is \( \mathcal{F}_1 \). This proves \( s(\mathcal{F}_1) \geq 5 \). On the other hand, the
The following matrix shows \( s(X_1 \times X_2) \leq 8 \):

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 2 \\
1 & 0 & 1 & 2
\end{array}
\]

(non-trivial!)

5. Closure relations with large \( s(\mathcal{C}) \)

In [7] and [8] it is proved that there is an \( \mathcal{C} \) satisfying

\[
s(\mathcal{C}) \geq s(\mathcal{X}(\mathcal{C})) \geq \frac{1}{n^2} \binom{n}{\lfloor n/2 \rfloor}.
\]

However, the proof is non-constructive; we are not able to find one with \( s(\mathcal{C}) \) more than \( \sqrt{2} \binom{n}{\lfloor n/2 \rfloor}^{1/2} \).

We pose here the analogous question for \( \mathcal{X} \subseteq \mathcal{C} \). Let

\[
f_k(n) = \max \{ s(\mathcal{X}) : \mathcal{X} \subseteq \mathcal{C}, |\mathcal{X}| = n \}.
\]

**Theorem 6.**

\[
f_k(n) \geq \sqrt{2} \left( \frac{2k-2}{k-1} \right)^{\lfloor n/(2k-2) \rfloor^{1/2}}
\]

**Proof.** Take a partition \( X = X_1 \cup \ldots \cup X_q \cup Y \), where \( q = \lfloor n/(2k-2) \rfloor \) and \( |X_i| = 2k-2 \) (1 \( i \leq q \)). \( \mathcal{X} \) is defined by \( \mathcal{X} = \{ K : |K| = k, K \subseteq X_i \text{ for some } i \} \). It is easy to see that

\[
\mathcal{X}^{-1} = \{ A : |A \cap X_i| = k-1 \text{ for all } i, |A \cap Y| = |Y| \}.
\]

Hence

\[
|\mathcal{X}^{-1}| = \left( \frac{2k-2}{k-1} \right)^{\lfloor n/(2k-2) \rfloor}
\]

follows and the theorem can be obtained by Lemma 3. \( \square \)

It is easy to see that \( f_1(n) = 2 \). Theorem 6 gives \( f_2(n) \geq \sqrt{2} 2^{\lfloor n/2 \rfloor^{1/2}} > 2^{n/4} \). It is surprising that such a 'simple' construction can have a big \( s(\mathcal{X}) \). However, we do not know the correct order of magnitude of \( f_2(n) \).
6. Open problems

Besides Conjectures 1–3 we would like to pose some other related problems:

**Problem 1.** Give sufficient conditions for equality in Theorem 5.

**Problem 2.** Give methods for lower estimates of $s(\mathcal{V})$ and $s(\mathcal{X})$ deeper than Lemmas 3, 4 and 5.

**Problem 3.** Determine $\max\{|x^{-1}| : \mathcal{X} \subseteq (\mathcal{X})^*, |X| = n\}$.

If $k = 2$, then $\mathcal{X}$ is a graph on $n$ vertices. $x^{-1}$ is the family of all maximal vertex-sets containing no edge of this graph. The problem asks for what graph is this family the largest. Moon and Moser [13] solved this graph-theoretical question. Our problem is its analogue for hypergraphs.

**References**


