The Norm of a Linear Functional with Respect to a Non-Negative Convex Functional Vanishing at the Origin

IVAN SINGER

National Institute for Scientific and Technical Creation, Bucharest, Romania

Submitted by Ky Fan

We give a generalization of the notion of the norm of a linear functional and some applications of this generalization.

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The extension of the methods of normed linear spaces to the study of convex functionals has led to useful results of convex analysis (for example, concerning differentiability of convex functionals, or optimization theory; for the latter, see [9] and the references therein). In this direction, in the present paper we want to give a generalization of the notion of the norm of a linear functional on a normed linear space $E$ to the case when the norm on $E$ is replaced by a finite non-negative convex functional $f$ on a linear space $E$, vanishing at the origin (or, sometimes, only at the origin). We shall also give some applications of this new notion to the study of extension of linear functionals. Namely, we shall show that the Hahn–Banach–Weston theorem (see [10, 11]) on the extension of linear functionals majorized by a finite convex functional admits an equivalent formulation as a theorem on extension with the same “norm,” and we shall show that there are some connections between the extension of linear functionals, with the same “norm” and best approximation of linear functionals in this “norm.” We shall conclude the paper with some remarks on the notion of “norm” of a functional with respect to a convex functional.

Throughout this paper we shall consider, without any special mention, only real linear spaces $E$. We shall denote by $E^*$ the algebraic dual of a linear space $E$ (i.e., the linear space of all linear functionals on $E$). We recall that for any convex functional $f$ on a linear space $E$ and any $x_0 \in E$, the subdifferential of $f$ at $x_0$ is the set defined by

$$
\partial f(x_0) = \{ \Phi \in E^* | \Phi(x) - \Phi(x_0) \leq f(x) - f(x_0) \ (x \in E) \}. \tag{1.1}
$$
In the sequel, for simplicity, we shall consider only finite convex functionals \( f \) on \( E \).

Finally, we recall (see, e.g., [3, 5]) that an asymmetric norm on a linear space \( E \) is a finite non-negative functional \( x \mapsto \|x\| \) on \( E \), such that (a) \( \|x\| = 0 \iff x = 0 \), (b) \( \|x + y\| \leq \|x\| + \|y\| \) \((x, y \in E)\), and (c) \( \|ax\| = a\|x\| \) \((x \in E, a \geq 0)\); property (c) is called the positive homogeneity of \( \| \cdot \| \). Thus, in particular, every norm in the usual sense, that is, every "symmetric" norm (i.e., such that \( \|ax\| = |a|\|x\| \) for all \( x \in E \) and \( a \in \mathbb{R} \), the real line) is an asymmetric norm. For a linear space \( E \) with an asymmetric norm \( \cdot \), we shall denote by \( E^* \) the space of all linear functionals \( \Phi \) on \( E \), such that \( \|\Phi\| = \sup_{x \in E, \|x\| < 1} \Phi(x) < +\infty \). For a locally convex space \( E \), we shall denote by \( E^* \) the algebraic–topological dual of \( E \) (the space of all continuous linear functionals on \( E \)).

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Let \( E \) be a linear space and let \( f \) be a finite convex functional on \( E \), such that

\[
\begin{align*}
\text{(2.1)} & \quad f(0) = 0, \\
\text{(2.2)} & \quad f(x) \geq 0 \quad (x \in E).
\end{align*}
\]

Then, by (2.1), we have

\[
\lambda \partial f(0) = \partial (\lambda f)(0) = \{ \Phi \in E^* | \Phi \leq \lambda f \} \quad (\lambda \geq 0), \quad (2.3)
\]

and hence, by (2.2), \( 0 \in \lambda \partial f(0) \) \((\lambda \geq 0)\).

**Definition 2.1.** Let \( E \) and \( f \) be as above. We shall call \( f \)-dual of \( E \) the set

\[
E' = \{ \Phi \in E^* | \exists \lambda > 0 \text{ such that } \Phi \leq \lambda f \}. \quad (2.4)
\]

**Remark 2.1.** By (2.3), we have

\[
E' = \{ \Phi \in E^* | \exists \lambda > 0 \text{ such that } \Phi \in \lambda \partial f(0) \}; \quad (2.5)
\]

thus, \( E' \) is the convex cone (with vertex 0) generated by \( \partial f(0) \). Clearly, \( 0 \in E' \) and, as shown by simple examples, it may also happen that \( E' = \{0\} \) (namely, this happens if and only if \( f \) is Gateaux differentiable at 0) or that \( E' = E^* \) (e.g., for \( f(x) = |x| \) on \( E = \mathbb{R} \)).
Remark 2.2. If \( E \) is a locally convex space and \( f \) is a continuous convex functional on \( E \), satisfying (2.1) and (2.2), we have

\[
E' = \{ \Phi \in E^* | \exists \lambda > 0 \text{ such that } \Phi \leq \lambda f \} = \{ \Phi \in E^* | \exists \lambda > 0 \text{ such that } \Phi \in \lambda \partial f(0) \};
\]

indeed, if \( \Phi \in E^* \) and \( \Phi \leq \lambda f \) for some \( \lambda > 0 \), then, since \( f \) is continuous, \( \lambda f \) and hence also \( \Phi \) are bounded on some neighborhood of 0, so \( \Phi \in E^* \).

(b) If \( E \) is a linear space with an asymmetric norm \( \| \cdot \| \) and \( f(x) = \| x \| \) (\( x \in E \)), then \( \partial f(0) = \{ \Phi \in E^* | \| \Phi \| \leq 1 \} \) (see e.g., (2.3)) and hence \( E' = E^* \).

**DEFINITION 2.2.** Let \( E \) and \( f \) be as in Definition 2.1 and let \( \Phi \in E^* \). We define the norm of \( \Phi \) with respect to \( f \), or, briefly, the \( f \)-norm of \( \Phi \), by

\[
\| \Phi \|_f = \begin{cases} 
\inf_{\lambda > 0} \frac{\lambda}{\phi} & \text{if } \Phi \in E' \\
\lambda & \text{if } \Phi \in E^* \setminus E' \\
\infty & \text{if } \Phi \in E^* \setminus E'.
\end{cases}
\]

**Remark 2.3.** By (2.3), we have

\[
\inf_{\Phi \in \partial f(0)} \| \Phi \|_f = \| \Phi \|_f = \sup_{x \in B, \| x \| \leq 1} \Phi(x) = \delta_B(\Phi),
\]

thus, \( \Phi \rightarrow \| \Phi \|_f \) is nothing else than the Minkowski functional (with respect to 0 \( \in \partial f(0) \)) of the set \( \partial f(0) \).

**Remark 2.4.** In the particular case when \( E \) is a linear space with an asymmetric norm \( \| \cdot \| \) and \( f(x) = \| x \| \) (\( x \in E \)), we have \( \| \Phi \|_f = \| \Phi \| = \sup_{x \in E, \| x \| \leq 1} \Phi(x) = \delta_B(\Phi) \), the support functional of \( B = \{ x \in E | \| x \| \leq 1 \} \). However, for an arbitrary \( f \) we have only the inequality \( \delta_B(\Phi) \leq \| \Phi \|_f \), where \( B = \{ x \in E | f(x) \leq 1 \} \) (see Proposition 2.1), which can be strict.

**PROPOSITION 2.1.** Let \( E \) and \( f \) be as in Definition 2.2. Then

\[
\Phi \leq \| \Phi \|_f f (\Phi \in E^*),
\]

so in (2.7) and (2.8) we can replace \( \inf \) by \( \min \). Furthermore, if \( \Phi \in E' \), then for each \( \varepsilon \) with \( 0 < \varepsilon < \| \Phi \|_f \) there exists \( x \in E \) such that

\[
\Phi(x) > (\| \Phi \|_f - \varepsilon) f(x).
\]

**Proof:** If \( \Phi \in E^* \), then either \( \| \Phi \|_f = +\infty \) or there exist \( \lambda_n > 0 \) with \( \Phi \leq \lambda_n f \) \((n = 1, 2,...)\), such that \( \lambda_n \rightarrow \| \Phi \|_f < +\infty \), from which we obtain (2.9). On the other hand, if \( 0 < \varepsilon < \| \Phi \|_f < +\infty \), then \( 0 < \| \Phi \|_f - \varepsilon < \| \Phi \|_f \).
from which by (2.7), \( \Phi \leq (\| \Phi \|_r - \epsilon) f \), so there exists \( x \in E \) satisfying (2.10), which completes the proof of Proposition 2.1.

**Remark 2.5.** For any \( x \in E \) as in Proposition 2.1 we have, clearly, \( x \neq 0 \) (since \( \Phi(0) = 0 = (\| \Phi \|_r - \epsilon) f(0) \), by (2.1)).

**PROPOSITION 2.2.** \( \Phi \to \| \Phi \|_r \) is a "generalized asymmetric norm" on \( E^* \), i.e., a non-negative functional on \( E^* \) which may take the value \( + \infty \), such that

\[
\| \Phi \|_r = 0 \iff \Phi \equiv 0, \tag{2.11}
\]

\[
\| \Phi_1 + \Phi_2 \|_r \leq \| \Phi_1 \|_r + \| \Phi_2 \|_r \quad (\Phi_1, \Phi_2 \in E^*). \tag{2.12}
\]

\[
\| \mu \Phi \|_r = \mu \| \Phi \|_r \quad (\Phi \in E^*, \mu \geq 0). \tag{2.13}
\]

**Proof.** Since \( 0 \in \partial f(0) \subset E^\prime \), we have \( \| 0 \|_r = \inf_{\lambda > 0, 0 \in \partial f(0)} \lambda = 0. \) Conversely, if \( \Phi \in E^* \) and \( \| \Phi \|_r = 0 \), then by (2.9), \( \Phi \leq 0 \), from which, since \( \Phi \in E^* \), we obtain \( \Phi = 0 \), which proves (2.11).

Now let \( \Phi_1, \Phi_2 \in E^\prime \). If either \( \Phi_1 \in E^\prime \) or \( \Phi_2 \in E^\prime \), then \( \| \Phi_1 \|_r + \| \Phi_2 \|_r = + \infty \), so (2.12) holds. Assume now that \( \Phi_1, \Phi_2 \in E^\prime \). Then, by (2.9), \( \Phi_1 \leq \| \Phi_1 \|_r f, \Phi_2 \leq \| \Phi_2 \|_r f \), from which \( \Phi_1 + \Phi_2 \leq (\| \Phi_1 \|_r + \| \Phi_2 \|_r) f \). Consequently, by (2.7), we obtain (2.12).

Finally, let \( \Phi \in E^\prime, \mu \geq 0 \). If \( \Phi \not\in E^\prime \), then also \( \mu \Phi \not\in E^\prime \), so (2.13) holds. Assume now that \( \Phi \in E^\prime \). Then, clearly, \( \mu \Phi \in E^\prime \) and, by (2.7), we obtain

\[
\| \mu \Phi \|_r = \inf_{\lambda > 0, \lambda \Phi \in \partial f} \lambda = \inf_{\lambda > 0, \lambda \Phi \in \partial f} \mu \lambda = \mu \| \Phi \|_r,
\]

which completes the proof of Proposition 2.2.

**Remark 2.6.** It is easy to show that a functional \( x \to \| x \| \) on a linear space \( E \), with values in \( [0, + \infty] \), is a generalized asymmetric norm if and only if it is a "gauge" in the sense of Rockafellar ([7, p. 128]), such that \( \| x \| > 0 \) \( (x \in E \setminus \{0\}) \).

3

In the general case, the following property is often a useful replacement of positive homogeneity of the asymmetric norm functionals:

**LEMMA 3.1.** Let \( E \) be a linear space and \( f \) a finite convex functional on \( E \), satisfying (2.1). Then

\[
f(\lambda x) \leq \lambda f(x) \quad (x \in E, 0 \leq \lambda \leq 1), \tag{3.1}
\]

\[
f(\lambda x) \geq \lambda f(x) \quad (x \in E, \lambda \geq 1). \tag{3.2}
\]

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Proof. If \( x \in E \) and \( 0 \leq \lambda \leq 1 \), then, by (2.1),

\[
f(\lambda x) - f(\lambda x + (1 - \lambda) 0) \leq \lambda f(x) + (1 - \lambda) f(0) - \lambda f(x).
\]

On the other hand, if \( x \in E \) and \( \lambda \geq 1 \), then \( 0 < 1/\lambda \leq 1 \), from which, by (3.1),

\[
f(x) = f\left(\frac{1}{\lambda} (\lambda x)\right) \leq \frac{1}{\lambda} f(\lambda x),
\]

which implies (3.2). This completes the proof of Lemma 3.1.

Using this lemma, we shall give now other expressions for the \( f \)-norm, in the case when \( f \) is vanishing only at the origin.

**Theorem 3.1.** Let \( E \) be a linear space and \( f \) a finite convex functional on \( E \), satisfying (2.1) and

\[
f(x) > 0 \quad (x \in E \setminus \{0\}). \tag{3.3}
\]

Then

\[
\|\Phi\|_f = \sup_{x \in E, x \neq 0} \frac{\Phi(x)}{f(x)} = \sup_{x \in E, 0 < f(x) \leq 1} \frac{\Phi(x)}{f(x)} \quad (\Phi \in E^*). \tag{3.4}
\]

**Proof.** By Proposition 2.1 and (3.3), we have

\[
\|\Phi\|_f \geq \sup_{x \in E, x \neq 0} \frac{\Phi(x)}{f(x)} \geq \sup_{x \in E, 0 < f(x) \leq 1} \frac{\Phi(x)}{f(x)} \quad (\Phi \in E^*), \tag{3.5}
\]

so it remains to prove the opposite inequalities.

Assume first that \( \Phi \in E^* \setminus E' \), so \( \|\Phi\|_f = +\infty \). Then for each \( \lambda > 0 \) there exists \( x_\lambda \in E \) such that \( \Phi(x_\lambda) > \lambda f(x_\lambda) \), from which, by (3.3), \( \Phi(x_\lambda)/f(x_\lambda) > \lambda \). Now, if \( 0 < f(x_\lambda) \leq 1 \), let \( z_\lambda = x_\lambda \). On the other hand, if \( f(x_\lambda) \geq 1 \), then \( 0 < 1/f(x_\lambda) \leq 1 \), from which, by Lemma 3.1, \( f(x_\lambda/f(x_\lambda)) \leq 1 \). Thus, for \( z_\lambda = x_\lambda/f(x_\lambda) \) we have \( f(z_\lambda) \leq 1 \) and

\[
\Phi(z_\lambda) = \Phi\left(\frac{x_\lambda}{f(x_\lambda)}\right) > \lambda \geq \lambda f(z_\lambda),
\]

from which, by (3.3), \( \Phi(z_\lambda)/f(z_\lambda) > \lambda \). Consequently,

\[
\sup_{x \in E, 0 < f(x) \leq 1} \frac{\Phi(x)}{f(x)} \geq \sup_{\lambda > 0} \frac{\Phi(z_\lambda)}{f(z_\lambda)} = +\infty = \|\Phi\|_f. \tag{3.6}
\]
Assume now that $\Phi \in E'$ and let $0 < \varepsilon < \| \Phi \|_f$. Then, by Proposition 2.1, there exists $x_\varepsilon \in E$ satisfying (2.10), from which, by (3.3), $\Phi(x_\varepsilon)/f(x_\varepsilon) > \| \Phi \|_f - \varepsilon$. Now, proceeding as above, with $\lambda$ replaced by $\| \Phi \|_f - \varepsilon$, we obtain (both when $0 < f(x_\varepsilon) \leq 1$ and when $f(x_\varepsilon) > 1$) an element $z_\varepsilon \in E$ with $0 < f(z_\varepsilon) \leq 1$, such that $\Phi(z_\varepsilon)/f(z_\varepsilon) > \| \Phi \|_f - \varepsilon$. Consequently,

$$\sup_{0 < f(x) < 1} \frac{\Phi(x)}{f(x)} \geq \sup_{0 < f(x) < f(z_\varepsilon)} \frac{\Phi(z_\varepsilon)}{f(z_\varepsilon)} \geq \sup_{0 < f(x) < f(z_\varepsilon)} (\| \Phi \|_f - \varepsilon) = \| \Phi \|_f, \quad (3.7)$$

which, together with (3.6) and (3.5), yields (3.4). This completes the proof of Theorem 3.1.

Remark 3.1. In the particular case when $f$ is a finite gauge (see Remark 2.6), the first equality in (3.4) has been observed by Rockafellar [7, p. 128], in the terminology of polar functionals (see Remark 6.1 below).

4

Let us consider now the problem of extension of linear functionals. If $E, f$ are as in Section 2 and if $G$ is a linear subspace of $E$ and $\varphi \in G^\sigma$, we shall write, for simplicity, $G', \| \varphi \|_f$ instead of $G'^G$ and $\varphi^{G'}$, respectively. Let us first observe that for any extension of $\varphi$ to a linear functional $\Phi$ on the whole space $E$ (i.e., $\Phi \in E''$, $\Phi|_G = \varphi$), we have

$$\| \varphi \|_f \leq \| \Phi \|_f. \quad (4.1)$$

Indeed, if $\Phi \in E'' \setminus E'$, then $\| \Phi \|_f = +\infty$, so (4.1) holds. On the other hand, if $\Phi \in E'$, then by $\Phi|_G = \varphi$ we have $\emptyset \neq \{ \lambda > 0 \mid \Phi \leq \lambda f \} \subset \{ \lambda > 0 \mid \varphi \leq \lambda f|_G \}$, from which, by Definition 2.2, we obtain (4.1).

Now we shall prove that every linear functional on a linear subspace $G$ can be extended, with the same $f$-norm, to the whole space $E$.

**Theorem 4.1.** Let $E$ be a linear space, $f$ a finite convex functional on $E$, satisfying (2.1) and (2.2), $G$ a linear subspace of $E$ and $\varphi \in G^\sigma$. Then there exists $\Phi \in E''$ such that

$$\Phi|_G = \varphi, \quad \| \Phi \|_f = \| \varphi \|_f. \quad (4.2)$$

**Proof.** Assume first that $\varphi \in G'' \setminus G'$, so $\| \varphi \|_f = +\infty$, and take any $\Phi \in E''$ such that $\Phi|_G = \varphi$; it is well known (see e.g. [1, Chap. I, Section 2]) that such a $\Phi$ exists. Then, by (4.1), $\| \Phi \|_f = +\infty = \| \varphi \|_f$.

Assume now that $\varphi \in G'$. Then, by Proposition 2.1,

$$\varphi(g) \leq \| \varphi \|_f f(g) \quad (g \in G),$$
so \( p \mid_G \leq p \mid_G \), where \( p \) is the finite convex functional on \( E \) defined by \( p = \| \varphi \|_f \). Hence, by the Hahn–Banach–Weston theorem (see [10, 11]), there exists \( \Phi \in E^* \) such that \( \Phi \mid_G = \varphi \) and \( \Phi \leq p = \| \varphi \|_f \). Then, by Definition 2.2, \( \| \Phi \|_f \leq \| \varphi \|_f \), which, together with (4.1), yields (4.2). This completes the proof of Theorem 4.1.

**Remark 4.1.** The particular case \( \varphi \in G' \) of Theorem 4.1 implies (and thus it is, essentially, equivalent to) the Hahn–Banach–Weston theorem used in its proof. Indeed, if \( \psi \in G^* \) and \( \psi \leq \varphi \mid_G \), where \( \varphi \) is a finite convex functional on \( E \), then \( \varphi \in G' \) and \( \| \varphi \|_f \leq 1 \), from which, by Theorem 4.1, there exists \( \Phi \in E^* \) satisfying \( \Phi \mid_G = \varphi \) and \( \| \Phi \|_f = \| \varphi \|_f \leq 1 \). Consequently, by Proposition 2.1, \( \Phi \leq \| \varphi \|_f f \leq f \), which completes the proof.

5

Using the concept of norm of a linear functional with respect to a convex functional, one can show that there are some connections between the extension of linear functionals on a linear subspace \( G \), to the whole space \( E \), with the same generalized asymmetric norm \( \| \cdot \|_f \) and best approximation in \( E^* \), in the generalized asymmetric norm \( \| \cdot \|_f \), by means of the elements of the linear subspace

\[
G^* = \{ \Psi \in E^* \mid \Psi(g) = 0 \ (g \in G) \};
\]  

(5.1)

for the particular case of norm functionals \( f \), such connections have been shown by Phelps [6] and later in [2] (see also [8]). In order to prove here a result of this type in the general case, it will be convenient to give first

**Definition 5.1.** We shall say that a linear subspace \( G \) of a linear space \( E \) has the unique Hahn–Banach extension property with respect to a finite convex functional \( f \) on \( E \) satisfying (2.1) and (2.2) (or, briefly, the \( f \)-UHBE property), if for each \( \varphi \in G' \) there exists a unique \( \Phi \in E^* \) satisfying (4.2) of Theorem 4.1 (hence \( \Phi \in E^f \)).

**Theorem 5.1.** For a linear subspace \( G \) of a linear space \( E \) and a finite convex functional \( f \) on \( E \), satisfying (2.1) and (2.2), the following statements are equivalent:

1. \( G \) has the \( f \)-UHBE property.
2. There do not exist functionals \( \Phi \in E' \) and \( \Phi_1, \Phi_2 \in G^* \) with \( \Phi_1 \neq \Phi_2 \), such that

\[
\| \Phi - \Phi_1 \|_f = \| \Phi - \Phi_2 \|_f = \inf_{\Psi \in G^*} \| \Phi - \Psi \|_f.
\]  

(5.2)
Proof. Assume that (2) is not satisfied, so there exist $\Phi \in E'$ and $\Phi_1, \Phi_2 \in G^+$ as in (2). Let

$$\varphi = (\Phi - \Phi_1)|_G = (\Phi - \Phi_2)|_G = \Phi|_G. \quad (5.3)$$

Then, since $\Phi \in E'$, we have $\varphi \in G'$. Furthermore, by (4.1),

$$\|\varphi\|_r = \|(\Phi - \Phi_1)|_G\|_r \leq \|\Phi - \Phi_i\|_r \quad (i = 1, 2). \quad (5.4)$$

On the other hand, by Theorem 4.1 there exists $\chi \in E^\sigma$ such that $\chi|_G = \varphi$, $\|\chi\|_r = \|\varphi\|_r$. Then $\Phi - \chi \in G^+$, from which, by (5.2),

$$\|\Phi - \Phi_i\|_r \leq \|\Phi - (\Phi - \chi)\|_r = \|\chi\|_r = \|\varphi\|_r \quad (i = 1, 2). \quad (5.5)$$

From (5.4) and (5.5) it follows that

$$\|\Phi - \Phi_1\|_r = \|\Phi - \Phi_2\|_r = \|\varphi\|_r. \quad (5.6)$$

and hence, by (5.3) and $\Phi_1 \neq \Phi_2$, $G$ does not have the $f$-UHBE property. Thus, (1) $\Rightarrow$ (2).

Conversely, assume now that (1) is not satisfied, so there exist $\varphi \in G'$, $\Phi_1, \Phi_2 \in E^\sigma$, $\Phi_1 \neq \Phi_2$, such that

$$\Phi_1|_G = \Phi_2|_G = \varphi, \quad \|\Phi_1\|_r = \|\Phi_2\|_r = \|\varphi\|_r. \quad (5.7)$$

Then, since $\varphi \in G'$, we have $\Phi_1, \Phi_2 \in E'$. Let

$$\Phi = \Phi_1 + \Phi_2. \quad (5.8)$$

Then, by Proposition 2.2, we have $\Phi \in E'$ and

$$\|\Phi\|_r \leq \|\Phi_1\|_r + \|\Phi_2\|_r = 2 \|\varphi\|_r, \quad (5.9)$$

$$\|\Phi - (\Phi_1 - \Phi_2)\|_r = 2\|\Phi_2\|_r = 2 \|\varphi\|_r. \quad (5.10)$$

On the other hand, for any $\Psi \in G^+$,

$$\|\Phi - \Psi\|_r \geq \|(\Phi - \Psi)|_G\|_r = \|(\Phi_1 + \Phi_2)|_G\|_r = 2\|\varphi\|_r = 2 \|\varphi\|_r. \quad (5.11)$$

From (5.9)–(5.11) it follows that the functionals $0 \in G^+$ and $\Phi_1 - \Phi_2 \in G^+$, $\Phi_1 - \Phi_2 \neq 0$ satisfy

$$\|\Phi - 0\|_r = \|\Phi - (\Phi_1 - \Phi_2)\|_r = \inf_{\Psi \in G^+} \|\Phi - \Psi\|_r,$$

so (2) does not hold. Thus, (2) $\Rightarrow$ (1), which completes the proof of Theorem 5.1.
Remark 5.1. (a) The conditions $\Phi \in E^f$ and (5.2) imply that in (2) we must have $\Phi - \Phi_i \in E^f$ (since $\|\Phi - \Phi_i\|_f = \inf_{\psi \in G^i} \|\Phi - \psi\|_f < \|\Phi - 0\|_f < +\infty$), for $i = 1, 2$.

(b) For $\Phi \in E^f$ we have $\|\Phi\|_f = \|\Phi\|_{G^f}$ if and only if $\|\Phi\|_f = \min_{\psi \in G^f} \|\Phi - \psi\|_f$ (since $\|\Phi\|_{G^f} = \min_{\psi \in G^f} \|\Phi - \psi\|_f$, by (4.1) and Theorem 4.1).

(c) A different concept of unique Hahn-Banach extension property with respect to a continuous convex functional $f$ on a locally convex space (which, in the particular case of norm functionals $f$, amounts to the requirement that every continuous linear functional which attains its norm on the subspace admits a unique norm-preserving extension to the whole space) and some characterizations of subspaces with that property have been given by Holmes [4].

6

We conclude with some remarks related to the notion of norm of a functional with respect to a convex functional.

Remark 6.1. For a gauge $f$ on $E = R^n$ (see Remark 2.6 above), Rockafellar [7, p. 128] has defined the polar functional $f^0$ of $f$ by

$$f^0(\Phi) = \inf_{\lambda \neq 0} \lambda (\Phi \in E^0);$$

since for a gauge $f \neq 0$ we have $E^f \neq \{0\}$, it follows that

$$f^0(\Phi) = \|\Phi\|_f \quad (\Phi \in E^0).$$

However, for a finite convex functional $f$ on $E = R^n$, satisfying (2.1) and (2.2), Rockafellar [7, p. 136] has defined the polar functional $f^0$ of $f$ by

$$f^0(\Phi) = \inf_{\lambda > 0} \lambda (\Phi \in E^0)$$

and has observed that for a gauge $f$ this reduces, by the positive homogeneity of $f$, to (6.1). Clearly, in general (e.g., for $f(x) = x^2$ on $E = R$ and $\Phi \neq 0$), (6.3) is different from (6.1) and hence from $\|\Phi\|_f$ as well.

Remark 6.2. One can also define a symmetric norm with respect to a finite convex functional $f$ on $E$ satisfying (2.1) and (2.2), by considering the set

$$E^f_\delta = \{\Phi \in E^f \mid \exists \lambda > 0 \text{ such that } |\Phi| \leq \lambda f\}$$
and by putting
\[
\|\Phi\|_r^r = \begin{cases} 
\inf_{\lambda > 0} \lambda & \text{if } \Phi \in E_1^r \\
\|\Phi\|_r^r & \text{if } \Phi \in E^* \setminus E_1^r.
\end{cases} \tag{6.5}
\]

Then, clearly,
\[
\|\Phi\|_r \leq \|\Phi\|_r^r \quad (\Phi \in E^*). \tag{6.6}
\]

In the particular case when \((E, \|\|)\) is a normed linear space and \(f(x) = \|x\| (x \in E)\), we have \(\|\Phi\|_r^r = \|\Phi\|\) in the usual sense. One can prove results similar to Propositions 2.1 and 2.2 and Theorem 3.1, with (2.9), (2.10), (2.13), and (3.4) replaced respectively by
\[
\Phi \leq \|\Phi\|_r^r f \quad (\Phi \in E^*), \tag{6.7}
\]
\[
|\Phi(x) - f(x)| > (\|\Phi\|_r^r - \varepsilon) f(x), \tag{6.8}
\]
\[
\|\mu\Phi\|_r^r = |\mu|\|\Phi\|_r^r \quad (\Phi \in E^*, \mu \in \mathbb{R}). \tag{6.9}
\]
\[
\|\Phi\|_r^r = \sup_{x \in E, x \neq 0} \frac{|\Phi(x)|}{f(x)} = \sup_{x \in E, 0 < f(x) < 1} \frac{|\Phi(x)|}{f(x)} \quad (\Phi \in E^*). \tag{6.10}
\]

Remark 6.3. One can replace \(E^*\) by larger linear subspaces of the linear space of all functionals \(E \to \mathbb{R}\), for example, by the space \(E^* = E^* + \mathbb{R}\) of all affine functionals on \(E\) (i.e., all functionals of the form \(\Phi + c\), where \(\Phi \in E^*\) and \(c \in \mathbb{R}\)). If we do this in (1.1) and denote the resulting set by \(\partial_a f(x_o)\), then, whenever \(\partial f(x_0) \neq \emptyset\), we have
\[
\partial_a f(x_0) = \partial f(x_0) + R = \{\Phi + c | \Phi \in \partial f(x_0), c \in \mathbb{R}\}. \tag{6.11}
\]

Furthermore, clearly
\[
E_1^a = \{\Phi \in E^a | \exists \lambda > 0 \text{ such that } \Phi \leq \lambda f\} \neq \{0\}, \tag{6.12}
\]
and the fact that a closed convex \(f\) on \(E = \mathbb{R}\) is the supremum of all \(\Phi \in E^a\) majorized by \(f\) (see, e.g. [7, Theorem 12.1]) can be also expressed in the form
\[
f(x) = \sup_{\Phi \in E_1^a} \Phi(x) \quad (x \in E). \tag{6.13}
\]

Remark 6.4. Let us note, concerning the assumptions (2.1) and (2.2), that if \(f_0\) is any finite convex functional satisfying \(f_0(x_0) = \min_{x \in E} f_0(x) = c \in \mathbb{R}\), then
\[
f(x) = (f_0 - c)(x + x_0) = f_0(x + x_0) - c \quad (x \in E) \tag{6.14}
\]
is a finite convex functional satisfying (2.1) and (2.2).
Note added in proof. We have shown, jointly with Michael Wriedt, that for each \( f \) as in Section 2, \( \| \Phi \|_f = \| \Phi \|_{h_f} (\Phi \in E^\#) \), where \( h_f(x) = \sup_{\Phi \in \mathcal{L}_0} \Phi(x) \) (\( x \in E \)) is a finite gauge with \( h_f \leq f \), and we have obtained some results on \( h_f \).

REFERENCES