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Quasitriangular and Differential Structures on Bicrossproduct Hopf Algebras

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Let $X = GM$ be a finite group factorisation. It is shown that the quantum double $D(H)$ of the associated bicrossproduct Hopf algebra $H = kM \triangleright\!\!\blacktriangleleft k(G)$ is itself a bicrossproduct $kX \triangleright\!\!\blacktriangleleft k(Y)$ associated to a group YX, where $Y = G \times M^{\mathrm{op}}$. This provides a class of bicrossproduct Hopf algebras which are quasitriangular. We also construct a subgroup $Y^\theta X^\theta$ associated to every order-reversing automorphism θ of X. The corresponding Hopf algebra $kX^\theta \triangleright\!\!\blacktriangleleft k(Y^\theta)$ has the same coalgebra as H. Using related results, we classify the first order bicovariant differential calculi on H in terms of orbits in a certain quotient space of X.

1. INTRODUCTION

The quantum double [1] of the bicrossproduct Hopf algebra $H = kM \triangleright\!\!\blacktriangleleft k(G)$ associated to a finite group factorisation $X = GM$ has been studied recently in [2]. Here we continue this study with further results on

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the same topic, including a concrete application to the classification of the bicovariant differential calculi on a bicrossproduct.

The bicrossproduct Hopf algebras were introduced in [3, 4] and have been extensively studied since then. Factorisations of groups abound in mathematics, so these Hopf algebras, which are non-commutative and non-cocommutative, are quite common. In the context of [4] they are viewed as systems which combine quantum mechanical ideas with geometry in a unified way. To develop this idea, a natural step is to compute and study the algebra of differential forms (the so-called differential calculus) over them, which we do here. Another open problem is the close connection which one may expect between these Hopf algebras and the method of inverse scattering in soliton theory. We make a connection of this type also in this paper. Our results, however, will be in an algebraic setting with finite groups and finite-dimensional Hopf algebras.

We recall two principal results from [2]. One is that $D(H)$ is isomorphic to a twist (i.e., up to a categorical equivalence of representations) of the much simpler double $D(X)$ of the group algebra kX. We provide in Section 2 a variant of this, namely that $D(H)$ is also isomorphic to a bicrossproduct $kX \triangleright\!\blacktriangleleft k(Y)$ where Y is the "dressing group" $G \times M^{\mathrm{op}}$ (the terminology comes from soliton theory [5]). The bicrossproduct is associated to a certain double cross product group $Y \bowtie X$ which factorises into X, Y. Our new result answers affirmatively the question: Can a bicrossproduct Hopf algebra be quasitriangular? Among non-commutative and non-cocommutative Hopf algebras, the quasitriangular [1] ones have a special place with remarkable properties.

The second principal result in [2] is that associated to every order-reversing isomorphism θ of X is a Hopf algebra isomorphism Θ of the quantum double. In Section 3 we provide more results about Θ. We then construct two new groups Y^θ, X^θ forming a subgroup $Y^\theta \bowtie X^\theta$ of $Y \bowtie X$. We study the associated Hopf algebra $kX^\theta \triangleright\!\blacktriangleleft k(Y^\theta)$. It has an isomorphic coalgebra to that of our original bicrossproduct H associated to the factorisation GM. However, we give a finite group example where these are not isomorphic as algebras.

In Section 4 we study a certain Hilbert space representation of H and find that our various constructions over $\mathbb{C}$ respect the $*$-structures. This is also one of the motivations behind the main isomorphism in Section 2.

In Section 5 we turn to a specific application of the quantum double, namely to the construction of first order bicovariant differential calculi. By definition a first order differential calculus over an algebra A means an A-bimodule Ω^1 over A and a map d: $A \to \Omega^1$ obeying the Leibniz rule (which makes sense using the bimodule structure). When A is a Hopf algebra the calculus is said to be bicovariant when d intertwines the left and right regular coactions of A with a bicomodule structure on Ω^1. It is known that first order bicovariant calculi are related to the representation theory of the quantum double [6], allowing us to apply our previous results [2] when $A = k(M)\triangleright\!\!\blacktriangleleft kG$. We find that the irreducible bicovariant calculi correspond to the choice of an orbit in a certain quotient space of the group X along with an irreducible subrepresentation of the isotropy subgroup associated to the orbit. We actually classify the bicovariant quantum tangent spaces using the techniques developed in [7] and obtain the corresponding 1-forms later by dualisation. The result is a constructive method which provides the entire moduli space of bicovariant calculi on a bicrossproduct, as we demonstrate on some nontrivial examples. On the other hand, armed with a choice of such differential calculus, one may proceed to non-commutative geometry such as gauge theory (i.e., bundles and connections) [9] on bicrossproduct Hopf algebras, to be considered elsewhere.

Preliminaries. We use the notation and conventions of [2], which are also the notation and conventions in the text [8]. Briefly, let $X = GM$ be a group which factorises into two subgroups. Then each group acts on the other through left and right actions $\triangleright$, $\triangleleft$ defined by $su = (s \triangleright u)(s \triangleleft u)$ for all $s \in M$ and $u \in G$. Conversely, given two actions $(\triangleright, \triangleleft)$ obeying certain matching conditions

$$\begin{aligned} &s \triangleleft e = s, (s \triangleleft u) \triangleleft v = s \triangleleft (uv); \\ &e \triangleleft u = e, (st) \triangleleft u = (s \triangleleft (t \triangleright u))(t \triangleleft u); \\ &e \triangleright u = u, s \triangleright (t \triangleright u) = (st) \triangleright u; \\ &s \triangleright e = e, s \triangleright (uv) = (s \triangleright u)((s \triangleleft u) \triangleright v), \end{aligned} \tag{1}$$

we can build a double cross product group on $G \times M$ with

$$\begin{aligned} &(u,s)(v,t) = (u(s \triangleright v),(s \triangleleft v)t), \qquad e = (e,e), \\ &(u,s)^{-1} = (s^{-1} \triangleright u^{-1}, s^{-1} \triangleleft u^{-1}). \end{aligned} \tag{2}$$

The associated bicrossproduct Hopf algebra $H = kM \triangleright\!\!\blacktriangleleft k(G)$ has the smash product algebra structure by the induced action of M and the smash coproduct coalgebra structure by the induced coaction of G. Explicitly,

$$\begin{aligned} (s \otimes \delta_u)(t \otimes \delta_v) &= \delta_{u, t \triangleright v}(st \otimes \delta_v), \\ \Delta(s \otimes \delta_u) &= \sum_{xy=u} s \otimes \delta_x \otimes s \triangleleft x \otimes \delta_y, \\ 1 &= \sum_u e \otimes \delta_u, \qquad \epsilon(s \otimes \delta_u) = \delta_{u,e}, \\ S(s \otimes \delta_u) &= (s \triangleleft u)^{-1} \otimes \delta_{(s \triangleright u)^{-1}}. \end{aligned} \tag{3}$$

We work over a general ground field k. There is also a natural $*$-algebra structure $(s\otimes\delta_u)^*=s^{-1}\otimes\delta_{s\triangleright u}$ when the ground field has an involution. This happens over $\mathbb{C}$, but can also be supposed for any field with $\bar{\lambda}=\lambda$ for all $\lambda\in k$. The dual H^* has a similar structure $k(M)\blacktriangleright\!\!\triangleleft kG$ on the dual basis,

$$
\begin{aligned}
(\delta_s\otimes u)(\delta_t\otimes v)&=\delta_{s\triangleleft u,t}(\delta_s\otimes uv),\\
\Delta(\delta_s\otimes u)&=\sum_{ab=s}\delta_a\otimes b\triangleright u\otimes\delta_b\otimes u,\\
1=\sum_s\delta_s\otimes e,&\qquad \epsilon(\delta_s\otimes u)=\delta_{s,e},\\
S(\delta_s\otimes u)&=\delta_{(s\triangleleft u)^{-1}}\otimes(s\triangleright u)^{-1},
\end{aligned}
\tag{4}
$$

and $(\delta_s\otimes u)^*=\delta_{s\triangleleft u}\otimes u^{-1}$ when the ground field has an involution. The quantum double [1] is a general construction $D(H)=H^{*\mathrm{op}}\bowtie H$ built on $H^*\otimes H$ with a double cross product algebra structure and tensor product coalgebra structure. In our case the cross relations between $H, H^{*\mathrm{op}}$ are [2]

$$
\begin{aligned}
(1\otimes t\otimes\delta_v)(\delta_s\otimes u\otimes 1)&=\delta_{t's(t\triangleleft vu^{-1})^{-1}}\\
&\quad\otimes(t\triangleleft vu^{-1})\triangleright u\otimes t'\otimes\delta_{(s\triangleright u)vu^{-1}},
\end{aligned}
\tag{5}
$$

where $t'=t\triangleleft(s\triangleright u)^{-1}$.

2. MORE ABOUT $D(H)$

Here we extend results about the quantum double associated to a bicrossproduct in [2]. For our first observation, it is known that to every factorisation $X=GM$ there is a "double factorisation" YX where Y is also $G\times M$ as a set and the action of X is the adjoint action viewed as an action on Y [8]. Here we give a similar but different double factorisation more suitable for our needs.

PROPOSITION 2.1. *Let $Y=G\times M^{\mathrm{op}}$ with group law $(us).(vt)=uvts$. Then there is a double cross product group $Y\bowtie X$ (factorising into Y, X) defined by the actions*

$$
\begin{aligned}
us\,\tilde{\triangleleft}\,vt&=\big((s\triangleleft v)ts^{-1}\triangleright u^{-1}\big)^{-1}(s\triangleleft v),\\
us\,\tilde{\triangleright}\,vt&=us(vt)(us)^{-1}\\
&=u(s\triangleright v)\big((s\triangleleft v)ts^{-1}\triangleright u^{-1}\big)\big((s\triangleleft v)ts^{-1}\triangleleft u^{-1}\big).
\end{aligned}
$$

The second line is the adjoint action on X which we view as an action on the set Y.

Proof. We show that these actions are matched in the required sense (see [8]). Thus

$$us \,\tilde{\triangleright}\, ((vt).(wr)) = (us \,\tilde{\triangleright}\, vt).\big((us \,\tilde{\triangleleft}\, vt) \,\tilde{\triangleright}\, wr\big)$$

holds as follows. We have $us \,\tilde{\triangleright}\, vt = u(s \triangleright v)yp$ where $y = (s \triangleleft v)ts^{-1} \triangleright u^{-1}$ and $p = (s \triangleleft v)ts^{-1} \triangleleft u^{-1}$. Meanwhile, from $us \,\tilde{\triangleleft}\, vt = y^{-1}(s \triangleleft v)$ we calculate

$$\begin{aligned}(us \,\tilde{\triangleleft}\, vt) \,\tilde{\triangleright}\, wr & \\ &= y^{-1}((s \triangleleft v) \triangleright w)\big((s \triangleleft vw)r(s \triangleleft v)^{-1} \triangleright y\big) \\ &\quad \times\big((s \triangleleft vw)r(s \triangleleft v)^{-1} \triangleleft y\big),\end{aligned}$$

and on applying the rules for multiplication in Y we find

$$\begin{aligned}(us \,\tilde{\triangleright}\, vt).\big((us \,\tilde{\triangleleft}\, vt) \,\tilde{\triangleright}\, wr\big) & \\ &= u(s \triangleright v)((s \triangleleft v) \triangleright w)\big((s \triangleleft vw)r(s \triangleleft v)^{-1} \triangleright y\big) \\ &\quad \times\big((s \triangleleft vw)r(s \triangleleft v)^{-1} \triangleleft y\big)p \\ &= u(s \triangleright vw)((s \triangleleft vw)rts^{-1} \triangleright u^{-1})((s \triangleleft vw)rts^{-1} \triangleleft u^{-1}) \\ &= us \,\tilde{\triangleright}\, ((vt).(wr)),\end{aligned}$$

as required. On the other side, we show that

$$((wr).(us)) \,\tilde{\triangleleft}\, vt = \big(wr \,\tilde{\triangleleft}\, (us \,\tilde{\triangleright}\, vt)\big).(us \,\tilde{\triangleleft}\, vt),$$

where

$$\begin{aligned}wr \,\tilde{\triangleleft}\, (us \,\tilde{\triangleright}\, vt) &= wr \,\tilde{\triangleleft}\, (u(s \triangleright v)yp) \\ &= \big((r \triangleleft u(s \triangleright v)y)pr^{-1} \triangleright w^{-1}\big)^{-1}(r \triangleleft u(s \triangleright v)y),\end{aligned}$$

and the definition of multiplication in X gives

$$\begin{aligned}\big(wr \,\tilde{\triangleleft}\, (us \,\tilde{\triangleright}\, vt)\big).(us \,\tilde{\triangleleft}\, vt) & \\ &= \big((r \triangleleft u(s \triangleright v)y)pr^{-1} \triangleright w^{-1}\big)^{-1}(r \triangleleft u(s \triangleright v)y)y^{-1}(s \triangleleft v)\end{aligned}$$

$$
\begin{aligned}
&= \big((r \triangleleft u(s \triangleright v)y)pr^{-1} \triangleright w^{-1}\big)^{-1} \\
&\quad \times \big((r \triangleleft u(s \triangleright v)y) \triangleright y^{-1}\big)(r \triangleleft u(s \triangleright v))(s \triangleleft v) \\
&= \big((r \triangleleft u(s \triangleright v)y)pr^{-1} \triangleright w^{-1}\big)^{-1} \\
&\quad \times ((r \triangleleft u(s \triangleright v)) \triangleright y)^{-1}((r \triangleleft u)s \triangleleft v) \\
&= \big[((r \triangleleft u(s \triangleright v)) \triangleright y)\big((r \triangleleft u(s \triangleright v)y)pr^{-1} \triangleright w^{-1}\big)\big]^{-1} \\
&\quad \times ((r \triangleleft u)s \triangleleft v) \\
&= \big[(r \triangleleft u(s \triangleright v)) \triangleright y\big(pr^{-1} \triangleright w^{-1}\big)\big]^{-1}((r \triangleleft u)s \triangleleft v) \\
&= \big[(r \triangleleft u(s \triangleright v))(s \triangleleft v)ts^{-1} \triangleright u^{-1}(r^{-1} \triangleright w^{-1})\big]^{-1} \\
&\quad \times ((r \triangleleft u)s \triangleleft v) \\
&= \big[((r \triangleleft u)s \triangleleft v)ts^{-1} \triangleright u^{-1}(r^{-1} \triangleright w^{-1})\big]^{-1}((r \triangleleft u)s \triangleleft v),
\end{aligned}
$$

where the last two equalities come from

$$
\begin{aligned}
&y\big(pr^{-1} \triangleright w^{-1}\big) \\
&\qquad = \big((s \triangleleft v)ts^{-1} \triangleright u^{-1}\big)\Big(\big((s \triangleleft v)ts^{-1} \triangleleft u^{-1}\big) \triangleright (r^{-1} \triangleright w^{-1})\Big) \\
&\qquad = (s \triangleleft v)ts^{-1} \triangleright u^{-1}(r^{-1} \triangleright w^{-1}).
\end{aligned}
$$

Meanwhile,

$$
\begin{aligned}
((wr).(us)) \mathbin{\tilde{\triangleleft}} vt &= (w(r \triangleright u)(r \triangleleft u)s) \mathbin{\tilde{\triangleleft}} vt \\
&= \big[((r \triangleleft u)s \triangleleft v)ts^{-1}(r \triangleleft u)^{-1} \triangleright (r \triangleright u)^{-1}w^{-1}\big]^{-1} \\
&\qquad \times ((r \triangleleft u)s \triangleleft v),
\end{aligned}
$$

as required on using the identity

$$
\begin{aligned}
&(r \triangleleft u)^{-1} \triangleright (r \triangleright u)^{-1}w^{-1} \\
&\qquad = \big((r \triangleleft u)^{-1} \triangleright (r \triangleright u)^{-1}\big)\Big(\big((r \triangleleft u)^{-1} \triangleleft (r \triangleright u)^{-1}\big) \triangleright w^{-1}\Big) \\
&\qquad = u^{-1}(r^{-1} \triangleright w^{-1}).
\end{aligned}
$$

∎

THEOREM 2.2. *$D(H) \cong kX \triangleright\!\blacktriangleleft k(Y)$ as Hopf algebras, by*

$$\psi: D(H) \to kX \triangleright\!\blacktriangleleft k(Y),$$
$$\psi(\delta_s \otimes u \otimes t \otimes \delta_v) = (s \triangleright u)^{-1} t \otimes \delta_{v(t \triangleleft v)^{-1} s^{-1} t}.$$

Over a field with involution, the map preserves the star operation.

Proof. The structure of $D(H)$ in the basis used is in [2]. We check that the linear map ψ is an algebra isomorphism to the smash product induced by $\tilde{\triangleright}$. Start with $\alpha = \delta_x \otimes q \otimes t \otimes \delta_v$ and $\beta = \delta_s \otimes u \otimes r \otimes \delta_w$ in $D(H)$, and multiply them together to get

$$\alpha\beta = \delta_{s' \triangleleft u', x} \delta_{v', r \triangleright w} (\delta_{s'} \otimes u'q \otimes t'r \otimes \delta_w),$$

where

$$t' = t \triangleleft (s \triangleright u)^{-1}, \qquad v' = (s \triangleright u) v u^{-1},$$
$$s' = t's(t \triangleleft vu^{-1})^{-1}, \qquad u' = (t \triangleleft vu^{-1}) \triangleright u.$$

Now we can calculate

$$\begin{aligned}
\psi(\alpha\beta) &= \delta_{s' \triangleleft u', x} \delta_{v', r \triangleright w} \Big((s' \triangleright u'q)^{-1} t'r \otimes \delta_{w(t'r \triangleleft w)^{-1} s'^{-1} t'r} \Big) \\
&= \delta_{s' \triangleleft u', x} \delta_{v', r \triangleright w} \Big(((s' \triangleleft u') \triangleright q)^{-1} (s' \triangleright u')^{-1} \\
&\qquad \times t'r \otimes \delta_{w(r \triangleleft w)^{-1} (t' \triangleleft (r \triangleright w))^{-1} s'^{-1} t'r} \Big) \\
&= \delta_{t(s \triangleleft u)(t \triangleleft v)^{-1}, x} \delta_{(s \triangleright u) v u^{-1}, r \triangleright w} \Big((x \triangleright q)^{-1} (t's \triangleright u)^{-1} \\
&\qquad \times t'r \otimes \delta_{w(r \triangleleft w)^{-1} (t' \triangleleft v')^{-1} s'^{-1} t'r} \Big) \\
&= \delta_{t(s \triangleleft u)(t \triangleleft v)^{-1}, x} \delta_{(s \triangleright u) v u^{-1}, r \triangleright w} \Big((x \triangleright q)^{-1} (t' \triangleleft (s \triangleright u)) \\
&\qquad \times (s \triangleright u)^{-1} r \otimes \delta_{w(r \triangleleft w)^{-1} s^{-1} r} \Big) \\
&= \delta_{t(s \triangleleft u)(t \triangleleft v)^{-1}, x} \delta_{(s \triangleright u) v u^{-1}, r \triangleright w} \Big((x \triangleright q)^{-1} t (s \triangleright u)^{-1} \\
&\qquad \times r \otimes \delta_{w(r \triangleleft w)^{-1} s^{-1} r} \Big).
\end{aligned}$$

Here we have used

$$\begin{aligned}
s' \lhd u' &= \left(t \lhd (s \rhd u)^{-1}\right)s(t \lhd vu^{-1})^{-1} \lhd \left((t \lhd vu^{-1}) \rhd u\right)\\
&= \left(\left(t \lhd (s \rhd u)^{-1}\right)s \lhd u\right)\left((t \lhd vu^{-1})^{-1} \lhd \left((t \lhd vu^{-1}) \rhd u\right)\right)\\
&= \left(t \lhd (s \rhd u)^{-1} \lhd (s \rhd u)\right)(s \lhd u)(t \lhd vu^{-1} \lhd u)^{-1}\\
&= t(s \lhd u)(t \lhd v)^{-1}.
\end{aligned}$$

Conversely, we can calculate the product in $kX \rhd\!\blacktriangleleft k(Y)$ as

$$\begin{aligned}
&\psi(\alpha)\psi(\beta)\\
&\quad= \left((x \rhd q)^{-1}t \otimes \delta_{v(t \lhd v)^{-1}x^{-1}t}\right)\left((s \rhd u)^{-1}r \otimes \delta_{w(r \lhd w)^{-1}s^{-1}r}\right)\\
&\quad= \delta_{v(t \lhd v)^{-1}x^{-1}t,\,(s \rhd u)^{-1}r \,\tilde{\rhd}\, w(r \lhd w)^{-1}s^{-1}r}(x \rhd q)^{-1}\\
&\qquad\times t(s \rhd u)^{-1}r \otimes \delta_{w(r \lhd w)^{-1}s^{-1}r}\\
&\quad= \delta_{v(t \lhd v)^{-1}x^{-1}t,\,(s \rhd u)^{-1}(r \rhd w)u(s \lhd u)^{-1}}(x \rhd q)^{-1}\\
&\qquad\times t(s \rhd u)^{-1}r \otimes \delta_{w(r \lhd w)^{-1}s^{-1}r}\\
&\quad= \delta_{v,\,(s \rhd u)^{-1}(r \rhd w)u}\,\delta_{(t \lhd v)^{-1}x^{-1}t,\,(s \lhd u)^{-1}}(x \rhd q)^{-1}\\
&\qquad\times t(s \rhd u)^{-1}r \otimes \delta_{w(r \lhd w)^{-1}s^{-1}r},
\end{aligned}$$

which is as required. That ψ preserves the unit is more immediate.

To compare the coproducts, we use the coproduct of $D(H)$, which is the tensor product

$$\begin{aligned}
&\Delta_{D(H)}(\delta_s \otimes u \otimes t \otimes \delta_v)\\
&\qquad= \sum_{ab=s,\ xy=v} (\delta_a \otimes b \rhd u \otimes t \otimes \delta_x) \otimes (\delta_b \otimes u \otimes t \lhd x \otimes \delta_y).
\end{aligned}$$

Applying $\psi \otimes \psi$ to this, we find the following expression for $(\psi \otimes \psi)\Delta_{D(H)}(\delta_s \otimes u \otimes t \otimes \delta_v)$:

$$\begin{aligned}
\sum_{ab=s,\ xw=v} &\left((s \rhd u)^{-1}t \otimes \delta_{x(t \lhd x)^{-1}a^{-1}t}\right)\\
&\otimes\left((b \rhd u)^{-1}(t \lhd x) \otimes \delta_{w(t \lhd v)^{-1}b^{-1}(t \lhd x)}\right).
\end{aligned}$$

Alternatively, we can calculate

$$\begin{aligned}
&\Delta_{kX \triangleright\!\blacktriangleleft k(Y)} \psi(\delta_s \otimes u \otimes t \otimes \delta_v)\\
&\quad = \Delta_{kX \triangleright\!\blacktriangleleft k(Y)}\big((s \triangleright u)^{-1} t \otimes \delta_{v(t \triangleleft v)^{-1} s^{-1} t}\big)\\
&\quad = \sum_{yz = v(t \triangleleft v)^{-1} s^{-1} t} (s \triangleright u)^{-1} t \otimes \delta_y \otimes (s \triangleright u)^{-1} t \,\tilde{\triangleleft}\, y \otimes \delta_z,
\end{aligned}$$

where $y, z \in Y$, which is the smash coproduct for the coaction induced by the back-reaction $\tilde{\triangleleft}$. We begin with the calculation

$$\big(x(t \triangleleft x)^{-1} a^{-1} t\big).\big(w(t \triangleleft v)^{-1} b^{-1} (t \triangleleft x)\big) = xw(t \triangleleft v)^{-1} b^{-1} a^{-1} t,$$

which shows that if we replace y by $x(t \triangleleft x)^{-1} a^{-1} t$ and z by $w(t \triangleleft v)^{-1} b^{-1} (t \triangleleft x)$, then the conditions of the summations are the same. It now remains to calculate

$$\begin{aligned}
(s \triangleright u)^{-1} t \,\tilde{\triangleleft}\, y &= (s \triangleright u)^{-1} t \,\tilde{\triangleleft}\, x(t \triangleleft x)^{-1} a^{-1} t = (a^{-1} s \triangleright u)^{-1} (t \triangleleft x)\\
&= (b \triangleright u)^{-1} (t \triangleleft x),
\end{aligned}$$

as required.

Next we consider the counit,

$$\begin{aligned}
\epsilon_{kX \triangleright\!\blacktriangleleft k(Y)}\big(\psi(\delta_s \otimes u \otimes t \otimes \delta_v)\big) &= \epsilon_{kX \triangleright\!\blacktriangleleft k(Y)}\big((s \triangleright u)^{-1} t \otimes \delta_{v(t \triangleleft v)^{-1} s^{-1} t}\big)\\
&= \delta_{v(t \triangleleft v)^{-1} s^{-1} t, e}.
\end{aligned}$$

The last δ-function splits into $\delta_{v,e} \delta_{(t \triangleleft v)^{-1} s^{-1} t, e}$, which is equal to $\delta_{v,e} \delta_{s,e}$, which is in turn equal to $\epsilon_{D(H)}(\delta_s \otimes u \otimes t \otimes \delta_v)$. Finally, we consider the antipode

$$\begin{aligned}
&S_{kX \triangleright\!\blacktriangleleft k(Y)}\big(\psi(\delta_s \otimes u \otimes t \otimes \delta_v)\big)\\
&\quad = S_{kX \triangleright\!\blacktriangleleft k(Y)}\big((s \triangleright u)^{-1} t \otimes \delta_{v(t \triangleleft v)^{-1} s^{-1} t}\big)\\
&\quad = \big((s \triangleright u)^{-1} t \,\tilde{\triangleleft}\, v(t \triangleleft v)^{-1} s^{-1} t\big)^{-1} \otimes \delta_{((s \triangleright u)^{-1} t \,\tilde{\triangleright}\, v(t \triangleleft v)^{-1} s^{-1} t)^{-1}}\\
&\quad = \big(u^{-1}(t \triangleleft v)\big)^{-1} \otimes \delta_{((s \triangleright u)^{-1} (t \triangleright v) s^{-1} (s \triangleright u))^{-1}}\\
&\quad = (t \triangleleft v)^{-1} u \otimes \delta_{((s \triangleright u)^{-1} (t \triangleright v) u (s \triangleleft u)^{-1})^{-1}}\\
&\quad = (t \triangleleft v)^{-1} u \otimes \delta_{u^{-1} (t \triangleright v)^{-1} (s \triangleright u)(s \triangleleft u)},
\end{aligned}$$

where we remember in the last line to take the inverse for the Y group operation, and compare this with

$$
\begin{aligned}
&\psi S_{D(H)}(\delta_s \otimes u \otimes t \otimes \delta_v)\\
&\quad = \psi\big((1 \otimes S(t \otimes \delta_v))(S^{-1}(\delta_s \otimes u) \otimes 1)\big)\\
&\quad = \psi\Big(\big(1 \otimes (t \triangleleft v)^{-1} \otimes \delta_{(t \triangleright v)^{-1}}\big)\big(\delta_{(s \triangleleft u)^{-1}} \otimes (s \triangleright u)^{-1} \otimes 1\big)\Big)\\
&\quad = \psi\Big(\delta_{t'\bar{s}(\bar{t} \triangleleft \bar{v}\bar{u}^{-1})^{-1}} \otimes (\bar{t} \triangleleft \bar{v}\bar{u}^{-1}) \triangleright \bar{u} \otimes t' \otimes \delta_{(\bar{s} \triangleright \bar{u})\bar{v}\bar{u}^{-1}}\Big),
\end{aligned}
$$

where $\bar{t} = (t \triangleleft v)^{-1}$, $\bar{v} = (t \triangleright v)^{-1}$, $\bar{s} = (s \triangleleft u)^{-1}$, $\bar{u} = (s \triangleright u)^{-1}$, and $t' = \bar{t} \triangleleft (\bar{s} \triangleright \bar{u})^{-1}$. Applying the definition of ψ, and using the fact that $\bar{s} \triangleright \bar{u} = u^{-1}$, we find

$$
\begin{aligned}
\psi S_{D(H)}(\delta_s \otimes u \otimes t \otimes \delta_v) &= (t'\bar{s} \triangleright \bar{u})^{-1} t'\\
&\qquad \otimes \delta_{(\bar{s} \triangleright \bar{u})\bar{v}\bar{u}^{-1}(t' \triangleleft (\bar{s} \triangleright \bar{u})\bar{v}\bar{u}^{-1})^{-1}(\bar{t} \triangleleft \bar{v}\bar{u}^{-1})\bar{s}^{-1}t'^{-1}t'}\\
&= (t' \triangleright u^{-1})^{-1} t' \otimes \delta_{u^{-1}\bar{v}\bar{u}^{-1}\bar{s}^{-1}}\\
&= (t' \triangleleft u^{-1}) u \otimes \delta_{u^{-1}\bar{v}(s \triangleright u)(s \triangleleft u)}\\
&= (t \triangleleft v)^{-1} u \otimes \delta_{u^{-1}(t \triangleright v)^{-1} s u},
\end{aligned}
$$

again as required. This concludes the proof of the Hopf algebra isomorphism. Now we show that the star operation is preserved:

$$
\begin{aligned}
&\psi * (\delta_s \otimes u \otimes t \otimes \delta_v)\\
&\quad = \psi\big((I \otimes t^{-1} \otimes \delta_{t \triangleright v})(\delta_{s \triangleleft u} \otimes u^{-1} \otimes I)\big)\\
&\quad = \psi\Big(\delta_{t'(s \triangleleft u)(t^{-1} \triangleleft (t \triangleright v)u)^{-1}} \otimes \big(t^{-1} \triangleleft (t \triangleright v)u\big) \triangleright u^{-1}\\
&\qquad\qquad \otimes t' \otimes \delta_{((s \triangleleft u) \triangleright u^{-1})(t \triangleright v)u}\Big);
\end{aligned}
$$

here $t' = t^{-1} \triangleleft (s \triangleright u)$. Applying the definition of ψ, we get

$$
\begin{aligned}
&\psi * (\delta_s \otimes u \otimes t \otimes \delta_v)\\
&\quad = (t'(s \triangleleft u) \triangleright u^{-1})^{-1} t'\\
&\qquad \otimes \delta_{(s \triangleright u)^{-1}(t \triangleright v)u(t^{-1} \triangleleft (t \triangleright v)u)^{-1}(t^{-1} \triangleleft (t \triangleright v)u)(s \triangleleft u)^{-1}t'^{-1}t'}\\
&\quad = ((t^{-1}s \triangleleft u) \triangleright u^{-1})^{-1} t' \otimes \delta_{(s \triangleright u)^{-1}(t \triangleright v)u(s \triangleleft u)^{-1}}\\
&\quad = t^{-1}(s \triangleright u) \otimes \delta_{(s \triangleright u)^{-1}(t \triangleright v)u(s \triangleleft u)^{-1}},
\end{aligned}
$$

$$\begin{aligned}
*\psi(\delta_s\otimes u\otimes t\otimes\delta_v)
&= *\Big((s\triangleright u)^{-1}t\otimes\delta_{v(t\triangleleft v)^{-1}s^{-1}t}\Big)\\
&= t^{-1}(s\triangleright u)\otimes\delta_{(s\triangleright u)^{-1}tv(t\triangleleft v)^{-1}s^{-1}tt^{-1}(s\triangleright u)}\\
&= t^{-1}(s\triangleright u)\otimes\delta_{(s\triangleright u)^{-1}(t\triangleright v)u(s\triangleleft u)^{-1}},
\end{aligned}$$

again as required. ∎

This means that $kX \triangleright\!\blacktriangleleft k(Y)$ inherits many of the nice properties of $D(H)$. In particular, it has a quasitriangular structure $\mathcal{R}$ and associated elements $Q=\mathcal{R}_{21}\mathcal{R}$ (the "quantum inverse Killing form") and $\mathrm{u}=\sum(S\mathcal{R}^{(2)})\mathcal{R}^{(1)}$ (the element which implements the square of the antipode) in Drinfeld's general theory of quasitriangular Hopf algebras [1].

COROLLARY 2.3. *The bicrossproduct $kX\triangleright\!\blacktriangleleft k(Y)$ is quasitriangular, with*

$$\mathcal{R}=\sum_{u,s,v,t} v^{-1}\otimes\delta_{us}\otimes s^{-1}\otimes\delta_{(s\triangleright v)t}.$$

The "quantum inverse Killing form" is

$$\mathcal{R}_{21}\mathcal{R}=\sum_{u,v\in G,\ s,p\in M} s^{-1}u^{-1}\otimes\delta_{u(s\triangleright v)(p\triangleleft u)u^{-1}}\otimes v^{-1}p^{-1}\otimes\delta_{pusp^{-1}}$$

and is nondegenerate as a bilinear functional on $k(X)\blacktriangleright\!\triangleleft kY$. The element u *is the canonical element* $\mathrm{u}=\sum_{x\in X}x\otimes\delta_x$ *in $kX\triangleright\!\blacktriangleleft k(Y)$, and is central.*

Proof. The computation is straightforward. Thus

$$\begin{aligned}
(\psi\otimes\psi)(\mathcal{R}) &= \sum\psi(\delta_s\otimes u\otimes e\otimes\delta_v)\otimes\psi(\delta_t\otimes e\otimes s\otimes\delta_u)\\
&= \sum(s\triangleright u)^{-1}\otimes\delta_{vs^{-1}}\otimes s\otimes\delta_{u(s\triangleleft u)^{-1}t^{-1}s}\\
&= \sum(s^{-1}\triangleright v)^{-1}\otimes\delta_{us}\otimes s^{-1}\otimes\delta_{vt},
\end{aligned}$$

which yields the formula shown on a change of variables v to $s\triangleright v$. We then compute $\mathcal{R}_{21}\mathcal{R}$ using the product in $kX\triangleright\!\blacktriangleleft k(Y)$:

$$\begin{aligned}
\mathcal{R}_{21}\mathcal{R} &= \Big(\sum_{u,s,v,t} s^{-1}\otimes\delta_{(s\triangleright v)t}\otimes v^{-1}\otimes\delta_{us}\Big)\\
&\qquad\times\Big(\sum_{u',s',v',t'} v'^{-1}\otimes\delta_{u's'}\otimes s'^{-1}\otimes\delta_{(s'\triangleright v')t'}\Big)\\
&= \sum_{u,s,v,t,u',s',v',t'}\Big(s^{-1}\otimes\delta_{(s\triangleright v)t}\Big)\Big(v'^{-1}\otimes\delta_{u's'}\Big)\\
&\qquad\otimes\Big(v^{-1}\otimes\delta_{us}\Big)\Big(s'^{-1}\otimes\delta_{(s'\triangleright v')t'}\Big)
\end{aligned}$$

$$= \sum_{u,s,v,t,u',s',v',t'} \delta_{(s\triangleright v)t,\, v'^{-1} \tilde{\triangleright} u's'} \delta_{us,\, s'^{-1} \tilde{\triangleright} (s'\triangleright v')t'} s^{-1} v'^{-1}$$

$$\otimes\ \delta_{u's'} \otimes v^{-1} s'^{-1} \otimes \delta_{(s'\triangleright v')t'}$$

$$= \sum_{u,s,v,t,u',s',v',t'} \delta_{(s\triangleright v)t,\, v'^{-1}u'(s'\triangleright v')(s'\triangleleft v')} \delta_{us,\, v'(s'\triangleleft v')^{-1}t's'} s^{-1} v'^{-1}$$

$$\otimes\ \delta_{u's'} \otimes v^{-1} s'^{-1} \otimes \delta_{(s'\triangleright v')t'}.$$

From the δ-functions here we can read off $s \triangleright v = v'^{-1}u'(s' \triangleright v')$, $t = s' \triangleleft v'$, $u = v'$, and $s = (s' \triangleleft v')^{-1}t's'$. If we rewrite these as $v' = u$, $s' = t \triangleleft u^{-1}$, $u' = u(s \triangleright v)(t \triangleright u^{-1})$, and $t' = ts(t \triangleleft u^{-1})^{-1}$, then, on substituting $p = t \triangleleft u^{-1}$,

$$\mathscr{R}_{21}\mathscr{R} = \sum_{u,v\in G,\, s,p\in M} s^{-1}u^{-1} \otimes \delta_{u(s\triangleright v)(p\triangleleft u)u^{-1}} \otimes v^{-1}p^{-1} \otimes \delta_{pusp^{-1}}.$$

Nondegeneracy of $\mathscr{R}_{21}\mathscr{R}$ as a linear map $D(H)^* \to D(H)$ is the so-called factorisability property holding for any quantum double [8]. Hence it carries over in our case to a linear isomorphism $k(X) \blacktriangleright\!\triangleleft\, kY \to kX \triangleright\!\blacktriangleleft k(Y)$ or, equivalently, to a nondegenerate bilinear functional on $k(X) \blacktriangleright\!\triangleleft\, kY$. Also,

$$\psi(\mathrm{u}) = \sum \psi\Big(\delta_s \otimes u \otimes (s \triangleleft u)^{-1} \otimes \delta_{(s\triangleright u)^{-1}}\Big)$$

$$= \sum (s \triangleright u)^{-1}(s \triangleleft u)^{-1} \otimes \delta_{(s\triangleright u)^{-1}((s\triangleleft u)^{-1} \triangleleft (s\triangleright u)^{-1})^{-1}s^{-1}(s\triangleleft u)^{-1}}$$

$$= \sum (s \triangleright u)^{-1}(s \triangleleft u)^{-1} \otimes \delta_{(s\triangleright u)^{-1}(s\triangleleft u)^{-1}} = \sum us \otimes \delta_{us}$$

on a change of summation variables in which $s \triangleright u$ is replaced by s^{-1} and $s \triangleright u$ by u^{-1}. Finally, the element u in any quasitriangular Hopf algebra implements the square of the antipode. But for any bicrossproduct, the antipode is involutive; hence u here is central. ∎

COROLLARY 2.4. *Over a field with involution, $kX \triangleright\!\blacktriangleleft k(Y)$ is antireal-quasitriangular in the sense* $(* \otimes *)(\mathscr{R}) = \mathscr{R}^{-1}$.

Proof. This is known for the quantum double $D(H)$ of any Hopf $*$-algebra [10], and hence follows from Theorem 2.2. ∎

As an application, the finite-dimensional modules of any quasitriangular Hopf algebra have a natural "quantum dimension" $\underline{\dim}$ defined as the trace of u in the representation. The modules of $\overline{kX \triangleright\!\blacktriangleleft k(Y)}$, as a cross product algebra, are just the Y-graded X-modules V such that $|x \triangleright v| = x \tilde{\triangleright} |v|$ for all $v \in V$ homogeneous of degree $|\ |$.

PROPOSITION 2.5. *The quantum dimension of a general $kX \triangleright\!\blacktriangleleft k(Y)$-module V is*

$$\underline{\dim}(V) = \sum_{y \in Y} \mathrm{trace}_{V_y} \pi(y),$$

where V_y is the subspace of degree y and $\pi(y)\colon V_y \to V_y$ is the restriction to V_y of the action of y viewed as an element of X.

Proof. We write $V = \oplus_y V_y$ for our Y-graded X-module. The action of $f \in k(Y)$ is $f(y)$ on V_y. A general element $x \in X$ acting on V sends $V_y \to V_{x \tilde{\triangleright} y}$. Hence, in particular, y viewed in X sends $V_y \to V_y$ as $y \tilde{\triangleright} y = y$ from Proposition 2.1. This is the operator on V_y denoted $\pi(y)$. Let $\{e_a^{(y)}\}$ be a basis of V_y, with dual basis $\{f^{a(y)}\}$. Then

$$\begin{aligned}\mathrm{Tr}(u) &= \sum_{y \in Y,\, x \in X} \sum_a \left\langle f^{a(y)}, (x \otimes \delta_x).e_a^{(y)} \right\rangle \\ &= \sum_{y \in Y} \sum_a \left\langle f^{a(y)}, y.e_a^{(y)} \right\rangle = \sum_{y \in Y} \mathrm{Tr}_{V_y} \pi(y).\end{aligned}$$

∎

For example, we may take the natural representation in $k(Y)$ by left multiplication of $k(Y)$ and the left action of X induced by its action on Y. This is the so-called Schrödinger representation of any cross product algebra. The spaces V_y are one-dimensional with basis $\{\delta_y\}$ and $\pi(y)\delta_y = \delta_y(y^{-1} \tilde{\triangleright} (\ )) = \delta_{y \tilde{\triangleright} y} = \delta_y$ is the identity. So $\underline{\dim}(k(Y)) = |Y| = \dim k(Y)$, where $|Y|$ is the order of group Y. So for this representation the quantum dimension is the usual dimension.

EXAMPLE 2.6. *We consider the factorisation of the group S_3 into a subgroup of order 3 and a subgroup of order 2. The quantum double of the associated bicrossproduct is $kS_3 \triangleright\!\blacktriangleleft k(C_6)$.*

Proof. Consider a factorisation of the group S_3 of permutations of three objects, which we label 1, 2, and 3. Let G be the subgroup consisting of the 3-cycles and the identity, and let the subgroup M consist of the transposition $(1,2)$ and the identity. Then, in the notation of this section, $X = S_3$, and $Y = GM^{\mathrm{op}}$ is a cyclic group of order 6. The left action of X on Y is the adjoint action of the group S_3 on the set S_3, and the right action of Y on X is given by

$$u \tilde{\triangleleft} v = u, \qquad u \tilde{\triangleleft} v(1,2) = u^{-1}, \qquad u(1,2) \tilde{\triangleleft} v = u(1,2),$$
$$u(1,2) \tilde{\triangleleft} v(1,2) = u^{-1}(1,2),$$

where u and v are any 3-cycles or the identity. This leads to a quasitriangular structure $\mathscr{R}$ on $kX \triangleright\!\blacktriangleleft k(Y)$, given by

$$\mathscr{R} = \sum_{u,v\in G,\, t\in M} v^{-1} \otimes \delta_u \otimes e \otimes \delta_{vt} + \sum_{u,v\in G,\, t\in M} v^{-1} \otimes \delta_{u(1,2)} \otimes (1,2) \otimes \delta_{v^{-1}t}.$$

The group $Y \bowtie X$ is of order 36 and is easily seen to have no center and 15 elements of order 2; hence it is isomorphic to $S_3 \times S_3$. ∎

3. SUBFACTORISATION FROM AN ORDER-REVERSING ISOMORPHISM

Let θ be an automorphism of X which reverses its factors GM (i.e., $\theta(G) = M$ and $\theta(M) = G$). It is shown in [2] that θ induces a semi-skew automorphism of $D(H)$ (i.e., an algebra antiautomorphism and coalgebra automorphism), which we denote Θ:

$$\Theta(\delta_s \otimes u \otimes t \otimes \delta_v) = \delta_{\theta(t\triangleright v)} \otimes \theta(t \triangleleft v) \otimes \theta(s \triangleright u) \otimes \delta_{\theta(s\triangleleft u)}. \quad (6)$$

Via Theorem 2.2, we may view this as a semi-skew automorphism of $kX \triangleright\!\blacktriangleleft k(Y)$. When the ground field is equipped with an involution, we may follow Θ by the star operation and obtain an antilinear Hopf algebra automorphism $*\Theta$.

LEMMA 3.1. *If θ is a factor-reversing automorphism of X then the induced antilinear automorphism of $kX \triangleright\!\blacktriangleleft k(Y)$ is given by*

$$*\Theta(x \otimes \delta_y) = \theta(x) \otimes \delta_{\theta(y)^{-1}}$$

when y is viewed in X (and the inverse is also in X).

Proof. We define $*\Theta$ via ψ and (6). Thus,

$$\begin{aligned}
*\psi&\left(\delta_{\theta(t\triangleright v)} \otimes \theta(t \triangleright v) \otimes \theta(s \triangleright u) \otimes \delta_{\theta(s\triangleleft u)}\right)\\
&= *\Big(\big(\theta(t \triangleright v) \triangleright \theta(t \triangleleft v)\big)^{-1}\theta(s \triangleright u)\\
&\qquad \otimes \delta_{\theta(s\triangleleft u)(\theta(s\triangleright u)\triangleleft\theta(s\triangleleft u))^{-1}\theta(t\triangleright v)^{-1}\theta(s\triangleright u)}\Big)\\
&= *\left(\theta(t^{-1}(s \triangleright u)) \otimes \delta_{\theta((s\triangleleft u)u^{-1}(t\triangleright v)^{-1}(s\triangleright u))}\right)\\
&= \theta\left((s \triangleleft u)^{-1}t\right) \otimes \delta_{\theta(t^{-1}s(t\triangleleft v)^{-1}v^{-1})}\\
&= *\Theta\psi(\delta_s \otimes u \otimes t \otimes \delta_v) = *\Theta\left((s \triangleright u)^{-1}t \otimes \delta_{v(t\triangleleft v)^{-1}s^{-1}t}\right),
\end{aligned}$$

as required after changing variables to general elements of X, Y. ∎

We observe that $*\Theta$-invariant basis elements $x\otimes\delta_y$ are characterized by the property that $\theta x = x$ and $\theta(y) = y^{-1}$ (computed in X).

PROPOSITION 3.2. *There is a subgroup X^θ of X consisting of those elements x for which $\theta x = x$, and a subset Y^θ of Y consisting of those elements y for which $\theta y = y^{-1}$ (inverse in X). The actions $\tilde{\triangleright}$, $\tilde{\triangleleft}$ restrict to X^θ, Y^θ, forming a double cross product group $Y^\theta \bowtie X^\theta$ factorising into Y^θ, X^θ. The corresponding bicrossproduct $kX^\theta \triangleright\!\blacktriangleleft k(Y^\theta)$ Hopf algebra has an isomorphic coalgebra to that of $kM \triangleright\!\blacktriangleleft k(G)$.*

Proof. The proof that the $\tilde{\triangleright}$ action restricts is immediate. If we take $x\in X^\theta$ and $y\in Y^\theta$, then $x\,\tilde{\triangleright}\, y = xyx^{-1}$ (adjoint action in the X multiplication). If we apply θ to this, then

$$\theta\big(x\,\tilde{\triangleright}\, y\big) = \theta\big(xyx^{-1}\big) = \theta(x)\theta(y)\theta(x^{-1}) = xy^{-1}x^{-1},$$

which in the inverse (in X) of $x\,\tilde{\triangleright}\, y = xyx^{-1}$, so $x\,\tilde{\triangleright}\, y\in Y^\theta$. The proof for the other action is rather more difficult, and we first find formulae for the elements of X^θ and Y^θ. If $y = vt \in Y^\theta$, then $\theta y = y^{-1}$ (inverse in X), so we can substitute $\theta(y) = \theta(v)\theta(t)$ and $y^{-1} = t^{-1}v^{-1}$ and use uniqueness of factorisation to say that $\theta(v) = t^{-1}$. Then we can write $y = Y(v) = v\theta(v)^{-1}$. Now we can write a simple formula for the multiplication in Y^θ as

$$\begin{aligned} y.y' = Y(v).Y(v') &= \big(v\theta(v)^{-1}\big).\big(v'\theta(v')^{-1}\big)\\ &= vv'\theta(v')^{-1}\theta(v)^{-1} = vv'\theta(vv')^{-1} = Y(vv'). \end{aligned}$$

This shows that Y^θ is actually isomorphic to G. Meanwhile, if $x = us\in X^\theta$, then $\theta x = x$, so we can substitute

$$\theta(x) = \theta(u)\theta(s) = x = us = \big(s^{-1}\triangleleft u^{-1}\big)^{-1}\big(s^{-1}\triangleright u^{-1}\big)^{-1}$$

and use uniqueness of factorisation to say that $\theta(s) = (s^{-1}\triangleright u^{-1})^{-1}$. Then $u^{-1} = s\triangleright\theta(s)^{-1}$, so we can write $x = X(s) = (s\triangleright\theta(s)^{-1})^{-1}s$. This shows that X^θ is bijective as a set with M. Finally, we may consider the right action

$$\begin{aligned} X(s)\,\tilde{\triangleleft}\, Y(v) &= \big(s\triangleright\theta(s)^{-1}\big)^{-1} s\,\tilde{\triangleleft}\, v\theta(v)^{-1}\\ &= \Big((s\triangleleft v)\theta(v)^{-1}s^{-1}\triangleright\big(s\triangleright\theta(s)^{-1}\big)\Big)^{-1}(s\triangleleft v)\\ &= \big((s\triangleleft v)\theta(v)^{-1}\triangleright\theta(s)^{-1}\big)^{-1}(s\triangleleft v)\\ &= \big((s\triangleleft v)\triangleright\theta(s\triangleleft v)^{-1}\big)^{-1}(s\triangleleft v) = X(s\triangleleft v). \end{aligned}$$

In particular, this shows that the result is in X^θ. We therefore have a bicrossproduct Hopf algebra $kX^\theta \triangleright\!\blacktriangleleft k(Y^\theta)$. Its coalgebra is determined by the action of Y^θ on X^θ and the group structure of Y^θ; hence we see that it is isomorphic via the maps X, Y to the coalgebra of $kM \triangleright\!\blacktriangleleft k(G)$. ∎

Also, by construction, we see that if we equip k with the trivial involution then

$$kX^\theta \triangleright\!\blacktriangleleft k(Y^\theta) \subseteq (kX \triangleright\!\blacktriangleleft k(Y))^{*\Theta}$$

as algebras. The right-hand side denotes the fixed point subalgebra under the algebra automorphism $*\Theta$. The inclusion is clear from Lemma 3.1 and the inclusion $X^\theta \subset X$ as subgroups and $k(Y^\theta) \subset k(Y)$ (extension by zero) as X^θ-module algebras. That $*\Theta$ is also a coalgebra automorphism tells us further that the coproduct of $kX \triangleright\!\blacktriangleleft k(Y)$ applied to elements of the fixed subalgebra yields elements invariant under $*\Theta \otimes *\Theta$. It is natural to ask to what extent the quasitriangular structure of $kX \triangleright\!\blacktriangleleft k(Y)$ is likewise invariant.

PROPOSITION 3.3. *The quasitriangular structure of $kX \triangleright\!\blacktriangleleft k(Y)$ obeys*

$$(\Theta \otimes \Theta)(\mathscr{R}) = \mathscr{R}_{21}.$$

When the field has an involution, we have $(*\Theta \otimes *\Theta)(\mathscr{R}) = \mathscr{R}_{21}^{-1}$. Moreover, if $\theta^2 = \mathrm{id}$ then $\Theta^2 = \mathrm{id}$ and $(*\Theta)^2 = \mathrm{id}$.

Proof. It is easier to do the first computations in $D(H)$. There, we have

$$\begin{aligned}(\Theta \otimes \Theta)(\mathscr{R}) &= \sum_{u,v,s,t} \Theta(\delta_s \otimes u \otimes e \otimes \delta_v) \otimes \Theta(\delta_t \otimes e \otimes s \otimes \delta_u) \\ &= \sum_{u,v,s,t} \left(\delta_{\theta(e \triangleright v)} \otimes \theta(e \triangleleft v) \otimes \theta(s \triangleright u) \otimes \delta_{\theta(s \triangleleft u)}\right) \\ &\quad \otimes \left(\delta_{\theta(s \triangleright u)} \otimes \theta(s \triangleleft u) \otimes \theta(t \triangleright e) \otimes \delta_{\theta(t \triangleleft e)}\right) \\ &= \sum_{u,s} \left(1 \otimes \theta(s \triangleright u) \otimes \delta_{\theta(s \triangleleft u)}\right) \otimes \left(\delta_{\theta(s \triangleright u)} \otimes \theta(s \triangleleft u) \otimes 1\right),\end{aligned}$$

where the sums over v, t are replaced by sums over $t' = \theta(v)$, $v' = \theta(t)$ and give the unit elements of $k(M)$ and $k(G)$, respectively. Then we change variables from u, s to $s' = \theta(s \triangleright u)$, $u' = \theta(s \triangleleft u)$, to recognise $\mathscr{R}_{21}$ in $D(H)$. Hence the same result applies for $kX \triangleright\!\blacktriangleleft k(Y)$. This combines with Corollary 2.4 to obtain the corresponding property for $*\Theta$.

Also, Θ in (6) clearly obeys

$$\Theta^2(\delta_s \otimes u \otimes t \otimes \delta_v) = \delta_{\theta^2(s)} \otimes \theta^2(u) \otimes \theta^2(t) \otimes \delta_{\theta^2(v)},$$

and hence $\Theta^2 = \mathrm{id}$ when $\theta^2 = \mathrm{id}$. The same feature for $*\Theta$ is immediate from Lemma 3.1. ∎

Hence $\mathscr{R}$ is not invariant in the usual sense (unless G, M are trivial), due to the nondegeneracy of $\mathscr{R}_{21}\mathscr{R}$ in Corollary 2.3. Rather, one should note that for any quasitriangular Hopf algebra, $\mathscr{R}_{21}^{-1}$ defines a second "conjugate" quasitriangular structure. It corresponds in topological applications to reversing braid crossings; we see that our quasitriangular structure is invariant up to conjugation in this sense. Although $kX^\theta \triangleright\!\blacktriangleleft k(Y^\theta)$ does not in general inherit a quasitriangular structure from $\mathscr{R}$, its inclusion as a fixed point subalgebra in a quasitriangular Hopf algebra equipped with such an automorphism might be a useful substitute. Of course, it may still happen that $kX^\theta \triangleright\!\blacktriangleleft k(Y^\theta)$ is quasitriangular for some other reason.

Another natural question, in view of Proposition 3.2, is whether $kX^\theta \triangleright\!\blacktriangleleft k(Y^\theta)$ is in fact isomorphic as a Hopf algebra to our original bicrossproduct $kM \triangleright\!\blacktriangleleft k(G)$. The following example also demonstrates that it is not necessarily isomorphic to it.

EXAMPLE 3.4. *We consider the example in* [2] *of the double cross product of two cyclic groups of order* 6 (C_6) *which gives the product of two symmetric groups* $S_3 \times S_3$. *In this case* $kX^\theta \triangleright\!\blacktriangleleft k(Y^\theta)$ *is isomorphic to* $kS_3 \triangleright\!\blacktriangleleft k(C_6)$ *in Example* 2.6 *and hence quasitriangular.*

Proof. Consider the group $X = S_3 \times S_3$ as the permutations of six objects labelled 1 to 6, where the first factor leaves the last three objects unchanged, and the second factor leaves the first three objects unchanged. We take G to be the cyclic group of order 6 generated by the permutation $1_G = (123)(45)$, and M to be the cyclic group of order 6 generated by the permutation $1_M = (12)(456)$. Our convention is that permutations act on objects on their right; for example, 1_G applied to 1 gives 2. The intersection of G and M is just the identity permutation, and counting elements shows that $GM = MG = S_3 \times S_3$. We write each cyclic group additively; for example, $G = \{0_G, 1_G, 2_G, 3_G, 4_G, 5_G\}$. The action of the element 1_M on G is seen to be given by the permutation $(1_G, 5_G)(2_G, 4_G)$, and that of 1_G on M is given by the permutation $(1_M, 5_M)(2_M, 4_M)$. The factor-reversing automorphism θ can be taken to be conjugation by the permutation $(1,4)(2,5)(3,6)$. Then if we split the elements of X into $S_3 \times S_3$, we see that the elements of X^θ are of the form $\sigma \times \sigma$, for $\sigma \in S_3$, and that the elements of Y^θ are of the form $\sigma \times \sigma^{-1}$. Then X^θ is isomorphic to S_3, and the action of X^θ on Y^θ is the adjoint action of the group S_3 on the set

S_3. Finally, deleting the points $4,5,6$ gives the explicit correspondence of $kX^\theta \bowtie k(Y^\theta)$ with the bicrossproduct in Example 2.6; we have the maps $\sigma \times \sigma \mapsto \sigma$ for X^θ and $\sigma \times \sigma^{-1} \mapsto \sigma$ for Y^θ, corresponding to the subsets of S_3 used in Example 2.6. ∎

4. A *-REPRESENTATION OF $D(H)$ ON A HILBERT SPACE

In this section we provide a Hilbert space representation of $D(H)$, which is one of the motivations behind Theorem 2.2. Recall that it was shown in [2] that representations of $D(H)$ are $G-M$-bigraded bicrossed $G-M$-bimodules. We shall use $|w|$ for the G-grading and $\langle w \rangle$ for the M-grading of a homogeneous element w of the representation.

PROPOSITION 4.1. *There is a representation of $D(H)$ on a vector space E with basis $\{\eta_{s,u} | s \in M,\ u \in G\}$, with gradings*

$$|\eta_{s,u}| = u, \qquad \langle \eta_{s,u} \rangle = s$$

and with group actions

$$t \triangleright \eta_{s,u} = \eta_{ts(t \triangleleft u)^{-1}, t \triangleright u}, \qquad \eta_{s,u} \triangleleft v = \eta_{s \triangleleft v, (s \triangleright v)^{-1} uv}.$$

The corresponding action of $D(H)$ is

$$(\delta_s \otimes u \otimes t \otimes \delta_v) \triangleright \eta_{r,w} = \delta_{v,w} \delta_{t^{-1}s(t \triangleleft v), r} \eta_{s \triangleleft u, (s \triangleright u)^{-1}(t \triangleright w)u}.$$

Proof. The definition of the group actions is made precisely so that the matching conditions in [2] are true. The corresponding actions of the Hopf algebras H^* and H are

$$(t \otimes \delta_v) \triangleright \eta_{s,u} = \delta_{v,u} t \triangleright \eta_{s,u}, \qquad \eta_{s,u} \triangleleft (\delta_t \otimes v) = \delta_{s,t} \eta_{s,u} \triangleleft v,$$

and the formula $(a \otimes h) \triangleright w = (h \triangleright w) \triangleleft a$ gives the action of $D(H)$ as

$$(I \otimes t \otimes \delta_v) \triangleright \eta_{s,u} = \delta_{v,u} \eta_{ts(t \triangleleft u)^{-1}, t \triangleright u},$$

$$(\delta_t \otimes v \otimes I) \triangleright \eta_{s,u} = \delta_{t,s} \eta_{s \triangleleft v, (s \triangleright v)^{-1} uv},$$

which gives the formula stated. ∎

As far as the original group double cross product is concerned, the E representation is more symmetric than the standard [8] "Schrödinger" representation of $D(H)$ in H, as we do not have to decide to take the group algebra of one factor subgroup and the function algebra of the other. The E representation is one of the motivations behind the isomorphism ψ in Theorem 2.2, when one compares it with the action of the type characteristic for a bicrossproduct.

Also, over $\mathbb{C}$, it is easy to verify that

$$(\eta_{s,u}, \eta_{r,w}) = \delta_{s,r}\delta_{u,w} \tag{7}$$

defines an inner product (η, ζ) on E (conjugate linear on the first variable and linear on the second) and

$$(\alpha \triangleright \eta, \zeta) = (\eta, \alpha^* \triangleright \zeta) \tag{8}$$

for any $\alpha \in D(H)$. Thus $D(H)$ with its natural $*$-structure is represented as a $*$-algebra.

PROPOSITION 4.2. *E with the coalgebra structure*

$$\Delta_E(\eta_{s,u}) = \sum_{ab=s,\, wz=u} \eta_{a,w} \otimes \eta_{b,z}, \qquad \epsilon(\eta_{s,u}) = \delta_{s,e}\delta_{u,e}$$

becomes a left-module coalgebra under the action of $D(H)$.

Proof. That E forms a coalgebra as stated is trivial. For the module coalgebra property

$$\Delta_E(\alpha \triangleright \eta) = \sum \alpha_{(1)} \triangleright \eta_{(1)} \otimes \alpha_{(2)} \triangleright \eta_{(2)}$$

we let $\alpha = 1 \otimes t \otimes \delta_v$ and leave the other case to the reader. We set $\eta = \eta_{s,u}$. Then

$$\begin{aligned} &\sum \alpha_{(1)} \triangleright \eta_{(1)} \otimes \alpha_{(2)} \triangleright \eta_{(2)} \\ &\qquad = \delta_{v,u} \sum_{xy=u,\, ab=s} \eta_{ta(t\triangleleft x)^{-1},\, t\triangleright x} \otimes \eta_{(t\triangleleft x)b(t\triangleleft u)^{-1},\, (t\triangleleft x)\triangleright y}. \end{aligned}$$

On the other hand, $\alpha \triangleright \eta = \delta_{v,u}\eta_{ts(t\triangleleft u)^{-1},\, t\triangleright u}$, and hence

$$\Delta_E(\alpha \triangleright \eta) = \delta_{v,u} \sum_{x'y'=t\triangleright u,\, a'b'=ts(t\triangleleft u)^{-1}} \eta_{a',x'} \otimes \eta_{b',y'},$$

as required on using the correspondences $a' = ta(t \triangleleft x)^{-1}$, $b' = (t \triangleleft x)b(t \triangleleft u)^{-1}$, $x' = t \triangleright x$, and $y' = (t \triangleleft x) \triangleright y$. We also need to show that $\epsilon(\alpha \triangleright \eta) = \epsilon(\alpha)\epsilon(\eta)$. Using $\alpha = \delta_s \otimes u \otimes t \otimes \delta_v$ and $\eta = \eta_{r,w}$, we have

$$\begin{aligned} \epsilon(\alpha \triangleright \eta) &= \delta_{v,w}\delta_{t^{-1}s(t\triangleleft v),r}\,\epsilon(\eta_{s\triangleleft u,(s\triangleright u)^{-1}(t\triangleright w)u}) \\ &= \delta_{v,w}\delta_{t^{-1}s(t\triangleleft v),r}\delta_{s\triangleleft u,e}\delta_{(s\triangleright u)^{-1}(t\triangleright w)u,e} \\ &= \delta_{v,w}\delta_{t^{-1}s(t\triangleleft v),r}\delta_{s,e}\delta_{(s\triangleright u)^{-1}(t\triangleright w)u,e} \\ &= \delta_{v,w}\delta_{t^{-1}(t\triangleleft v),r}\delta_{s,e}\delta_{u^{-1}(t\triangleright w)u,e} \end{aligned}$$

$$= \delta_{v,w}\delta_{t^{-1}(t\triangleleft v),r}\delta_{s,e}\delta_{(t\triangleright w),e} = \delta_{v,w}\delta_{t^{-1}(t\triangleleft v),r}\delta_{s,e}\delta_{w,e}$$
$$= \delta_{v,e}\delta_{t^{-1}(t\triangleleft v),r}\delta_{s,e}\delta_{w,e} = \delta_{v,e}\delta_{e,r}\delta_{s,e}\delta_{w,e}$$
$$= \epsilon(\delta_s \otimes u \otimes t \otimes \delta_v)\epsilon(\eta_{r,w}). \qquad \blacksquare$$

Over $\mathbb{C}$, it is easy to see that

$$(\Delta_E\eta, \Delta_E\zeta) = |X|(\eta, \zeta), \tag{9}$$

where we use the tensor product inner product on $E \otimes E$. Finally, suppose that θ is an order-reversing isomorphism of the double cross product group $X = GM$. As previously mentioned, from [2] we have an antilinear Hopf algebra automorphism $*\Theta$: $D(H) \to D(H)$.

PROPOSITION 4.3. *Over $\mathbb{C}$, there is an antilinear map $\hat{\theta}$: $E \to E$ defined by*

$$\hat{\theta}(\eta_{s,u}) = \eta_{\theta(u),\,\theta(s)},$$

which obeys

$$\hat{\theta}(\alpha \triangleright \eta) = (*\Theta\alpha) \triangleright \hat{\theta}(\eta), \qquad \forall\alpha \in D(H).$$

Proof. As usual we shall only prove this in the case $\alpha = I \otimes t \otimes \delta_v$ and leave the other case to the reader. We begin with

$$*\Theta(I \otimes t \otimes \delta_v) = *\big(\delta_{\theta(t\triangleright v)} \otimes \theta(t \triangleleft v)\big) \otimes I = \delta_{\theta(v)} \otimes \theta(t \triangleleft v)^{-1} \otimes I,$$

where we have used the equation $\theta(t \triangleright v) \triangleleft \theta(t \triangleleft v) = \theta(v)$. Now

$$\Big(\delta_{\theta(v)} \otimes \theta(t \triangleleft v)^{-1} \otimes I\Big) \triangleright \eta_{\theta(u),\,\theta(s)}$$
$$= \delta_{\theta(v),\,\theta(u)}\eta_{\theta(u)\triangleleft\theta(t\triangleleft v)^{-1},\,(\theta(u)\triangleright\theta(t\triangleleft v)^{-1})^{-1}\theta(s)\theta(t\triangleleft v)^{-1}}.$$

Since θ is 1–1, we can replace $\delta_{\theta(v),\,\theta(u)}$ by $\delta_{v,u}$. Also we calculate

$$\theta(u) \triangleright \theta(t \triangleleft v)^{-1} = \theta\big((t \triangleleft v) \triangleleft u^{-1}\big)^{-1},$$
$$\theta(u) \triangleleft \theta(t \triangleleft v)^{-1} = \theta\big((t \triangleleft v) \triangleright u^{-1}\big)^{-1},$$

so the equation above becomes

$$\Big(\delta_{\theta(v)} \otimes \theta(t \triangleleft v)^{-1} \otimes I\Big) \triangleright \eta_{\theta(u),\,\theta(s)}$$
$$= \delta_{v,u}\eta_{\theta((t\triangleleft v)\triangleright u^{-1})^{-1},\,\theta((t\triangleleft v)\triangleleft u^{-1})\theta(s)\theta(t\triangleleft v)^{-1}}.$$

Now, using the $\delta_{v,u}$ to put $v = u$ in the equation,

$$\left(\delta_{\theta(v)} \otimes \theta(t \triangleleft v)^{-1} \otimes I\right) \triangleright \eta_{\theta(u),\,\theta(s)}$$
$$= \delta_{v,u}\eta_{\theta(t \triangleright u),\,\theta(t)\theta(s)\theta(t\triangleleft u)^{-1}} = \delta_{v,u}\eta_{\theta(t\triangleright u),\,\theta(ts(t\triangleleft u)^{-1})},$$

which is the formula for $\hat{\theta}(\alpha \triangleright \eta_{s,u})$ as required. ∎

5. FIRST ORDER BICOVARIANT DIFFERENTIAL CALCULI ON H

In this section, we regard the Hopf algebra $A = H^* = k(M) \blacktriangleright\!\!\triangleleft kG$ associated to a group factorisation GM as a "coordinate ring" of some non-commutative geometric phase space. This is the point of view introduced in [4], where H^* is an algebraic model for the quantization of a particle on M moving along orbits under the action of G. Here we develop some of the "non-commutative geometry" associated to this point of view. To simplify reducibility questions we will assume throughout the section that k is algebraically closed and of characteristic zero.

First, on any algebra A, one may define a first order differential calculus or "cotangent space" Ω^1 in a standard way (cf. [11]):

1. Ω^1 is an A-bimodule.

2. d: $A \to \Omega^1$ is a linear map obeying the Leibniz rule $d(ab) = (da)b + a\,db$.

3. $A \otimes A \to \Omega^1$ given by $a \otimes b \mapsto a\,db$ is surjective.

When A is a Hopf algebra, we add left and right covariance (bicovariance) under A. Thus [6]

4. Ω^1 is an A-bicomodule and the bimodule structures and d are bicomodule maps.

Here $\Omega^1 \otimes A$ and $A \otimes \Omega^1$ have the induced tensor product bicomodule structures, where A is a bicomodule under its coproduct.

From [12] one knows that compatible bimodules and bicomodules (Hopf bimodules) are of the form (say) $\Omega^1 = V \otimes A$ for some left-crossed A-module V. The latter in our finite-dimensional setting means nothing more than left modules of the Drinfeld quantum double $D(A)$. A particular module is $\ker \epsilon \subset A$, a restriction of the canonical Schrödinger representation of $D(A)$ on A (by multiplication and the coadjoint action; see [8]). As observed in [6], the further conditions for Ω^1, d amount to requiring V to be a quotient of $\ker \epsilon$ as a quantum double module. Then

$$da = (\pi_V \otimes \mathrm{id})\big(a_{(1)} \otimes a_{(2)} - 1 \otimes a\big), \qquad \forall a \in A, \tag{10}$$

where $\pi_V\colon \ker \epsilon \to V$ is the quotient map. The right (co)module structure on $V \otimes A$ is by right (co)multiplication in A; the left (co)module structure is the tensor product of that of V and left (co)multiplication in A.

In the finite-dimensional setting which concerns us, one may equally well work in the dual picture in terms of $H = kM \triangleright\!\blacktriangleleft k(G)$ and $L = V^*$. By definition (cf. [6]), a *bicovariant quantum tangent space* L for H is a submodule of $\ker \epsilon \subset H$ under the quantum double $D(H)$. Here $D(H)$ acts on $\ker \epsilon$ by

$$h \triangleright g = h_{(1)} g S h_{(2)},$$
$$a \triangleright g = \langle a, g_{(1)} \rangle g_{(2)} - \langle a, g \rangle 1, \forall h \in H, \forall g \in \ker \epsilon \subset H, a \in H^*$$

as a projection to $\ker \epsilon$ of the Schrödinger representation. A quantum tangent space is *irreducible* if L is irreducible as a quantum double module. It corresponds to Ω^1 having no bicovariant quotients. This dual point of view has been used recently in [7], where the irreducible bicovariant quantum tangent spaces over a general quasitriangular Hopf algebra have been classified under the assumptions of $\mathcal{R}_{21}\mathcal{R}$ nondegenerate and a Peter–Weyl decomposition for the regular representation. In the same manner, we now classify the irreducible bicovariant quantum tangent spaces L when H is a bicrossproduct. The corresponding Ω^1 will be given as well.

The canonical or Schrödinger representation of $D(H)$ on $H = kM \triangleright\!\blacktriangleleft k(G)$ has been computed in [2]. Moreover, it is shown there that $D(H)$ is a coproduct twisting of $D(X)$, the double of the group algebra of X. Since $D(X) = k(X) \rtimes kX$ *as an algebra*, this will make it easier to decompose representations into irreducibles. One can also use the isomorphism in Theorem 2.2 to transfer to an action of $kX \triangleright\!\blacktriangleleft k(Y)$, but this appears to be less natural for the present purpose.

LEMMA 5.1. *The canonical action of $D(H)$ in* [2] *corresponds to an action of $k(X) \rtimes kX$ on H given by*

$$us \triangleright (t \otimes \delta_v) = (s'' t s^{-1}) \triangleleft u^{-1} \otimes \delta_{u(s \triangleright v)},$$
$$\delta_{us} \triangleright (t \otimes \delta_v) = \delta_{u, v((t \triangleleft v)^{-1} \triangleright v^{-1})} \delta_{s \triangleleft v, (t \triangleleft v)^{-1}} t \otimes \delta_v,$$

where $s'' = s \triangleleft v(t \triangleright v)^{-1}$.

Proof. Using the description of $D(H)$ modules as $M-G$-bicrossed bimodules (as recalled in Section 4), the canonical action is [2]

$$
\begin{gathered}
|t\otimes\delta_v| = (t\triangleright v)v^{-1}, \qquad \langle t\otimes\delta_v\rangle = t,\\
s\triangleright(t\otimes\delta_v) = sts'^{-1}\otimes\delta_{s'\triangleright v}, \qquad (t\otimes\delta_v)\triangleleft u = t\triangleleft u\otimes\delta_{u^{-1}v},
\end{gathered}
\tag{11}
$$

where $s' = s\triangleleft(t\triangleright v)v^{-1}$. The corresponding X-grading $\|\ \|$ and X-action can be computed from this $M-G$-bicrossed bimodule structure as $\|w\| = \langle w\rangle^{-1}|w|$ and $us\triangleright w = ((s\triangleleft|w|^{-1})\triangleright w)\triangleleft u^{-1}$ for all w in the module. Hence

$$
\begin{aligned}
\|t\otimes\delta_v\| &= t^{-1}(t\triangleright v)v^{-1} = v\big(t^{-1}\triangleleft(t\triangleright v)\big)v^{-1} = v(t\triangleleft v)^{-1}v^{-1},\\
us\triangleright(t\otimes\delta_v) &= \big(s''\triangleright(t\otimes\delta_v)\big)\triangleleft u^{-1}\\
&= \big(s''ts^{-1}\otimes\delta_{s\triangleright v}\big)\triangleleft u^{-1} = (s''ts^{-1})\triangleleft u^{-1}\otimes\delta_{u(s\triangleright v)}.
\end{aligned}
$$

∎

Motivated by the form of $\|t\otimes\delta_v\|$ in the proof of the preceding lemma, we chose new bases for the vector space on which $D(X)$ acts.

LEMMA 5.2. *Let $\phi_{vt} = t^{-1}\triangleleft v^{-1}\otimes\delta_v$. Here $\{\phi_{vt}\}$ is a basis of H labelled by $vt\in X$. Then the action in Lemma* 5.1 *is*

$$
us\triangleright\phi_{vt} = \phi_{usvt(s\triangleleft v)^{-1}}, \qquad \delta_{us}\triangleright\phi_{vt} = \delta_{us,\,vtv^{-1}}\phi_{vt}.
$$

Proof. Here $\|\phi_{vt}\| = vtv^{-1} = v(t\triangleright v^{-1})(t\triangleleft v^{-1})$ gives the action of δ_{us} by evaluation against the degree. This can be written more explicitly as $\delta_{us}\triangleright\phi_{vt} = \delta_{u,\,v(t\triangleright v^{-1})}\delta_{s,\,t\triangleleft v^{-1}}\phi_{vt}$ and is thereby equivalent to the action in Lemma 5.1. Moreover,

$$
\begin{aligned}
s''(t^{-1}\triangleleft v^{-1})s^{-1} &= \Big(s\triangleleft v\big((t^{-1}\triangleleft v^{-1})\triangleright v\big)^{-1}\Big)(t^{-1}\triangleleft v^{-1})s^{-1}\\
&= \big(s\triangleleft v(t^{-1}\triangleright v^{-1})\big)(t^{-1}\triangleleft v^{-1})s^{-1}\\
&= \big(((s\triangleleft v)t^{-1})\triangleleft v^{-1}\big)s^{-1}\\
&= \big((s\triangleleft v)t^{-1}(s\triangleleft v)^{-1}\big)\triangleleft(s\triangleright v)^{-1}\\
&= \big((s\triangleleft v)t(s\triangleleft v)^{-1}\big)^{-1}\triangleleft(s\triangleright v)^{-1}
\end{aligned}
$$

as $(s\triangleleft v)^{-1}\triangleright(s\triangleright v)^{-1} = v^{-1}$ and $(s\triangleleft v)^{-1}\triangleleft(s\triangleright v)^{-1} = s^{-1}$ for any matched pair of groups. Then

$$us \triangleright \phi_{vt} = us \triangleright \left(t^{-1} \triangleleft v^{-1} \otimes \delta_v\right) = \left(s''(t^{-1} \triangleleft v^{-1})s^{-1}\right) \triangleleft u^{-1} \otimes \delta_{u(s \triangleright v)}$$

$$= \left((s \triangleleft v)t(s \triangleleft v)^{-1}\right)^{-1} \triangleleft \left(u(s \triangleright v)\right)^{-1} \otimes \delta_{u(s \triangleright v)}$$

$$= \phi_{u(s \triangleright v)(s \triangleleft v)t(s \triangleleft v)^{-1}} = \phi_{usvt(s \triangleleft v)^{-1}}.$$

∎

Our task is to decompose $\ker \epsilon \subset H$ into irreducibles under this action of $k(X) \rtimes kX$. We begin by decomposing the action in Lemma 5.2 into irreducibles and afterwards projecting to $\ker \epsilon$. Note that Lemma 5.2 tells us that when we identify $H \cong kX$ as linear spaces by the above basis, the action of X is the linear extension of a certain group action of X on itself.

PROPOSITION 5.3. *Let X act on itself by the action $us \,\tilde{\triangleright}\, vt = usvt(s \triangleleft v)^{-1}$ as in Lemma* 5.2. *Let $\|vt\| \equiv \|\phi_{vt}\| = vtv^{-1}$ as an X-valued function on X.*

(i) *Let $\sim$ denote the equivalence relation on X defined by $vt \sim us$ if and only if $\|us\| = \|vt\|$. Then $\tilde{\triangleright}$ descends to an action of X on the quotient space $X/\sim$.*

(ii) *Let $\Xi_{[vt]} \subseteq X$ denote the isotropy subgroup of an equivalence class $[vt] \in X/\sim$. Then*

$$\Xi_{[vt]} = \{us \in X \mid us\|vt\| = \|vt\|us\},$$

the centraliser of $\|vt\|$ in X.

Proof. (i) may be verified directly. However, it follows from Lemma 5.2 since an action of $k(X) \rtimes kX$ (where X acts on X by the adjoint action in the semidirect product) requires $\|us \triangleright \phi_{vt}\| = us\|\phi_{vt}\|(us)^{-1}$. In terms of the group X, this is $\|us \,\tilde{\triangleright}\, vt\| = us\|vt\|(us)^{-1}$. This also implies (ii) since the group $\Xi_{[vt]}$ consists of $us \in X$ such that $us \,\tilde{\triangleright}\, vt \sim vt$, i.e., such that $\|us \,\tilde{\triangleright}\, vt\| = \|vt\|$. ∎

We denote by $\mathscr{O}_{[vt]}$ the orbit containing the point $[vt]$ in $X/\sim$.

EXAMPLE 5.4. *We may restrict attention to orbits of the form $\mathscr{O}_{[s]}$, where $s \in M$. Then the elements of the equivalence class $[s]$ may be identified with the subset of M fixed under the action of s, $[s] = \{sv | s \triangleright v = v\}$. The stabiliser $\Xi_{[s]}$ consists of all elements of X which commute with s. The action of $\Xi_{[s]}$ on elements of the equivalence class $[s]$ is given by $ut \,\tilde{\triangleright}\, sv = su(t \triangleright v)$. In the particular case where $s = e$, we get $[e] = G$ and $\Xi_{[e]} = X$.*

Proof. This may seem to be a rather specialised example, but in fact any orbit $\mathscr{O}_{[us]}$ in $X/\sim$ contains a point of the given form, since $[s] \in \mathscr{O}_{[us]}$. We compute

$$\begin{aligned}[s] &= \{vt|vtv^{-1} = s\} = \{vt|v(t \triangleright v^{-1}) = e, t \triangleleft v^{-1} = s\} \\ &= \{v(s \triangleleft v)|s \triangleright v = v\}.\end{aligned}$$

This can be simplified if we note that if $s \triangleright v = v$, then $v(s \triangleleft v) = (s \triangleright v)(s \triangleleft v) = sv$, giving the result stated. The action of $\Xi_{[s]}$ is computed as

$$\begin{aligned}ut \,\tilde{\triangleright}\, sv &= ut \,\tilde{\triangleright}\, v(s \triangleleft v) = utv(s \triangleleft v)(t \triangleleft v)^{-1} \\ &= utsv(t \triangleleft v)^{-1} = sutv(t \triangleleft v)^{-1} = su(t \triangleright v)\end{aligned}$$

as stated. ∎

For $\chi \in X/\sim$, define $\mathscr{S}_\chi = kp^{-1}(\chi) \subset kX$, where p is the canonical projection to $X/\sim$. Here $\mathscr{S}_\chi$ is the linear span of the elements of χ viewed as a leaf in X and is a linear Ξ_χ representation since, by definition, the action of Ξ_χ sends $p^{-1}(\chi)$ to itself.

PROPOSITION 5.5. *Let $\mathscr{O}$ be an orbit in $X/\sim$ under the action of X. Then $\mathscr{M}_{\mathscr{O}} = \oplus_{\chi \in \mathscr{O}} \mathscr{S}_\chi \subset kX$ is a subrepresentation under the action of $k(X) \rtimes kX$ in Lemma 5.2. Moreover, $kX = \oplus_{\mathscr{O}} \mathscr{M}_{\mathscr{O}}$ is a decomposition of kX into subrepresentations.*

Proof. The action of $\delta_{usu^{-1}} \in k(X)$ on kX in Lemma 5.2 is the projection operator

$$\delta_{usu^{-1}} \triangleright \phi_{vt} = \delta_{usu^{-1}, vtv^{-1}} \phi_{vt}.$$

This is evident since the action of $k(X)$ is by evaluation against $\|\phi_{vt}\|$ (or, explicitly, put $\delta_{usu^{-1}} = \delta_{u(s \triangleright u^{-1})(s \triangleleft u^{-1})}$ into Lemma 5.2). We see that $\pi_{[us]} \equiv \delta_{usu^{-1}} \triangleright$ projects kX onto the subspace $\mathscr{S}_{[us]}$ and

$$kX = \bigoplus_\chi \mathscr{S}_\chi = \bigoplus_{\mathscr{O}} \left(\bigoplus_{\chi \in \mathscr{O}} \mathscr{S}_\chi \right) = \bigoplus_{\mathscr{O}} \mathscr{M}_{\mathscr{O}} \tag{12}$$

as vector spaces. Then the operator

$$\pi_{\mathscr{O}} = \sum_{\chi \in \mathscr{O}} \pi_\chi$$

commutes with the action of $k(X) \rtimes kX$ and is a projection to $\mathscr{M}_{\mathscr{O}}$. To see that $\pi_{\mathscr{O}}$ does commute with the action of the algebra, we can calculate

$$\pi_\chi \circ (\delta_x \otimes y \triangleright) = (\delta_x \otimes y \triangleright) \circ \pi_{y^{-1} \tilde{\triangleright} \chi},$$

and note that the operation $y^{-1} \tilde{\triangleright}$ is a 1–1 correspondence on the set $\mathscr{O}$. ∎

In what follows, we fix an orbit $\mathscr{O} \subset X/\sim$ and a base point χ_0 on it. We denote by Ξ the isotropy subgroup at χ_0, and $\mathscr{S} = \mathscr{S}_{\chi_0}$.

PROPOSITION 5.6. *Let $\mathscr{S} = \mathscr{S}_1 \oplus \cdots \oplus \mathscr{S}_n$ be a decomposition into irreducibles under the action of Ξ. Let $\chi.\mathscr{S}_i = us \triangleright \mathscr{S}_i$ when $\chi = us \tilde{\triangleright} \chi_0$ (this is independent of the choice of us). Then*

$$\mathscr{M}_i = \bigoplus_{\chi \in \mathscr{O}} \chi.\mathscr{S}_i \subset \mathscr{M}_{\mathscr{O}}$$

are irreducible subrepresentations under $k(X) \rtimes kX$. Moreover, $\mathscr{M}_{\mathscr{O}} = \oplus_i \mathscr{M}_i$ is a decomposition of $\mathscr{M}_{\mathscr{O}}$ into irreducibles.

Proof. Let $x_\chi \in X$ be a choice of us such that $us \tilde{\triangleright} \chi_0 = \chi$. Define $\chi.\mathscr{S}_i = x_\chi \triangleright \mathscr{S}_i$. Now if we take any x so that $x \tilde{\triangleright} \chi_0 = \chi$, then $x_\chi^{-1} x \in \Xi$ so

$$x \triangleright \mathscr{S}_i = x_\chi \triangleright \left(x_\chi^{-1} x \triangleright \mathscr{S}_i\right) = x_\chi \triangleright \mathscr{S}_i = \chi.\mathscr{S}_i,$$

as $\mathscr{S}_i$ is a representation of Ξ. Moreover, $\chi.\mathscr{S}_i \cap \eta.S_i = \{0\}$ for χ, η distinct, so $\mathscr{M}_i$ spanned as shown is a direct sum.

Let $P_i\colon \mathscr{S} \to \mathscr{S}$ be a Ξ-map which projects to $\mathscr{S}_i \subset \mathscr{S}$, with all the other $\mathscr{S}_j$ contained in its kernel. Now define the map $Q_i\colon \mathscr{M}_{\mathscr{O}} \to \mathscr{M}_{\mathscr{O}}$ by the formula

$$Q_i = \sum_{\chi \in \mathscr{O}} (x_\chi \triangleright) \circ P_i \circ \left(x_\chi^{-1} \triangleright\right) \circ \pi_\chi,$$

and observe that Q_i is a projection to $\mathscr{M}_i$. We now show that it is a $k(X) \rtimes kX$ map. Begin with the equations

$$\begin{aligned} Q_i \circ (\delta_x \otimes y \triangleright) &= \sum_{\chi \in \mathscr{O}} (x_\chi \triangleright) \circ P_i \circ \left(x_\chi^{-1} \triangleright\right) \circ \pi_\chi \circ (\delta_x \otimes y \triangleright) \\ &= \sum_{\chi \in \mathscr{O}} (x_\chi \triangleright) \circ P_i \circ \left(x_\chi^{-1} \triangleright\right) \circ (\delta_x \otimes y \triangleright) \circ \pi_{y^{-1} \tilde{\triangleright} \chi}. \end{aligned}$$

Now set $\eta = y^{-1} \tilde{\triangleright} \chi$ and write

$$\left(x_\chi^{-1} \triangleright\right) \circ (\delta_x \otimes y \triangleright) \circ \pi_{y^{-1} \tilde{\triangleright} \chi} = \left(\delta_{x_\chi^{-1} x x_\chi} \otimes x_\chi^{-1} y x_\eta \triangleright\right) \circ \left(x_\eta^{-1} \triangleright\right) \circ \pi_\eta,$$

where $x_\chi^{-1} y x_\eta \in \Xi$ and so commutes with P_i. Also $\delta_{x_\chi^{-1} x x_\chi}$ is either zero or the identity on all of $\mathscr{S}$ and so commutes with P_i too. Then we see that

$$\begin{aligned} Q_i \circ (\delta_x \otimes y \triangleright) &= \sum_{\chi \in \mathscr{O}} (x_\chi \triangleright) \circ \left(\delta_{x_\chi^{-1} x x_\chi} \otimes x_\chi^{-1} y x_\eta \triangleright\right) \circ P_i \circ \left(x_\eta^{-1} \triangleright\right) \circ \pi_\eta \\ &= \sum_{\chi \in \mathscr{O}} (\delta_x \otimes y \triangleright) \circ (x_\eta \triangleright) \circ P_i \circ \left(x_\eta^{-1} \triangleright\right) \circ \pi_\eta = (\delta_x \otimes y \triangleright) \circ Q_i. \end{aligned}$$

Then each $\mathscr{M}_i$ is a representation of the algebra and, since $Q_i Q_j = 0$ for $i \neq j$, we see that $\mathscr{M}_\mathscr{O} = \oplus_i \mathscr{M}_i$.

We now prove irreducibility. Let $m \in \mathscr{M}_i$ be nonzero. Then there is some χ such that the projection m_χ to $\chi.\mathscr{S}_i$ is nonzero and then $x_\chi^{-1} \triangleright m_\chi$ is a nonzero element of $\mathscr{S}_i$. Since $\mathscr{S}_i$ is irreducible under $\Xi \subseteq X$, we know that vectors of the form $\xi x_\chi^{-1} \triangleright m_\chi$, for $\xi \in \Xi$, span all of $\mathscr{S}_i$. Since the projection is itself the action of an element of $k(X) \rtimes kX$, we see that $\mathscr{S}_i$ is contained in the space spanned by the action of this algebra on m. Moreover, by using $x_\eta \triangleright \mathscr{S}_i = \eta.\mathscr{S}_i$ we see that every $\eta.\mathscr{S}_i$ is contained in the image of m under $k(X) \rtimes kX$. Hence $\mathscr{M}_i = (k(X) \rtimes kX).m$; i.e., $\mathscr{M}_i$ is irreducible. ∎

These two propositions give a total decomposition of kX into irreducibles. In particular, we obtain irreducible subrepresentations for every choice of orbit and every irreducible subrepresentation of the associated isotropy group. The converse also holds by similar arguments to those in the preceding proposition.

PROPOSITION 5.7. *Let $\mathscr{M} \subset kX$ be an irreducible subrepresentation under $k(X) \rtimes kX$ in Lemma* 5.2. *Then as a vector space $\mathscr{M}$ is of the form*

$$\mathscr{M} = \bigoplus_{\chi \in \mathscr{O}} \chi.\mathscr{M}_0$$

for some orbit $\mathscr{O}$ (with base point χ_0) and some irreducible subrepresentation $\mathscr{M}_0 \subset \mathscr{S}$ under Ξ. (Here $\chi.\mathscr{M}_0 = us \triangleright \mathscr{M}_0$ when $\chi = us \tilde{\triangleright} \chi_0$.)

Proof. Consider $\mathscr{M} \subset kX$ and let $\mathscr{M}_\chi = \pi_\chi(\mathscr{M})$ for any $\chi \in X/\sim$. Choose a χ_0 so that $\mathscr{M}_{\chi_0}$ is nonzero, write $\mathscr{M}_0 = \mathscr{M}_{\chi_0}$, and let Ξ be the stabiliser of χ_0. Now $\mathscr{M}_0$ must be a representation of Ξ. If $\mathscr{S}_1$ is an irreducible subrepresentation of $\mathscr{M}_0$ under Ξ, then, by the previous proposition,

$$\bigoplus_{\chi \in \mathscr{O}} \chi . \mathscr{S}_1 \subset \mathscr{M}$$

is an (irreducible) representation of the algebra, where $\mathscr{O}$ is the orbit containing χ_0. Since $\mathscr{M}$ is irreducible, this representation must be $\mathscr{M}$, and in particular $\mathscr{M}_0$ is irreducible. ∎

It is now a short step to obtain the classification for subrepresentations of $\ker \epsilon \subset H$. Note that every Hopf algebra is a direct sum $k1 \oplus \ker \epsilon$ as vector spaces, with associated projection $\Pi(h) = h - 1\epsilon(h)$. In our case, $\ker \epsilon$ is spanned by the projected basis elements

$$\bar{\phi}_{vt} \equiv \Pi(\phi_{vt}) = \phi_{vt} - \delta_{v,e} \sum_{u \in G} \phi_{ue}$$

and the action in Lemma 5.2 projects to the action

$$us \triangleright \bar{\phi}_{vt} = \bar{\phi}_{us \tilde{\triangleright} vt}, \qquad \delta_{us} \triangleright \bar{\phi}_{vt} = \delta_{us, vtv^{-1}} \bar{\phi}_{vt} \tag{13}$$

on $\ker \epsilon$ (such that Π: $kX \to \ker \epsilon$ is an intertwiner).

THEOREM 5.8. *The irreducible quantum tangent spaces* $L \subset \ker \epsilon$ *are all given by the following two cases*:

(a) *For an orbit* $\mathscr{O} \neq \{|e|\}$ *in* $X/\sim$, *choose a base point* $\chi_0 \in \mathscr{O}$. *For each irreducible subrepresentation* $\mathscr{M}_0 \subset \mathscr{S}$ *of* Ξ, *we have an irreducible quantum double subrepresentation* $\mathscr{M} = \oplus_{\chi \in \mathscr{O}} \chi . \mathscr{M}_0 \subset kX$ *and an isomorphic quantum double subrepresentation* $L = \Pi(\mathscr{M}) \subset \ker \epsilon$.

(b) *For the orbit* $\mathscr{O} = \{|e|\}$ *in* $X/\sim$, *choose the base point* $\chi_0 = [e] \in \mathscr{O}$. *For any irreducible subrepresentation* $\mathscr{M}_0 \subset \mathscr{S}$ *of* Ξ *other than the trivial one* (*multiples of* 1), *we have an irreducible quantum double subrepresentation* $\mathscr{M} = \oplus_{\chi \in \mathscr{O}} \chi . \mathscr{M}_0 \subset kX$ *and an isomorphic quantum double subrepresentation* $L = \Pi(\mathscr{M}) \subset \ker \epsilon$. *Here* $\mathscr{S} = kG$ *and* $\Xi = X$, *as in Example* 5.4.

Proof. In the cases above, $\mathscr{M} = \oplus_{\chi \in \mathscr{O}} \chi . \mathscr{M}_0$ is an irreducible representation of the unprojected action. By the above remarks, Π: $\mathscr{M} \to L$ is a map of representations, where $L \subset \ker \epsilon$ uses the projected representation. The map is onto, and if it is 1–1 then the two representations are isomorphic, and hence L is also irreducible.

The only case where the map Π is not 1–1 is where $1 \in \mathscr{M}$. Since $k1$ is a representation and $\mathscr{M}$ is assumed irreducible, this is just the nontriviality exclusion stated in the theorem. We have shown that the cases described lead to irreducible subrepresentations of $\ker \epsilon$.

Now suppose that we have L, an irreducible subrepresentation of $\ker\epsilon$. Then its inverse image $\Pi^{-1}L \subset H$ is a representation of $k(X) \rtimes kX$ and contains the subrepresentation $k1$. if $L \neq \{0\}$, then there is at least one other irreducible subrepresentation $\mathscr{M} \subset H$ so that $k1 \oplus \mathscr{M} \subset \Pi^{-1}L$. Then $\mathscr{M}$ must be of the form described earlier, and by irreducibility $\Pi\mathscr{M} = L$. ∎

We are left only with the standard problem of finding the irreducible subrepresentations of $\mathscr{S}$ in Theorem 5.8 under the group Ξ. It is well known (and follows easily from Schur's lemma) that if $\mathscr{M}_i$ are inequivalent irreducible representations of a group Ξ, irreducible subrepresentations of

$$\mathscr{M}_1^{\oplus n_1} \oplus \cdots \oplus \mathscr{M}_k^{\oplus n_k}$$

are of the form

$$\left\{0 \oplus \cdots \oplus \left(\lambda_1 w \oplus \cdots \oplus \lambda_{n_j} w\right) \oplus \cdots \oplus 0 \in \mathscr{N}_1^{\oplus n_1} \oplus \cdots \oplus \mathscr{N}_k^{\oplus n_k} | w \in \mathscr{N}_j\right\}$$

for some j and some $(\lambda_1, \ldots, \lambda_{n_j}) \in \mathbb{P}(k^{n_j})$.

Thus, given an orbit $\mathscr{O}$ with base point χ_0, we decompose $\mathscr{S}$ into irreducibles. The full data set for a quantum tangent space is then $(\mathscr{O}, \mathscr{M}_0, \lambda)$, where $\mathscr{M}_0$ is an irreducible representation of Ξ occurring in $\mathscr{S}$ with multiplicity $n > 0$, and $\lambda \in \mathbb{P}(k^n)$.

Bicrossproducts interpolate between group algebras and group function algebras. As a check, we recover the seemingly quite disparate results known separately for these two cases.

COROLLARY 5.9. *Suppose that G is trivial. Then $H = kM$. In this case $\sim$ is the same as equality and $X/\sim\, = X$. In this case the equivalence classes are singletons corresponding to points in M and $s \,\tilde{\triangleright}\, t = sts^{-1}$ is conjugation. Hence the orbits $\mathscr{O}$ are conjugacy classes in M. The action of the isotropy group is trivial and hence these are the only data. We recover the result* [13, 7] *that the irreducible quantum tangent spaces are the projected spans of the nontrivial conjugacy classes.*

COROLLARY 5.10. *Suppose that M is trivial. Then $H = k(G)$. In this case X is one entire equivalence class and $X/\sim\, = \{[e]\}$. Then there is only one orbit $\mathscr{O} = [e]$ and Theorem* 5.8 *reduces to the classification of irreducible quantum tangent spaces in* $\ker\epsilon \subset k(G)$, *which has been obtained in* [7] *as irreducible subrepresentations of the left regular representation. This gives the classification as pairs (V, λ) where V is a nontrivial irreducible representation and $\lambda \in P(V^*)$.*

We can also keep M, G nontrivial but let $\triangleright$ or $\triangleleft$ be trivial in the matched pair so that X is a semidirect product. In one case H is a tensor product algebra, while in the other it is a cross product $kM \ltimes k(G)$ where M acts by group automorphisms of G.

COROLLARY 5.11. *Suppose that the action* $\triangleright$ *is trivial. If we start with an orbit* $\mathscr{O}$ *containing* $[s]$, *then* $[s] = sG$ *and* $\Xi_{[s]} = \{ut \mid tst^{-1} = s \triangleleft u\}$. *The canonical projection* $ut \mapsto u$ *is a group homomorphism* $\Xi \to G$ *and, identifying* $\mathscr{S} = kG$, *the action of* Ξ *is the pull back along this of the left regular representation.*

COROLLARY 5.12. *Suppose that the action* $\triangleleft$ *is trivial. If we start with an orbit* $\mathscr{O}$ *containing* $[s]$, *then* $[s] = \{sv \mid sv = vs\}$ *may be identified with* G_s, *the intersection with* G *of the centraliser of* $s \in X$. *The isotropy group is* $\Xi_{[s]} = \{ut \mid su = us \text{ and } st = ts\} = G_s \times M_s$, *where* M_s *is the centraliser of* $s \in M$. *Identifying* $\mathscr{S} = kG_s$, *the action of* Ξ *is with* M_s *acting by* $\triangleright$ *and* G_s *acting by the left regular representation.*

As well as recovering these known special cases, we may now classify the quantum tangent spaces for nontrivial bicrossproducts such as those occurring earlier in the paper.

EXAMPLE 5.13. *We completely solve the problem by hand for Example* 2.6 *where* $X = S_3$. *We decompose* $k(X)$ *into irreducible subspaces* $\mathscr{M}$ *under* (13) *and project to* $\ker \epsilon$. *We verify that the result agrees with the above theory in this special case.*

The Orbit Decomposition. First we find the allowed orbits:

Orbit 1. $\mathscr{O} = \{[e]\}$, choosing base point $[e]$. Then $[e] = \{e, (123), (321)\}$ and $\Xi = S_3$. The action of Ξ on $[e]$ is given by the formula $ut \,\tilde{\triangleright}\, v = utvt^{-1}$. We have the eigenspaces of the G action $\mathscr{M}_0 = \langle e + (123) + (321)\rangle$, $\mathscr{M}_1 = \langle e + \omega(123) + \omega^2(321)\rangle$, and $\mathscr{M}_2 = \langle e + \omega^2(123) + \omega(321)\rangle$. Here ω is a primitive third root of unity. Now $\mathscr{M}_0$ forms a one-dimensional representation of Ξ, but this is annihilated by Π. The action of $(12) \in \Xi$ is to swap $\mathscr{M}_1$ and $\mathscr{M}_2$, so we get a two-dimensional irreducible representation $\mathscr{M}_1 \oplus \mathscr{M}_2$ of Ξ, giving a two-dimensional irreducible subrepresentation in $\ker \epsilon$.

Orbit 2. $\mathscr{O} = \{[(12)], [(13)], [(23)]\}$, choosing base point $[(12)]$. Then $[(12)] = \{(12)\}$ and $\Xi = M$. The action of Ξ on $[(12)]$ is the trivial one-dimensional representation, giving a three-dimensional irreducible representation in $\ker \epsilon$.

The Direct Approach. The space $\ker \epsilon$ is spanned by the vectors $\{\overline{\phi}_x \mid x \in S_3\}$, where $\overline{\phi}_x = \Pi \phi_x$, and there is the linear relation $\overline{\phi}_e + \overline{\phi}_{(123)} + \overline{\phi}_{(321)} = 0$. We use the relation to rewrite $\overline{\phi}_{(321)} = -\overline{\phi}_e - \overline{\phi}_{(123)}$, giving a basis consisting of five elements. This is then split into two parts by the action of X:

(1) The space spanned by the elements $\{\overline{\phi}_e, \overline{\phi}_{(123)}\}$. This has the X action $us \triangleright \overline{\phi}_e = \overline{\phi}_u$ and $us \triangleright \overline{\phi}_{(123)} = \overline{\phi}_{us(123)s^{-1}}$, where we remember to rewrite $\overline{\phi}_{(321)} = -\overline{\phi}_e - \overline{\phi}_{(123)}$. This gives a two-dimensional irreducible representation.

(2) The space spanned by the elements $\{\overline{\phi}_{(12)}, \overline{\phi}_{(13)}, \overline{\phi}_{(23)}\}$. This has the X action $us \triangleright \overline{\phi}_{v(12)} = \overline{\phi}_{usvs^{-1}(12)}$ for all $v \in G$. This gives a three-dimensional irreducible representation.

EXAMPLE 5.14. *We apply the theory above to Example* 3.4 *where* $X = C_6 \bowtie C_6$ *and exhibit the final result. This is rather complicated to do directly* (*hence justifying our methods*). *It is one of the simplest examples* [2] *of a true bicrossproduct Hopf algebra.*

First we identify the possible values of $\|\cdot\|$ and the orbits of these values under $X = S_3 \times S_3$. Since $\|vt\| = vtv^{-1}$, the possible values and the orbits are simply given by looking at the conjugacy classes of the elements $t \in M$ in X. These are:

Orbit 0. $t = 0_M$ gives the conjugacy class consisting only of the identity. We choose $[0_M]$ as the base point for this orbit. Then $\Xi_{[0]} = X$, and the equivalence class is $[0_M] = G$ (as previously noted). The action of $\Xi_{[0]} = X$ on $[0_M] = G$ is given by the formula $ut \,\tilde{\triangleright}\, v = u(t \triangleright v)$.

Let us now look at the decomposition of $\mathscr{S} = kG$ into irreducibles under the action of $\Xi_{[0]} = X$. The element $1_G \in \Xi$ acts on any irreducible $\mathscr{M} \subset \mathscr{S}$, and its action diagonalises; that is, $\mathscr{M}$ is a sum of $\mathscr{M}_r (r \in C_6)$, where each $\mathscr{M}_r$ is zero or $\langle f_r \rangle_k$, where

$$f_r = 0_G + \omega^r 1_G + \omega^{2r} 2_G + \omega^{3r} 3_G + \omega^{4r} 4_G + \omega^{5r} 5_G$$

for ω a primitive sixth root of unity. The action of 1_M is to send f_r to f_{-r}, so we get the four irreducible representations $\mathscr{S}_1 = \langle f_0 \rangle$, $\mathscr{S}_2 = \langle f_1, f_5 \rangle$, $\mathscr{S}_3 = \langle f_2, f_4 \rangle$, and $\mathscr{S}_4 = \langle f_3 \rangle$. The two one-dimensional representations $\mathscr{S}_1$ and $\mathscr{S}_4$ are not equivalent, as $\mathscr{S}_1$ is the trivial representation and $\mathscr{S}_4$ is not trivial. To show that $\mathscr{S}_2$ and $\mathscr{S}_3$ are not equivalent, we use the trace of 1_G on the representations, which is $\omega + \omega^{-1}$ on $\mathscr{S}_2$ and $\omega^2 + \omega^{-2}$ on $\mathscr{S}_3$.

There are four inequivalent irreducible representations for this orbit, but one is annihilated by Π, leaving three irreducible representations in $\ker \epsilon$ on application of Π.

Orbit 1. $t = 1_M$ and $t = 5_M$ give the conjugacy class consisting of elements of the form (any 2-cycle in $1, 2, 3$) (any 3-cycle in $4, 5, 6$). We choose $[1_M]$ as the base point for this orbit. Then $\Xi_{[1]} = M$, and the equivalence class is $[1_M] = \{1_M v | 1_M \triangleright v = v\}$. Since 1_M acts on G by the permutation $(1_G, 5_G)(2_G, 4_G)$, we find that $[1_M] = \{1_M 0_G, 1_M 3_G\}$. The action of $\Xi_{[1]} = M$ on $[1_M]$ is given by the formula $t \triangleright 1_M v = 1_M(t \triangleright v)$, which is the trivial action since both 0_G and 3_G are fixed by the left action of M.

The decomposition of $\mathscr{S} = k\{1_M 0_G, 1_M 3_G\}$ into irreducibles under the action of $\Xi_{[1]} = M$ gives two copies of the trivial one-dimensional representation.

Orbit 2. $t = 2_M$ and $t = 4_M$ give the conjugacy class consisting of elements of the form (any 3-cycle in $4, 5, 6$). We choose $[2_M]$ as the base point for this orbit. Then $\Xi_{[2]} = S_3 \times C_3$, where C_3 is the group of permutations of $\{4, 5, 6\}$ consisting of the identity and the two 3-cycles. In terms of the factorisation, $\Xi_{[2]}$ consists of all elements of the form ut for $u \in \{0_G, 2_G, 4_G\}$ and $t \in M$. The equivalence class is $[2_M] = \{2_M v | 2_M \triangleright v = v\}$. Since 2_M has the trivial action on G, we find that $[2_M] = \{2_M 0_G, 2_M 1_G, 2_M 2_G, 2_M 3_G, 2_M 4_G, 2_M 5_G\}$. Under the $\Xi_{[2]}$ action $ut \tilde{\triangleright} sv = su(t \triangleright v)$, $[2_M]$ splits into two orbits, $\{2_M 0_G, 2_M 2_G, 2_M 4_G\}$ and $\{2_M 1_G, 2_M 3_G, 2_M 5_G\}$.

First we decompose $k\{2_M 0_G, 2_M 2_G, 2_M 4_G\}$ into irreducibles under the action of $\Xi_{[2]}$. The action of 2_G on this vector space diagonalises; that is, $k\{2_M 0_G, 2_M 2_G, 2_M 4_G\} = \mathscr{M}_0 \oplus \mathscr{M}_1 \oplus \mathscr{M}_2$, where $\mathscr{M}_0 = \langle 2_M 0_G + 2_M 2_G + 2_M 4_G \rangle_k$, $\mathscr{M}_1 = \langle 2_M 0_G + \omega 2_M 2_G + \omega^2 2_M 4_G \rangle_k$, and $\mathscr{M}_2 = \langle 2_M 0_G + \omega^2 2_M 2_G + \omega 2_M 4_G \rangle_k$ (ω being a primitive third root of unity). The effect of 1_M on these eigenspaces is to swap $\mathscr{M}_1$ and $\mathscr{M}_2$. The decomposition into irreducibles gives $\mathscr{S}_1 = \mathscr{M}_0$ (trivial one-dimensional) and $\mathscr{S}_2 = \mathscr{M}_1 \oplus \mathscr{M}_2$.

Next we decompose $k\{2_M 1_G, 2_M 3_G, 2_M 5_G\}$ into irreducibles under the action of $\Xi_{[2]}$. The action of 2_G on this vector space diagonalises; that is, $k\{2_M 1_G, 2_M 3_G, 2_M 5_G\} = \mathscr{N}_0 \oplus \mathscr{N}_1 \oplus \mathscr{N}_2$, where $\mathscr{N}_0 = \langle 2_M 1_G + 2_M 3_G + 2_M 5_G \rangle_k$, $\mathscr{N}_1 = \langle 2_M 1_G + \omega 2_M 3_G + \omega^2 2_M 5_G \rangle_k$, and $\mathscr{N}_2 = \langle 2_M 1_G + \omega^2 2_M 3_G + \omega 2_M 5_G \rangle_k$. The effect of 1_M on these eigenspaces is to swap $\mathscr{N}_1$ and $\mathscr{N}_2$. The decomposition into irreducibles gives $\mathscr{S}_3 = \mathscr{N}_0$ (trivial one-dimensional) and $\mathscr{S}_4 = \mathscr{N}_1 \oplus \mathscr{N}_2$.

In fact the two two-dimensional representations $\mathscr{S}_2$ and $\mathscr{S}_4$ are isomorphic, using the maps $2_M 0_G + \omega 2_M 2_G + \omega^2 2_M 4_G \mapsto 2_M 1_G + \omega 2_M 3_G + \omega^2 2_M 5_G$ and $2_M 0_G + \omega^2 2_M 2_G + \omega 2_M 4_G \mapsto \omega^2(2_M 1_G + \omega^2 2_M 3_G + \omega 2_M 5_G)$.

Orbit 3. $t = 3_M$ gives the conjugacy class consisting of elements of the form (any 2-cycle in $1, 2, 3$). We choose $[3_M]$ as the base point for this orbit. Then $\Xi_{[3]} = C_2 \times S_3$, where C_2 is the group of permutations of $\{1, 2, 3\}$ consisting of the identity and $(1, 2)$. In terms of the factorisation, $\Xi_{[3]}$ consists of all elements of the form ut for $u \in \{0_G, 3_G\}$ and $t \in M$. The equivalence class is $[3_M] = \{3_M v | 3_M \triangleright v = v\}$. Since 3_M acts on G by the permutation $(1_G, 5_G)(2_G, 4_G)$, we find that $[3_M] = \{3_M 0_G, 3_M 3_G\}$. The $\Xi_{[3]}$ action on $[3_M]$ is given by $ut \,\tilde{\triangleright}\, sv = su(t \triangleright v) = suv$.

Now decompose $\mathscr{S} = k\{3_M 0_G, 3_M 3_G\}$ into irreducibles under the action of $\Xi_{[3]}$. The action of 3_G on this vector space diagonalises; that is, $\mathscr{S} = \mathscr{M}_1 \oplus \mathscr{M}_2$, where $\mathscr{M}_1 = \langle 3_M 0_G + 3_M 3_G \rangle_k$ and $\mathscr{M}_1 = \langle 3_M 0_G - 3_M 3_G \rangle_k$. The decomposition into irreducibles gives $\mathscr{S}_1 = \mathscr{M}_1$ and $\mathscr{S}_2 = \mathscr{M}_2$, which are not equivalent.

This completes our classification of the bicovariant quantum tangent spaces on bicrossproducts. It remains to dualise the results to obtain the corresponding first order differential calculi, as outlined at the start of the section. Explicitly, the dual of the inclusion i_L: $L \to \ker \epsilon_H$ is a surjection i_L^*: $(\ker \epsilon_H)^* \to V$, where $V = L^*$. On the other hand, the inclusion j: $\ker \epsilon_H \to H$ dualises to a map j^*: $A \to (\ker \epsilon_H)^*$ where $A = H^*$. Since $\ker j^* = (\text{image } j)^\perp = (\ker \epsilon_H)^\perp = k1_A$, the restriction $j^*|_{\ker \epsilon_A}$: $\ker \epsilon_A \to (\ker \epsilon_H)^*$ is an isomorphism. Putting these together, we get a quotient map π_V: $\ker \epsilon_A \to V$. Recall also that we can describe a representation L of $D(H)$ as a left H- and right H^*-module, with actions obeying the compatibility condition

$$h \triangleright (x \triangleleft a) = \sum \big((h_{(1)} \triangleleft a_{(1)}) \triangleright x \big) \triangleleft \big(h_{(2)} \triangleright a_{(2)} \big)$$

for all $x \in L$, which can be further computed in terms of the mutual coadjoint actions. (We freely identify a left $H^{*\,\mathrm{op}}$-module as a right H^*-module). For a given $a \in H^*$, the action $\triangleleft a$: $L \to L$ dualises to $(\triangleleft a)^*$: $V \to V$ where $V = L^*$. Similarly for the operators $h \triangleright$. We obtain in this way a left action of $A = H^*$ and a right action of H on V, by $a \triangleright v = (\triangleleft a)^*(v)$ and $v \triangleleft h = (h \triangleright)^*(v)$, which make V into a $D(A)$ representation. This is a general observation about the dualisation of quantum double modules. Combined with the projection π_V and d in (10), we obtain the corresponding first order differential calculus $\Omega^1 = V \otimes A$.

To fully specify the resulting exterior differential d it is equivalent and more convenient to give its evaluations $\partial_v = (\langle v, \ \rangle \otimes \mathrm{id}) \circ d$ against all $v \in L$. These operators ∂_v: $A \to A$ are the *braided vector fields* associated to elements $v \in L$. The term is justified because they obey a braided version of the Leibniz rule [7]

$$\partial_x(ab) = (\partial_x a)b + \sum \Psi(a \otimes x)^{-(2)}\, \partial_{\Psi(a\otimes x)^{-(1)}} b, \qquad \forall a, b \in A,\ x \in L, \tag{14}$$

where Ψ: $L \otimes A \to A \otimes L$ is the braiding induced by an action of the quantum double, with inverse denoted explicitly by $\sum \Psi(a \otimes x)^{-(1)} \otimes \Psi(a$

$\otimes x)^{-(2)}$. Here

$$\Psi(x \otimes a) = \sum a_{(2)} \otimes Sa_{(1)} \triangleright x, \qquad \Psi^{-1}(a \otimes b) = \sum a_{(1)} \triangleright x \otimes a_{(2)}. \tag{15}$$

Since we obtain L from the map $\Pi\colon \mathscr{M} \to L$, we specify equivalently the operators

$$\partial_m = (\langle m, \ \rangle \otimes \mathrm{id})(\Pi^* \otimes \mathrm{id}) \circ d, \qquad \forall m \in \mathscr{M}. \tag{16}$$

PROPOSITION 5.15. *Let $\mathscr{O} = \mathscr{O}_{[s]}$ as in Example 5.4. Then*

$$\mathscr{S}_{[s]} = \mathrm{span}\{\phi_{sv} : s \triangleright v = v\}, \qquad \mathscr{M}_{\mathscr{O}} = \mathrm{span}\{ut \triangleright \phi_{sv} : s \triangleright v = v,\ ut \in X\}.$$

For $\|\phi_{sv}\| = s$ (i.e., $s \triangleright v = v$) and $\|ut \triangleright \phi_{sv}\| = uts(ut)^{-1} = \bar{s}\bar{v}$,

$$\partial_{ut \triangleright \overline{\phi}_{sv}}(\delta_r \otimes w) = \delta_{r \triangleright w, \bar{v}u(t \triangleright v)} \delta_{\bar{s}r} \otimes w - \delta_{e, \bar{v}u(t \triangleright v)} \delta_r \otimes w,$$

$$\Psi\Big(\big(ut \triangleright \overline{\phi}_{sv}\big) \otimes (\delta_c \otimes z)\Big) = \Big(\delta_{\bar{s}^{-1}(c \triangleleft z) \triangleleft z^{-1}} \otimes z\Big) \otimes (c \triangleright z) ut \triangleright \overline{\phi}_{sv}.$$

Proof. First we calculate

$$\begin{aligned} ut \triangleright \phi_{sv} &= \phi_{utsv(t \triangleleft v)^{-1}} \\ &= \phi_{utst^{-1}(t \triangleright v)} = \phi_{u(ts \triangleright v)(tst^{-1} \triangleleft (t \triangleright v))} = ts^{-1}t^{-1} \triangleleft u^{-1} \otimes \delta_{u(ts \triangleright v)}, \end{aligned}$$

and hence

$$\begin{aligned} &(\langle ut \triangleleft \phi_{sv}, \ \rangle \otimes \mathrm{id})\Delta(\delta_r \otimes w) \\ &\qquad = \sum_{ab = r} \delta_{ts^{-1}t^{-1} \triangleleft u^{-1}, a} \delta_{u(ts \triangleright v), a^{-1}r \triangleright w} \delta_{a^{-1}r} \otimes w \\ &\qquad = \delta_{(ts^{-1}t^{-1} \triangleright u^{-1})^{-1}(t \triangleright v), r \triangleright w} \delta_{(ts^{-1}t^{-1} \triangleleft u^{-1})^{-1}r} \otimes w, \\ &(\langle ut \triangleright \phi_{sv}, \ \rangle \otimes \mathrm{id})(1 \otimes \delta_r \otimes w) \\ &\qquad = \delta_{(ts^{-1}t^{-1} \triangleright u^{-1}), (t \triangleright v)} \delta_r \otimes w. \end{aligned}$$

If we look at $\|ut \triangleright \phi_{sv}\| = uts(ut)^{-1} = \bar{s}\bar{v}$, it is apparent that $\bar{s} = (ts^{-1}t^{-1} \triangleleft u^{-1})^{-1}$ and $\bar{v} = (ts^{-1}t^{-1} \triangleright u^{-1})^{-1}u^{-1}$. This then gives $\partial_{ut \triangleright \phi_{sv}}$ as stated. The computation of the braiding from (15) is similar and left to the reader. One may use the formula for the Hopf algebra structure in [2]. ∎

EXAMPLE 5.16. *We provide the ∂ operators in a few cases from Example* 5.14.

We take orbit 0 in Example 5.14 and write

$$f_r = 0_M \otimes \delta_{0_G} + \omega^r 0_M \otimes \delta_{1_G} + \omega^{2r} 0_M \otimes \delta_{2_G} + \omega^{3r} 0_M \otimes \delta_{3_G} + \omega^{4r} 0_M \otimes \delta_{4_G} + \omega^{5r} 0_M \otimes \delta_{5_G},$$

where ω is a primitive sixth root of unity. A short calculation gives

$$\partial_{f_r}(\delta_t \otimes v) = \sum_{ab=t} \langle \delta_a \otimes b \triangleright v, f_r \rangle \delta_b \otimes v - \sum_a \langle \delta_a \otimes 0_G, f_r \rangle \delta_t \otimes v = (\omega^{r(t \triangleright v)} - 1) \delta_t \otimes v.$$

We use the allowed spaces for $\mathscr{M}$ for this orbit, which are $\langle f_3 \rangle_k$, $\langle f_1, f_5 \rangle_k$, and $\langle f_2, f_4 \rangle_k$.

Next we take a case from orbit 2, the irreducible representation given by $\mathscr{S}_2 = \langle g_1, g_2 \rangle_k$,

$$g_r = \phi_{2_M 0_G} + \omega^r \phi_{2_M 2_G} + \omega^{2r} \phi_{2_M 4_G} = 4_M \otimes \delta_{0_G} + \omega^r 4_M \otimes \delta_{2_G} + \omega^{2r} 4_M \otimes \delta_{4_G},$$

and ω is a primitive third root of unity. In this case the orbit consists of more than one point; in fact, $\mathscr{O} = \{[2_M], [4_M]\}$. We choose $x_{[4_M]} \in X$ so that $x_{[4_M]} \tilde{\triangleright} [2_M] = [4_M]$; for example, $x_{[4_M]} = 1_G$. Now we can add $1_G \tilde{\triangleright} g_1$ and $1_G \tilde{\triangleright} g_2$ to g_1 and g_2 to get a basis of the four-dimensional representation specified by $\mathscr{O}$ and $\mathscr{M}_0$, that is, $\mathscr{M} = \langle g_1, g_2, 1_G \tilde{\triangleright} g_1, 1_G \tilde{\triangleright} g_2 \rangle_k$, where

$$1_G \tilde{\triangleright} g_r = \phi_{1_G 2_M} + \omega^r \phi_{3_G 2_M} + \omega^{2r} \phi_{5_G 2_M} = 2_M \otimes \delta_{1_G} + \omega^r 2_M \otimes \delta_{3_G} + \omega^{2r} 2_M \otimes \delta_{5_G}.$$

Then we may calculate the evaluations

$$\partial_{g_r}(\delta_t \otimes v) = \left(\delta_{(t-4) \triangleright v, 0} + \omega^r \delta_{(t-4) \triangleright v, 2} + \omega^{2r} \delta_{(t-4) \triangleright v, 4}\right) \delta_{t-4} \otimes v - \delta_t \otimes v,$$

$$\partial_{1_G \tilde{\triangleright} g_r}(\delta_t \otimes v) = \left(\delta_{(t-2) \triangleright v, 1} + \omega^r \delta_{(t-2) \triangleright v, 3} + \omega^{2r} \delta_{(t-2) \triangleright v, 5}\right) \delta_{t-2} \otimes v.$$

A further general feature of the quantum tangent space is [6, 14] a "quantum Lie bracket" $L \otimes L \to L$, defined by $[x, y] = \mathrm{Ad}_x(y) = \sum x_{(1)} y S x_{(2)}$. This obeys a braided Jacobi identity

$$[x, [y, z]] = [[x, y], z] + [\ , [\ , z]] \circ \Psi(x \otimes y), \qquad \forall x, y, z \in L \tag{17}$$

for $\Psi\colon L \otimes L \to L \otimes L$ induced by the action of the quantum double. This is given by

$$\Psi(x \otimes y) = \mathrm{Ad}_{x_{(1)}}(y) \otimes x_{(2)} - [x, y] \otimes 1. \tag{18}$$

The quantum Lie bracket here extends in the quasitriangular case to a more complete theory of ‘‘braided Lie algebras’’ when viewed as a braided Lie bracket [15].

PROPOSITION 5.17. *The quantum Lie bracket and its braiding in the case* $\mathcal{O} = \mathcal{O}_{[s]}$ *are given by the formulae*

$$\left[ut \triangleright \overline{\phi}_{sv}, u't' \triangleright \overline{\phi}_{sv'}\right] = \delta_{w, u(t \triangleright v)}\left(\left(ts^{-1}t^{-1} \triangleleft (t \triangleright v)\right)u't'\right) \triangleright \overline{\phi}_{sv'}$$
$$- \delta_{u(t \triangleright v), e} u't' \triangleright \overline{\phi}_{sv'},$$
$$\Psi\left(ut \triangleright \overline{\phi}_{sv} \otimes u't' \triangleright \overline{\phi}_{sv'}\right) = \left((ts^{-1}t^{-1} \triangleleft u^{-1}w)u't'\right) \triangleright \overline{\phi}_{sv'} \otimes w^{-1}ut \triangleright \overline{\phi}_{sv},$$

where $w = (t's^{-1}t'^{-1} \triangleright u'^{-1})^{-1}u'^{-1}$.

Proof. From the Hopf algebra structure of the bicrossproduct one has easily

$$\begin{aligned}
\mathrm{Ad}_{\hat{s} \otimes \delta_{\hat{u}}}\left(\hat{t} \otimes \delta_{\hat{v}}\right) &= x_{(1)} y S(x_{(2)}) \\
&= \sum_{\hat{w}\hat{z} = \hat{u}} (\hat{s} \otimes \delta_{\hat{w}})\left(\hat{t} \otimes \delta_{\hat{v}}\right)\left((\hat{s} \triangleleft \hat{u})^{-1} \otimes \delta_{((\hat{s} \triangleleft \hat{w}) \triangleright \hat{z})^{-1}}\right) \\
&= \sum_{\hat{w}\hat{z} = \hat{u}} \delta_{\hat{w}, \hat{t} \triangleright \hat{v}}\left(\widehat{st} \otimes \delta_{\hat{v}}\right)\left((\hat{s} \triangleleft \hat{u})^{-1} \otimes \delta_{((\hat{s} \triangleleft \hat{w}) \triangleright \hat{z})^{-1}}\right) \\
&= \sum_{\hat{w}\hat{z} = \hat{u}} \delta_{\hat{w}, \hat{t} \triangleright \hat{v}} \delta_{\hat{v}, \hat{z}^{-1}} \widehat{st}(\hat{s} \triangleleft \hat{u})^{-1} \otimes \delta_{((\hat{s} \triangleleft \hat{w}) \triangleright \hat{z})^{-1}} \\
&= \delta_{(\hat{t} \triangleright \hat{v})\hat{v}^{-1}, \hat{u}} \widehat{st}(\hat{s} \triangleleft \hat{u})^{-1} \otimes \delta_{(\hat{s} \triangleright \hat{u})^{-1}(\widehat{st} \triangleright \hat{v})}.
\end{aligned}$$

Let $\tilde{h} = ts^{-1}t^{-1} \triangleleft u^{-1} \otimes \delta_{u(ts \triangleright v)}$ and $\tilde{m} = t's^{-1}t'^{-1} \triangleleft u'^{-1} \otimes \delta_{u'(t's \triangleright v')}$. Then set $h = ut \triangleright \overline{\phi}_{sv} = \tilde{h} - \epsilon(\tilde{h})1$ and $m = u't' \triangleright \overline{\phi}_{sv'} = \tilde{m} - \epsilon(\tilde{m})1$, so that $[h, m] = \tilde{h}_{(1)}\tilde{m}S\tilde{h}_{(2)} - \epsilon(\tilde{h})\tilde{m}$. To calculate the first term we set $\hat{s} = ts^{-1}t^{-1} \triangleleft u^{-1}$, $\hat{u} = u(ts \triangleright v)$, $\hat{t} = t's^{-1}t'^{-1} \triangleleft u'^{-1}$, and $\hat{v} = u'(t's \triangleright v')$ in the above expression for Ad. We would like to give its output in the same form as the input, namely as $\delta_{(\hat{t} \triangleright \hat{v})\hat{v}^{-1}, \hat{u}} u''t'' \triangleright \phi_{sv'}$ for some u'' and t'' which we have to find. Then

$$u''t'' \triangleright \phi_{sv'} = t''s^{-1}t''^{-1} \triangleleft u''^{-1} \otimes \delta_{u''(t'' \triangleright v')} = \widehat{st}(\hat{s} \triangleleft \hat{u})^{-1} \otimes \delta_{(\hat{s} \triangleright \hat{u})^{-1}(\widehat{st} \triangleright \hat{v})},$$

under the condition that $(\hat{t} \triangleright \hat{v})\hat{v}^{-1} = \hat{u}$. From this we calculate

$$
\begin{aligned}
t''s^{-1}t''^{-1} \triangleleft (t'' \triangleright v') &= \hat{s}\hat{t}(\hat{s} \triangleleft \hat{u})^{-1} \triangleleft (\hat{s} \triangleright \hat{u})^{-1}(\hat{s}\hat{t} \triangleright \hat{v}) \\
(t''s^{-1} \triangleleft v')(t'' \triangleleft v')^{-1} &= (\hat{s}\hat{t} \triangleleft \hat{u}^{-1})\hat{s}^{-1} \triangleleft (\hat{s}\hat{t} \triangleright \hat{v}) \\
(t'' \triangleleft v')(s^{-1} \triangleleft v')(t'' \triangleleft v')^{-1} &= \big(\hat{s}\hat{t} \triangleleft \hat{u}^{-1}(\hat{t} \triangleright \hat{v})\big)\big(\hat{s} \triangleleft (\hat{t} \triangleright \hat{v})\big)^{-1} \\
&= (\hat{s}\hat{t} \triangleleft \hat{v})\big(\hat{s} \triangleleft (\hat{t} \triangleright \hat{v})\big)^{-1} \\
&= \big(\hat{s} \triangleleft (\hat{t} \triangleright \hat{v})\big)(\hat{t} \triangleleft \hat{v})\big(\hat{s} \triangleleft (\hat{t} \triangleright \hat{v})\big)^{-1} \\
&= \big(\hat{s} \triangleleft (\hat{t} \triangleright \hat{v})\big)\big(t's^{-1}t'^{-1} \triangleleft (t' \triangleright v')\big) \\
&\quad \times \big(\hat{s} \triangleleft (\hat{t} \triangleright \hat{v})\big)^{-1} \\
&= \big(\hat{s} \triangleleft (\hat{t} \triangleright \hat{v})\big)(t' \triangleleft v')(s^{-1} \triangleleft v') \\
&\quad \times (t' \triangleleft v')^{-1}\big(\hat{s} \triangleleft (\hat{t} \triangleright \hat{v})\big)^{-1}.
\end{aligned}
$$

From this we can identify a possible value of t'' from $t'' \triangleleft v' = (\hat{s} \triangleleft (\hat{t} \triangleright \hat{v}))(t' \triangleleft v')$. However, instead of finding t'' on its own we combine it with the equation $u''(t'' \triangleright v') = (\hat{s} \triangleright \hat{u})^{-1}(\hat{s}\hat{t} \triangleright \hat{v})$ to write

$$
\begin{aligned}
u''(t'' \triangleright v')(t'' \triangleleft v') = u''t''v' &= (\hat{s} \triangleright \hat{u})^{-1}(\hat{s}\hat{t} \triangleright \hat{v})\big(\hat{s} \triangleleft (\hat{t} \triangleright \hat{v})\big)(t' \triangleleft v') \\
&= (\hat{s} \triangleright \hat{u})^{-1}\hat{s}(\hat{t} \triangleright \hat{v})(t' \triangleleft v') \\
u''t'' &= (\hat{s} \triangleleft \hat{u})\hat{u}^{-1}(\hat{t} \triangleright \hat{v})(t' \triangleleft v')v'^{-1},
\end{aligned}
$$

and on this we use the condition $(\hat{t} \triangleright \hat{v})\hat{v}^{-1} = \hat{u}$ to write

$$
\begin{aligned}
u''t'' &= (\hat{s} \triangleleft \hat{u})\hat{v}(t' \triangleleft v')v'^{-1} = (\hat{s} \triangleleft \hat{u})u'(t' \triangleright v')(t' \triangleleft v')v'^{-1} \\
&= (\hat{s} \triangleleft \hat{u})u't' = \big(ts^{-1}t^{-1} \triangleleft (t \triangleright v)\big)u't'.
\end{aligned}
$$

Now we just add $\epsilon(\tilde{h}) = \delta_{u(t \triangleright v), e}$ and the formulae $\hat{t} \triangleright \hat{v} = (t's^{-1}t'^{-1} \triangleright u'^{-1})^{-1}(t' \triangleright v')$ and $\hat{u}\hat{v} = u(t \triangleright v)u'(t' \triangleright v')$ in the condition $(\hat{t} \triangleright \hat{v})\hat{v}^{-1} = \hat{u}$ to get the result given for the quantum Lie bracket. The computation for the braiding follows similar lines and we omit it. ∎

Finally, once the quantum tangent space is known, one may recover the differential forms as $\Omega^1 = V \otimes A$ where $V = \ker \epsilon / Q$. Here the ideal Q is recovered as

$$
Q = \operatorname{span}\{q \in \ker \epsilon \mid \epsilon\, \partial_x q = 0, \forall x \in L\} \tag{19}
$$

since $\epsilon\,\partial_x a = \langle x, \pi_V(a - \epsilon(a))\rangle$ from the definition of ∂. Recall that in the construction of the tangent spaces we used an irreducible representation $\mathscr{S}_i \subset \mathscr{S}$. We write an element $\sigma \in \mathscr{S}_i$ as

$$\sigma = \sum_{v\,:\,s \triangleright v = v} \sigma_v \phi_{sv},$$

and the entire irreducible representation is spanned by translations of (i.e., $ut \triangleright$ applied to) elements of this form. For convenience, we assume that $\sigma_v = 0$ if $s \triangleright v \neq v$. We use these coefficients σ_v to describe Q.

PROPOSITION 5.18. *For $\mathscr{O} = \mathscr{O}_{[s]}$ as above, the ideal Q consists of all elements of the form*

$$q = \sum_{r,w} \mu_{rw}\,\delta_r \otimes w,$$

where the coefficients μ_{rw} obey the equations

$$\sum_{v\,:\,s \triangleright v = v} \sigma_v\,\mu_{tst^{-1} \triangleleft u^{-1},\, u(t \triangleright v)} = 0$$

for all u, t and all $\sigma \in \mathscr{S}_i$.

Proof. We take a homogeneous element of the tangent space

$$x = \sum_{v\,:\,s \triangleright v = v} \sigma_v ut \triangleright \overline{\phi}_{sv},$$

where ut is fixed, and we take the derivative in this direction of an element of the ideal $q \in Q$ given by

$$q = \sum_{r,w} \mu_{rw}\,\delta_r \otimes w$$

and then apply the counit to obtain

$$\epsilon\,\partial_x q = \epsilon\Bigg(\sum_{r,w,v\,:\,s \triangleright v = v} \sigma_v\,\mu_{rw}\Big(\delta_{r \triangleright w,\, \bar{v}u(t \triangleright v)}\delta_{\bar{s}r} \otimes w - \delta_{e,\, \bar{v}u(t \triangleright v)}\delta_r \otimes w\Big)\Bigg) = 0.$$

From the form of ϵ on the bicrossproduct, this is

$$\sum_{r,w,v\,:\,s \triangleright v = v} \sigma_v\,\mu_{rw}\Big(\delta_{r \triangleright w,\, \bar{v}u(t \triangleright v)}\delta_{\bar{s}r,\, e} - \delta_{e,\, \bar{v}u(t \triangleright v)}\delta_{r,\, e}\Big) = 0.$$

The first product of δ-functions in this expression can be combined in the form

$$\delta_{r^{-1}(r \triangleright w),\, \bar{s}\bar{v}u(t \triangleright v)} = \delta_{r^{-1}(r \triangleright w),\, utst^{-1}(t \triangleright v)} = \delta_{w(r \triangleleft w)^{-1},\, u(t \triangleright v)(tst^{-1} \triangleleft (t \triangleright v))}.$$

From this we can read that $w = u(t \triangleright v)$ and $r = \bar{s}^{-1} = tst^{-1} \triangleleft u^{-1}$. The second product of δ-functions in the expression is rather easier to deal with

$$\sum_{r,w,v:s\triangleright v=v} \sigma_v \mu_{rw} \delta_{e,\bar{v}u(t\triangleright v)} \delta_{r,e} = \Bigg(\sum_{v:s\triangleright v=v} \sigma_v \delta_{e,\bar{v}u(t\triangleright v)} \Bigg) \Bigg(\sum_w \mu_{ew} \Bigg) = 0,$$

as $\epsilon(q) = 0$ by the definition of Q. We are left with the following equation on the coefficients

$$\sum_{v:s\triangleright v=v} \sigma_v \mu_{tst^{-1}\triangleleft u^{-1}, u(t\triangleright v)} = 0,$$

which must be satisfied by a sufficient number of x to span the tangent space. ∎

Moreover, the left and right A-module structures on Ω^1 may be recovered from the directional derivatives as

$$(da \cdot b)(x) = (\partial_x a) b, \qquad (b \cdot da)(x) = b_{(2)}\big(\partial_{b_{(1)}\triangleright x} a\big), \qquad x \in L.$$

PROPOSITION 5.19. *For $\mathcal{O} = \mathcal{O}_{[s]}$ as above, let $a = \delta_r \otimes w$, $b = \delta_c \otimes z$, and $x = ut \triangleright \overline{\phi}_{sv}$. Then*

$$\begin{aligned}
&(da \cdot b)(x) \\
&\quad = \delta_{r\triangleright w,\bar{v}u(t\triangleright v)} \delta_{\bar{s}r\triangleleft w,c} \delta_{\bar{s}r} \otimes wz - \delta_{e,\bar{v}u(t\triangleright v)} \delta_{r\triangleleft w,c} \delta_r \otimes wz, \\
&(b \cdot da)(x) \\
&\quad = \delta_{(c\triangleright z)(r\triangleright w),\bar{v}u(t\triangleright v)} \delta_{c\triangleleft z,r} \delta_{\bar{s}c} \otimes zw - \delta_{c\triangleright z,\bar{v}u(t\triangleright v)} \delta_{\bar{s}c\triangleleft z,r} \delta_{\bar{s}c} \otimes zw,
\end{aligned}$$

where s, v, $\bar{s}$, and $\bar{v}$ are as in Proposition 5.15.

Proof. The right action of b is given directly by the multiplication in A and ∂ in Proposition 5.19. For the left action, we write $\Delta b = \sum_{gf=c} \delta_g \otimes f \triangleright z \otimes \delta_f \otimes z$, say, and from [2, (4.2)],

$$\begin{aligned}
\big(\delta_g \otimes f \triangleright z\big) \triangleright (ut \triangleright \phi_{sv}) &= \big(\bar{s}^{-1} \otimes \delta_{u(ts\triangleright v)}\big) \triangleleft \big(\delta_g \otimes f \triangleright z\big) \\
&= \delta_{\bar{s}^{-1},g} (f \triangleright z)^{-1} ut \triangleright \phi_{sv}.
\end{aligned}$$

From the δ-function here we can calculate $f = g^{-1}c = \bar{s}c$. Now we use the result of Proposition 5.19 again to calculate

$$\partial_{(f\triangleright z)^{-1}ut\triangleright\overline{\phi}_{sv}} a = \delta_{r\triangleright w,\hat{v}(f\triangleright z)^{-1}u(t\triangleright v)} \delta_{\hat{s}r} \otimes w - \delta_{e,\hat{v}(f\triangleright z)^{-1}u(t\triangleright v)} \delta_r \otimes w,$$

where $\hat{s}\hat{v} = (f \triangleright z)^{-1}\bar{s}\bar{v}(f \triangleright z)$. Now use the equations $\hat{s}\hat{v}(f \triangleright z)^{-1} = (f \triangleright z)^{-1}\bar{s}\bar{v}$ and $\bar{s}\bar{v}u = utst^{-1}$ to find $\hat{s}\hat{v}(f \triangleright z)^{-1}u = (f \triangleright z)^{-1}utst^{-1}$, or

$$\begin{aligned}\left(\hat{v}(f \triangleright z)^{-1}u\right)^{-1} &= ts^{-1}t^{-1} \triangleright u^{-1}(f \triangleright z)\\ &= \left(ts^{-1}t^{-1} \triangleright u^{-1}\right)(c \triangleright z) = u^{-1}\bar{v}^{-1}(c \triangleright z).\end{aligned}$$

Now we can rewrite

$$\begin{aligned}&\partial_{(f \triangleright z)^{-1}ut \triangleright \bar{\phi}_{sv}}a\\ &\quad= \delta_{(\hat{v}(f \triangleright z)^{-1}u)^{-1},(t \triangleright v)(r \triangleright w)^{-1}}\delta_{\hat{s}r} \otimes w - \delta_{(\hat{v}(f \triangleright z)^{-1}u)^{-1},(t \triangleright v)}\delta_r \otimes w\\ &\quad= \delta_{u^{-1}\bar{v}^{-1}(c \triangleright z),(t \triangleright v)(r \triangleright w)^{-1}}\delta_{\hat{s}r} \otimes w - \delta_{u^{-1}\bar{v}^{-1}(c \triangleright z),(t \triangleright v)}\delta_r \otimes w\end{aligned}$$

and also calculate the products

$$\begin{aligned}\left(\delta_f \otimes z\right)\left(\delta_{\hat{s}r} \otimes w\right) &= \delta_{f \triangleleft z,\hat{s}r}\delta_f \otimes zw,\\ \left(\delta_f \otimes z\right)\left(\delta_r \otimes w\right) &= \delta_{f \triangleleft z,r}\delta_f \otimes zw.\end{aligned}$$

Since $\hat{s}^{-1} = \bar{s}^{-1} \triangleleft (f \triangleright z)$, we can write $\delta_{f \triangleleft z,\hat{s}r} = \delta_{\hat{s}^{-1}(f \triangleleft z),r} = \delta_{c \triangleleft z,r}$, giving the result stated. ∎

EXAMPLE 5.20. *Explicit description of the* 1*-forms in the case of Example* 5.13.

In Example 5.13 we had $\mathscr{M} = \langle g_1, g_2, 1_G \tilde{\triangleright} g_1, 1_G \tilde{\triangleright} g_2\rangle_k$, with

$$g_n = \phi_{2_M 0_G} + \omega^n\phi_{2_M 2_G} + \omega^{2n}\phi_{2_M 4_G}.$$

First we shall find the ideal Q in this case. If we take a general element

$$\mu = \sum_{p,x}\mu_{px}\delta_p \otimes x$$

and calculate its directional derivatives, we find

$$\begin{aligned}\partial_{g_n}\mu = &\sum_p\left(\mu_{p-2,0} - \mu_{p,0}\right)\left(\delta_p \otimes 0\right)\\ &+ \sum_{p\text{ even}}\Big(\left(\omega^n\mu_{p-2,2} - \mu_{p,2}\right)\left(\delta_p \otimes 2\right)\\ &\qquad+\left(\omega^{2n}\mu_{p-2,4} - \mu_{p,4}\right)\left(\delta_p \otimes 4\right)\Big)\end{aligned}$$

$$+\sum_{p\text{ odd}}\Big(\big(\omega^{2n}\mu_{p-2,2}-\mu_{p,2}\big)\big(\delta_p\otimes 2\big)$$

$$+\big(\omega^{n}\mu_{p-2,4}-\mu_{p,4}\big)\big(\delta_p\otimes 4\big)\Big),$$

$$\partial_{1\triangleright g_n}\mu=\sum_{p\text{ even}}\big(\mu_{p+2,1}\delta_p\otimes 1+\omega^n\mu_{p+2,3}\delta_p\otimes 3+\omega^{2n}\mu_{p+2,5}\delta_p\otimes 5\big)$$

$$+\sum_{p\text{ odd}}\big(\mu_{p+2,5}\delta_p\otimes 5+\omega^n\mu_{p+2,3}\delta_p\otimes 3+\omega^{2n}\mu_{p+2,1}\delta_p\otimes 1\big).$$

If we now impose the conditions $\epsilon(\mu)=0$, $\epsilon(\partial_{g_n}\mu)=0$, and $\epsilon(\partial_{1\triangleright g_n}\mu)=0$ (for $n=1,2$), we get the following conditions on the coefficients, which define Q,

$$\sum_x \mu_{0x}=0$$

$$\mu_{40}+\omega^n\mu_{42}+\omega^{2n}\mu_{44}=\mu_{00}+\mu_{02}+\mu_{04},$$

$$\mu_{21}+\omega^n\mu_{23}+\omega^{2n}\mu_{25}=0.$$

Now we look for a left-module basis of the 1-forms. We define v_1 and v_2 by, for $n=1$ and 2,

$$v_n=\sum_{c=1}^{6}\sum_{z=2,4}p_{cz}(n)\big(\delta_c\otimes z\big)\cdot d\big(\delta_{c+2}\otimes(-z)\big),$$

where the coefficients $p_{cz}(n)$ are given by

$$p_{c2}(n)=\frac{1}{1-\omega^n},\qquad p_{c4}(n)=\frac{-\omega^n}{1-\omega^n}$$

if c is even and by

$$p_{c4}(n)=\frac{1}{1-\omega^n},\qquad p_{c2}(n)=\frac{-\omega^n}{1-\omega^n}$$

if c is odd. These have the property that v_1 and v_2 both give zero when evaluated in the directions $1_G\,\tilde{\triangleright}\,g_1$ and $1_G\,\tilde{\triangleright}\,g_2$, and also that $v_n(g_m)=\delta_{n,m}1$. We can also define v_3 and v_4 by the following formula, for $n=1$ and 2,

$$v_{n+2}=\sum_{c=1}^{6}\sum_{z=1,5}q_{cz}(n)\big(\delta_c\otimes z\big)\cdot d\big(\delta_{2-c}\otimes(-z)\big),$$

where the coefficients $q_{cz}(n)$ are given by

$$q_{c5}(n)=\frac{1}{\omega^n-\omega^{2n}},\qquad q_{c1}(n)=\frac{-\omega^n}{\omega^n-\omega^{2n}}$$

if c is even and by

$$q_{c1}(n) = \frac{1}{\omega^n - \omega^{2n}}, \qquad q_{c5}(n) = \frac{-\omega^n}{\omega^n - \omega^{2n}}$$

if c is odd. These have the property that v_3 and v_4 both give zero when evaluated in the directions g_1 and g_2, and also that $v_n(1_G \tilde{\triangleright} g_m) = \delta_{n-2,m}1$. In other words, the forms $\{v_1, v_2, v_3, v_4\}$ give the dual basis for the basis $\{g_1, g_2, 1_G \tilde{\triangleright} g_1, 1_G \tilde{\triangleright} g_2\}$ (in that order) of the tangent space. From this we can express any 1-form using the v_i as a left-module basis; that is, for α a 1-form

$$\alpha = v_1.\alpha(g_1) + v_2.\alpha(g_2) + v_3.\alpha\left(1_G \tilde{\triangleright} g_1\right) + v_4.\alpha\left(1_G \tilde{\triangleright} g_2\right).$$

Now we can examine the commutator $b \cdot da - da \cdot b$, where $a = \delta_r \otimes w$, $b = \delta_c \otimes z$, and write it in terms of the basis elements v_i. To do this, we give its evaluation against the basis of the quantum tangent space and use the formula above. Begin with the evaluations

$$\begin{aligned}
&(b \cdot da - da \cdot b)(\phi_{2v}) \\
&\quad = \left(\delta_{(c \triangleright z)(r \triangleright w), v}\delta_{c \triangleleft z, r} - \delta_{(c \triangleright z), v}\delta_{(c+2) \triangleleft z, r}\right)\delta_{c+2} \otimes (w + z) \\
&\qquad - \delta_{(r \triangleright w), v}\delta_{(r+2) \triangleleft w, c}\delta_{r+2} \otimes (w + z) \\
&\qquad + \delta_{0, v}\delta_{r \triangleleft w, c}\delta_r \otimes (w + z), \\
&(b \cdot da - da \cdot b)(1 \triangleright \phi_{2v}) \\
&\quad = \delta_{(c \triangleright z)(r \triangleright w), 1+v}\delta_{c \triangleleft z, r}\delta_{c+4} \otimes (w + z) \\
&\qquad - \delta_{(c \triangleright z), 1+v}\delta_{(c+4) \triangleleft z, r}\delta_{c+4} \otimes (w + z) \\
&\quad - \delta_{(r \triangleright w), 1+v}\delta_{(r+4) \triangleleft w, c}\delta_{r+4} \otimes (w + z).
\end{aligned}$$

Now sum these over v even with powers of ω to get

$$\begin{aligned}
&(b \cdot da - da \cdot b)(g_m) \\
&\quad = \left(\omega^{(((c \triangleright z)+(r \triangleright w))/2)m}\delta_{c \triangleleft z, r} - \omega^{((c \triangleright z)/2)m}\delta_{(c+2) \triangleleft z, r}\right)\delta_{c+2} \\
&\qquad \otimes (w + z) \\
&\qquad - \omega^{((r \triangleright w)/2)m}\delta_{(r+2) \triangleleft w, c}\delta_{r+2} \otimes (w + z) \\
&\qquad + \delta_{r \triangleleft w, c}\delta_r \otimes (w + z), \\
&(b \cdot da - da \cdot b)(1 \triangleright g_m) \\
&\quad = \omega^{(((c \triangleright z)+(r \triangleright w)-1)/2)m}\delta_{c \triangleleft z, r}\delta_{c+4} \otimes (w + z) \\
&\qquad - \omega^{(((c \triangleright z)-1)/2)m}\delta_{(c+4) \triangleleft z, r}\delta_{c+4} \otimes (w + z) \\
&\qquad - \omega^{(((r \triangleright w)-1)/2)m}\delta_{(r+4) \triangleleft w,c}\delta_{r+4} \otimes (w + z).
\end{aligned}$$

To interpret these formulae, remember that the right and left actions of C_6 on itself are given by even elements having the trivial action and odd elements acting by the additive inverse. To save writing many special cases, we have also used the convention that $\omega^{(k/2)m}$ is zero if k is odd and the usual value if k is even.

This rather complicated result can be broken up into special cases. The simplest case is to note that $b \cdot da - da \cdot b = 0$ if one of r, c is odd and the other is even. As an example of another case, let r, c, and z all be even and w odd. Then

$$(b \cdot da - da \cdot b)(g_m) = \left(\delta_{c+r,0} - \omega^{(z/2)m}\delta_{c+2,r}\right)\delta_r \otimes (w+z),$$

$$(b \cdot da - da \cdot b)(1 \triangleright g_m) = \omega^{((w-1)/2)m}\left(\omega^{(z/2)m}\delta_{c,r} - \delta_{c+r,2}\right)\delta_{r+4} \otimes (w+z),$$

and we can now use the v_i as a right basis to write

$$\begin{aligned} b \cdot da - da \cdot b = {} & \left(\delta_{c+r,0} - \omega^{z/2}\delta_{c+2,r}\right)v_1 \cdot \left(\delta_r \otimes (w+z)\right) \\ & + \left(\delta_{c+r,0} - \omega^{z}\delta_{c+2,r}\right)v_2 \cdot \left(\delta_r \otimes (w+z)\right) \\ & + \omega^{(w-1)/2}\left(\omega^{z/2}\delta_{c,r} - \delta_{c+r,2}\right)1 \triangleright v_1 \cdot \left(\delta_{r+4} \otimes (w+z)\right) \\ & + \omega^{w-1}\left(\omega^{z}\delta_{c,r} - \delta_{c+r,2}\right)1 \triangleright v_2 \cdot \left(\delta_{r+4} \otimes (w+z)\right). \end{aligned}$$

Finally, we can ask what the left-module structure is in terms of the right basis. If we start with $(\delta_k \otimes x)v_n$ for $n = 1$ or 2, then, applying our formula for v_n we get

$$(\delta_k \otimes x)v_n = \sum_{z=2,4} p_{(k \triangleleft x)z}(n)(\delta_k \otimes xz) \cdot d\left(\delta_{k \triangleleft x+2} \otimes (-z)\right),$$

and evaluating this against g_m,

$$\begin{aligned} (\delta_k \otimes x)v_n(g_m) = \sum_{z=2,4} p_{(k \triangleleft x)z}(n)\Big(& \omega^{(((k \triangleright xz)+(k \triangleright (-z)))/2)m}\delta_{k \triangleleft x, k \triangleleft x+2} \\ & - \omega^{((k \triangleright xz)/2)m}\delta_{(k+2) \triangleleft x, k \triangleleft x+2}\Big)\delta_{k+2} \otimes x. \end{aligned}$$

This vanishes if x is odd, and if x is even it gives

$$\begin{aligned} (\delta_k \otimes x)v_n(g_m) &= -\left(\sum_{z=2,4} p_{kz}(n)\,\omega^{((k \triangleright xz)/2)m}\right)\delta_{k+2} \otimes x \\ &= -\omega^{((k \triangleright x)/2)m}\left(\sum_{z=2,4} p_{kz}(n)\,\omega^{((k \triangleright z)/2)m}\right)\delta_{k+2} \otimes x. \end{aligned}$$

On evaluation against $1 \triangleright g_m$, we find

$$(\delta_k \otimes x) v_n (1 \triangleright g_m) = \sum_{z=2,4} p_{(k \triangleleft x)z}(n) \Big(\omega^{(((k \triangleright xz) + (k \triangleright (-z)) - 1)/2)m} \times \delta_{k \triangleleft x, k \triangleleft x + 2} - \omega^{(((k \triangleright xz) - 1)/2)m} \delta_{(k+4) \triangleleft x, k \triangleleft x + 2} \Big) \delta_{k+4} \otimes x.$$

This vanishes if x is even, and if x is odd it gives

$$\begin{aligned}(\delta_k \otimes x) v_n (1 \triangleright g_m) &= -\Big(\sum_{z=2,4} p_{(-k)z}(n)\, \omega^{(((k \triangleright xz) - 1)/2)m} \Big) \delta_{k+4} \otimes x \\ &= -\omega^{((k \triangleright x - 1)/2)m} \Big(\sum_{z=2,4} p_{(-k)z}(n)\, \omega^{((k \triangleright z)/2)m} \Big) \delta_{k+4} \otimes x.\end{aligned}$$

If we use the formulae for $p_{kz}(n)$ then we get, for $n = 1$ or 2,

$$\begin{aligned}(\delta_k \otimes x) v_n (g_m) &= \frac{(\omega^{n+m} - 1)\, \omega^m}{1 - \omega^n}\, \omega^{((k \triangleright x)/2)m} \delta_{k+2} \otimes x \\ &= \delta_{n,m}\, \omega^{((k \triangleright x)/2)m} \delta_{k+2} \otimes x, \\ (\delta_k \otimes x) v_n (1 \triangleright g_m) &= \frac{(\omega^{n+m} - 1)\, \omega^m}{1 - \omega^n}\, \omega^{((k \triangleright x - 1)/2)m} \delta_{k+4} \otimes x \\ &= \delta_{n,m}\, \omega^{((k \triangleright x - 1)/2)m} \delta_{k+4} \otimes x.\end{aligned}$$

We can do a similar calculation for v_3 and v_4 to get, for $n = 1$ or 2,

$$\begin{aligned}(\delta_k \otimes x) v_{n+2} (g_m) &= -\sum_{z=1,5} q_{(k \triangleleft x)z}(n)\, \omega^{((k \triangleright (x+z))/2)m} \delta_{k+2} \otimes x \\ &= -\omega^{((k \triangleright x + 1)/2)m} \Big(\sum_{z=1,5} q_{(-k)z}(n)\, \omega^{((k \triangleright z - 1)/2)m} \Big) \\ &\quad \times \delta_{k+2} \otimes x, \\ (\delta_k \otimes x) v_{n+2} (1 \triangleright g_m) &= -\sum_{z=1,5} q_{(k \triangleleft x)z}(n)\, \omega^{((k \triangleright (x+z) - 1)/2)m} \delta_{k+4} \otimes x \\ &= -\omega^{((k \triangleright x)/2)m} \Big(\sum_{z=1,5} q_{kz}(n)\, \omega^{((k \triangleright z - 1)/2)m} \Big) \\ &\quad \times \delta_{k+4} \otimes x.\end{aligned}$$

If we use the formulae for $q_{k_z}(n)$ then we get, for $n = 1$ or 2,

$$
\begin{aligned}
(\delta_k \otimes x) v_{n+2}(g_m) &= \frac{\omega^n - \omega^{2m}}{\omega^n - \omega^{2n}} \omega^{((k \triangleright x+1)/2)m} \delta_{k+2} \otimes x \\
&= \delta_{n,m} \omega^{((k \triangleright x+1)/2)m} \delta_{k+2} \otimes x, \\
(\delta_k \otimes x) v_{n+2}(1 \triangleright g_m) &= \frac{\omega^n - \omega^{2m}}{\omega^n - \omega^{2n}} \omega^{((k \triangleright x)/2)m} \delta_{k+4} \otimes x \\
&= \delta_{n,m} \omega^{((k \triangleright x)/2)m} \delta_{k+4} \otimes x.
\end{aligned}
$$

Now we can use these evaluations to write the result in terms of the right basis as

$$
\begin{aligned}
(\delta_k \otimes x) \cdot v_n &= \omega^{((k \triangleright x)/2)n} v_n \cdot (\delta_{k+2} \otimes x) \\
&\quad + \omega^{((k \triangleright x-1)/2)n} v_{n+2} \cdot (\delta_{k+4} \otimes x), \\
(\delta_k \otimes x) \cdot v_{n+2} &= \omega^{((k \triangleright x+1)/2)n} v_n \cdot (\delta_{k+2} \otimes x) \\
&\quad + \omega^{((k \triangleright x)/2)n} v_{n+2} \cdot (\delta_{k+4} \otimes x)
\end{aligned}
$$

for $n = 1, 2$. This is our final result for the commutation relations between functions and differentials in this example.

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