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Fluid Approximation of Point-Queue Model

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Abstract

Point-queue model is widely used in the dynamic user equilibrium (DUE) analysis in discrete-time or continuous-time form. In this paper, a continuous time point queue is proposed. In the former studies, the negativity of the queue length of the original point-queue model is shown and some improvement has been made. Based on the observation that the original point-queue model is actually a queuing model with a server and a buffer with infinite capacity, a fluid approximation (FA) model is proposed to interpret the original point-queue model. Three essential components are a flow balance function, an exit flow function and a time-dependent capacity utilization ratio function, which are all in continuous form. During the analysis, the theoretical proof and numerical study of the non-negativity of queue length are accomplished. With the first-order Taylor expansion, this paper applies the Gronwall's inequality to prove the non-negativity of queue-length. Through numerical testing different specific FA models in the solution scheme, we can show that the negativity of the queue length in the FA model is overcome and some differences between the FA and former studies are discussed. Based on the testing, the capability of our model in approximating the point-queue model is demonstrated.

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Keywords: Continuous point-queue model; Fluid Approximation; Gronwall's inequality; Non-negativity; Kendall form

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1. Introduction

During recent decades, dynamic traffic assignment (DTA) has been studied extensively, which can be used in the real-time traffic application, such as variable message signs (VMS), on-line traffic control and so on. Travel choice model and traffic flow component are the two essential parts of DTA and the later one is also called dynamic network loading (DNL) problem.

As a specific problem in DTA research, dynamic user equilibrium (DUE) assumes that some rational behavioral choices, such as to minimize travel time or the generalized costs. Methodologies for DTA/DUE can be generally categorized as simulation-based approach and the analytical approach. For a review, please refer to and Ran and Boyce (1996) and Peeta and Ziliaskopoulos (2001). Ben-Akiva et al. (2001) and Mahmassani (2001) fall into the former one. Among the analytical modeling methodologies used for DUE, including mathematical programming, optimal control and variational inequality (VI) (or equivalently Nonlinear Complementariy Problems (NCPs)) have been considered the most capable for modeling and computing DUE. (Akamatsu, 2001; Friesz et al., 1993; Peeta and Ziliaskopoulos, 2001; Ran and Boyce, 1996; Wie et al., 2002).

In DNL models, the whole link model and the point-queue models are the mostly used ones. The whole link model assumes that the link travel time is a function of link flow (the inflow and the outflow) as $\tau(t) = h(x(t), u(t), v(t))$, where $\tau(t)$ is the link travel time at time t, $x(t)$ is the number of vehicles on the link at time t, $u(t)$ and $v(t)$ are the inflow and outflow of the link at time t. A number of models are the special cases of this model, such as Ran et al., 1993; Ran and Boyce, 1994and 1996; Friesz et al., 1993; Carey and Subrahmanian, 2000; Carey and McCartney, 2002; Carey and Ge, 2007; Xu et al., 1999. Particularly, in Carey and Ge (2007), both the explicit and implicit discretization schemes of the continuous-time whole link model were studied and the explicit scheme may violate FIFO but the implicit scheme will always guarantee FIFO was concluded. Point queue model was first introduced in Vickrey (1969) and the assumption that traffic flow travels in the free flow speed for the entire length of a network link until it gets to the end of the link, where a queue may form is made in the point-queue mode. That is to say, the physical length of the queue is neglected. Therefore, the spillback can never be produced with the point-queue model, contrary to the spatial queue model (Nie et al., 2008). In this paper, the point-queue model is chosen motivated by Ban et al. (2012). Firstly, the discrete-time point-queue models have been studied extensively compared to the continuous-time ones. Secondly, the negativity of the queue length can be derived in the original point-queue model. Finally, the queue forming and discharging process can be captured in the point-queue model, which is more realistic than the whole link model.

A few efforts have been made to refine the negativity of the queue length in the original point-queue model, such as Nie and Zhang, 2005; Ban et al., 2012; Han et al., 2013a,b. Nie and Zhang (2005) have introduced a modification in discrete time to ensure the non-negativity of the queue length. However, the modification is quite *ad hoc*. Ban et al. (2012) have proposed a depth analysis of the original point-queue model and two modified models are presented. One is a Linear Complementarity System (LCS) and the other is an Ordinary Differential Equation (ODE) with a right-hand side that would be Lipschitz continuous if not for the possible discontinuity of the inflow rate function, which is called α -model. Ban et al. (2012) showed the former LCS formulation is connected with the model in Pang et al. (2011) and the model of Nie and Zhang (2005) and the latter one is an approximated ODE formulation of the former LCS formulation whose solutions are shown to converge to a solution of the LCS as the parameter in the ODE goes to infinity. In Han et al. (2013a,b), a key ingredient called a virtual spatial dimension $x \in [0, L]$, where L is the link length, was introduced and with this ingredient, the original point-queue model is described based on the conservation of mass.

Based on the observation that the original point-queue model is actually a queue model with a server and an infinite buffer at the downstream of the link, a fluid approximation (FA) model is proposed in this paper to interpret the Vickrey's model (1969). In the fields of electrical engineering and management science, a fluid approximation model has received increasing attention during the recent decades considering modeling time-dependent queuing systems, while in transportation, the FA is seldom used. In Green and Kolesar (1991), the time was divided into small time intervals and then stationary approximations were used to calculate performance measures based on the service rates and the time-dependent demand rates in each time interval. The remaining queue length from the last interval and the arrival rate for the current interval are considered in the analysis of demand rate. Their study also empirically examined the accuracy of this computationally efficient approximation in multi-server queuing systems with exponential service times and periodic Poisson arrival processes. Whitt (1991) further verified that model in Green and Kolesar (1991) is asymptotically correct as the service and arrival rates increase with fixed instantaneous traffic intensity. Wang et al. (1996) combined the point-wise stationary fluid flow approximation model with flow balance equations and obtained a system of nonlinear differential equations to numerically represent queue evolution in telecommunication networks. For other detailed theoretical and practical use of the FA model, please refer to Stolletz , 2008a,b; Agnew , 1976; Kramer and Lagenbach-Belz, 1976 and the reference therein.

In this paper, the main findings are as follows:

(1)With the observation that the Vickrey's point-queue is actually a queue model with a server at the downstream of the link and an infinite buffer, a fluid approximation (FA) model is used to interpret the original model.

(2)The arrival to the server is time-dependent with constant time delay, which is the free flow time of the link, while the service rate is constant whose value is the link capacity. With the different ensemble server utilization, the outflow is time-dependent, too.

(3)With the first-order Taylor expansion, the Gronwall's inequality is used to prove the non-negativity of the queue length in the FA model based on some mild assumpttions.

(4)With different arrival processes and service processes in the queue system, different FA models are obtained to approximate the queue length in the point-queue model. With the solution scheme presented in Section 2.5, we test different time step sizes and the queue length is non-negative.

The remainder of the paper is organized as follows. Section 2 reviews the original point-queue model and proposes the interpretation of original point queue model with fluid approximation (FA) model. And the theoretical proof and numerical solution scheme are also proposed in section 2. In section 3, we test different specific queuing forms to show the non-negativity of our model and in the last section, some conclusions are obtained.

2. Model Development

2.1. Original point queue model

The original point-queue model was proposed in Vickrey (1969) for a single destination on the link and later on some researchers put some efforts on it, such as Nie and Zhang (2005), Ban et al. (2012) and Han et al. (2013a,b). The model is defined for aggregated link variables by (i) introducing the free flow travel time t_0 and capacity $\bar{C}$ on a link, (ii) specifying the link flow by a point queue of vehicles considered as the queue length $q(t)$ at the exit node of the link, (iii) replacing the link flow dynamics by a queue dynamics, and (iv) expressing the link travel time $\tau(t)$ in terms of the average link queue flow relative to the (known) link capacity. We introduce this model as follows which is the same as the one in Nie and Zhang (2005) and Ban et al. (2012). Here $p(t)$ and $v(t)$ represent, respectively, the inflow rate at the start node of the link and the exit flow rate at the end node of the link at time instant t:

(a) the dynamics of the queue is: $\dot{q}(t) = \begin{cases} 0 & \text{if } t \in (0, t_0) \\ p(t-t_0) - v(t) & \text{if } t > t_0; \end{cases}$

(b) the initial queue is zero: $q(t_0) = 0$;

(c) for $t < t_0$, the exit flow rate is: $v(t) = 0$;

(d) for $t > t_0$, the exit flow rate is: $v(t) = \begin{cases} \min(\bar{C}, p(t-t_0)) & \text{if } q(t) = 0 \\ \bar{C} & \text{if } q(t) \neq 0; \end{cases}$

(e) for $t \in [0, T-t_0]$, the actual travel time is $\tau(t) = t_0 + (\bar{C})^{-1} q(t+t_0)$.

2.2. Fluid approximation model

Actually, the point-queue model can be treated as a queue model with a single server and infinite buffer as follows in Fig. 1. The arrival rate is non-stationary, or time-dependent with constant time delay, which is the free flow time and the service rate is constant, which is the capacity of the link. With different server utilization, the link outflow also changes to be time-dependent, which is the real situation. In this section, the notations are as prescribed above in order to reduce the abuse of the notations.

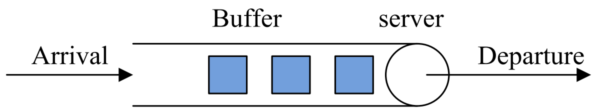


Fig. 1. Queue system of point-queue model

Let $\bar{C}$ denote the average queue service rate and $p(t)$ denote the ensemble average arrival rate at time t. We define $q(t)$ as the queue length at time t. Let $\dot{q}(t)=\frac{dq(t)}{dt}$ be the rate of change of the queue length respect to time. The fluid approximation model is comprised of three major components, namely, a flow balance function, an exit flow function and a time-dependent capacity utilization ratio function.

From the conservation principle, the rate of change of the average queue length is equal to the difference between the average arrival and departure rates. Let $f_{in}(t)$ and $f_{out}(t)$ denote the ensemble average flow in and flow out of the system at time t, respectively, which t_0 is the free flow travel time. The rate of change of the queue length can be related to the flow in and flow out by

$$\dot{q}(t)=-f_{out}(t)+f_{in}(t-t_0) \quad (1)$$

Definition (Utilization of server $\rho(t)$). During queuing description of point-queue model, the utilization of server is ratio of outflow and link capacity.

The flow out of the system $f_{out}(t)$ can be related to the ensemble average utilization of the server $\rho(t)$ by $f_{out}(t)=\bar{C}\rho(t)$. Especially, the physical length of the vehicle is neglected in the point-queue model, so the queue waiting space is infinite. That is to say, the buffer is infinite. Then the flow into the system is just the arrival rate, namely, $f_{in}(t-t_0)=p(t-t_0)$. With the introduction of exit flow function and utilization ratio function, the fluid flow model of Eq. (1) becomes

$$\dot{q}(t)=-\bar{C}\rho(t)+p(t) \quad (2)$$

The expression for $\rho(t)$ in Eq. (2) will depend on the specific queuing system under study. In general, it is difficult to determine an exact expression for $\rho(t)$, even for the simplest queues. Hence, an approximate method based on the FA method is adopted. The general idea is to determine the values for $\rho(t)$ at a particular instants of time by a point-wise mapping from the current value of $q(t)$ in to ρ using the steady state queuing relationships. The general form of fluid approximation is proposed in Wang et al. (1996) as follows.

$$\dot{q}(t)=-\bar{C}(G^{-1}(q(t)))+p(t) \quad (3)$$

And the general Kendall notations is the GI/G/1 Queue, which is as follows.

$$q\approx\rho+\frac{\rho^2\cdot(C_a^2+C_s^2)\cdot J(C_a^2,C_s^2,\rho)}{2(1-\rho)} \quad (4)$$

$$\text{where}: J(C_a^2, C_s^2, \rho) = \begin{cases} e^{-\frac{2(1-\rho)(1-C_a^2)^2}{3\rho(C_a^2+C_s^2)}} & C_a^2 \le 1 \\ e^{-\frac{(1-\rho)(C_a^2-1)}{C_a^2+C_s^2}} & C_a^2 > 1 \end{cases}$$

In Eq. (3), ρ is the server utilization, C_a^2 and C_s^2 represent the squared coefficients of variation of the inter-arrival and service process, respectively. In this paper, the arrival process is assumed to be exponentially distribution (M) and the departure process is assumed to be general distribution (G). Therefore, the Kendall form considered in this paper is as follows, which is M/G/1 form (Wang et al., 1996).

$$\dot{q} = -\bar{C}\left[\frac{q+1-\sqrt{q^2+2C_s^2 q+1}}{1-C_s^2}\right] + p(t-t_0) \tag{5}$$

With the introduction of M/G/1 form, we test the specific forms, which includes M/D/1, M/E_k/1 and M/M/1, where the arrival process is assumed to be exponentially distribution (M) and the departure process are assumed to be deterministic distribution (D), Erlang distribution (E_k) and exponentially distribution (M). The specific forms are shown in table 1 and the relationship between queue length and utilization ratio is in Fig. 2.

Table 1. Different FA models of M/G/1 type.

Queuing system	C_s^2	FA models
M/D/1	0	$\dot{q} = -\bar{C}\left[(q+1) - \sqrt{q^2+1}\right] + p(t-t_0)$
M/E_k/1	$1/k$	$\dot{q} = -\bar{C}\left[\frac{k(q+1)}{k-1} - \frac{\sqrt{k^2q^2+2kq+k^2}}{k-1}\right] + p(t-t_0)$
M/M/1	1	$\dot{q} = -\bar{C}\left[\frac{q}{q+1}\right] + p(t-t_0)$

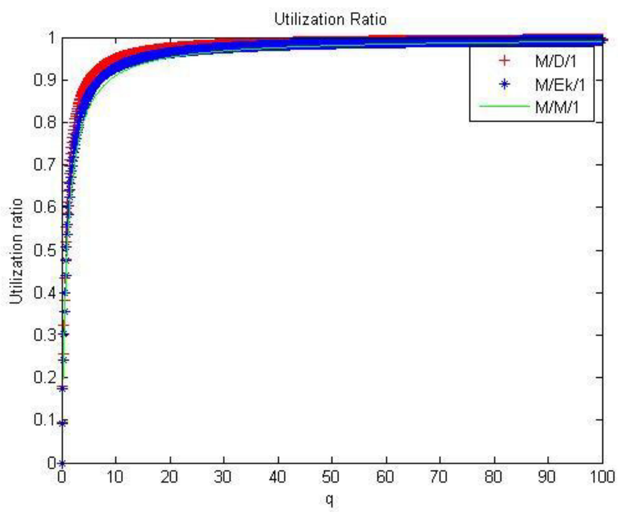


Fig. 2. Relationship between queue length and utilization ratio.

2.3. Interpretation of Vickrey Model in FA

With the introduction of the above FA model, the original Vickrey's point-queue model can be interpreted in the form of FA as follows:

(a) the dynamics of the queue is:

$$\dot{q}(t) = \begin{cases} 0 & \text{if } t \in (0, t_0) \\ -\bar{C}(G^{-1}(q(t))) + p(t - t_0) & \text{if } t > t_0 \end{cases}$$

(b) the initial queue is zero: $q(t_0) = 0$;

(c) for $t < t_0$, the exit flow rate is: $v(t) = 0$. Actually, in the FA model, it is $\rho(t) = 0$;

(d) for $t > t_0$, the exit flow rate is $v(t) = \bar{C}\rho(t)$;

(e) for $t \in [0, T - t_0]$, actual travel time is $\tau(t) = t_0 + (\bar{C}\rho(t + t_0))^{-1} q(t + t_0)$.

2.4. Theoretical Proof of Non-negativity of Queue Length

In the theoretical proof, we apply the Gronwall's inequality to show the non-negativity of queue length with the Eq. (5) for the M/G/1 type.

Lemma (Gronwall's inequality)

Let $g(t)$ be a non-negative function, such that, for $t \in [0, U]$, $g(t) \le k + K\int_0^t g(u)du$. For some constants *k* and *K*. Then, for $t \in [0, U]$, $g(t) \le ke^{Kt}$

For the proof, please see Bellman (1943).

Actually, the non-negative of function $g(t)$ can be relaxed during the proof of the Gronwall's inequality, which is the case in our proof.

Proposition 1. With the trivial observation that $p(t - t_0) \ge 0$ and assume the $q(t_0) = 0$, the following two statements can be hold.

(1) $q(t) \ge q(t')e^{-\bar{C}(t-t')}$ for all $t > t' > t_0$; moreover, if $q(t_0) > 0$ for some $t' \ge t_0$, then $q(t) > 0$ for all $t \ge t_0$.

(2) With the state-dependent exit-flow function, $v(t) \ge 0$.

Proof.

Assuming $q(t)$ is continuous derivative , through a first-order Taylor's series expansion of the square root of Eq. (5), we can get

$$\dot{q} \ge -\bar{C}q \tag{6}$$

especially with the specific forms in table 1. Apply the Gronwall's inequality to Eq. (6), the statement (1) can be easily obtained. Based on the deduction in Section 2.2, the non-negativity of exit-flow function is trivial based on the statement (1).

2.5. Numerical Solution Scheme of FA Model

The time interval $[t_0, T]$ is partitioned into $N_h + 1 \triangleq \frac{T - t_0}{h}$ (assumed to be an integer) time steps with a step size $h > 0$.

$$t_0 \triangleq t_{h,0} < t_{h,1} < \ldots < t_{h,N_h} < t_{h,N_{h+1}} \triangleq T$$

The arrival rate is assumed to be a constant over each time step $[t_{h,0}, t_{h,1}]$ (i.e., $\lambda(t) = \lambda(\frac{t_{h,1} - t_{h,0}}{2})$) for $t \in [t_{h,0}, t_{h,1}]$. Then the Eq. (4) can be numerically integrated for the value of the state variable at the end of the time interval, $q(t_{h,1})$. Note that in solving the fluid flow model over a small time interval, one may need to apply a numerical procedure to find $G^{-1}(q)$. The state variable value at the end of the time interval, $q(t_{h,1})$, then becomes the initial condition for the next time step $\left[t_{h,1}, t_{h,2}\right]$. We then adjust the arrival rate for the new time step. This procedure is repeated for each time interval in the time horizon. In the numerical integration to the differential Eqs used in this paper, the fourth order Runge-Kutta method is adopted.

The Runge-Kutta methods are an important family of iterative methods for the approximation of solutions of ODEs, that were developed around 1900 by the German mathematicians Runge and Kutta. Since then, many research have been done on the original Runge-Kutta method. See Huang et al. (2009), Jiang et al. (2011) and the reference therein.

The most popular formula of the Runge-Kutta methods is the fourth-order formula for the ordinary differential Eq. $y' = F(x, y)$ and we just simply cite here as follows:

$$y_{i+1} = y_i + \frac{1}{6}(K_0 + 2K_1 + 2K_2 + K_3)$$

where

$$K_0 = hF(x_i, y_i),$$
$$K_1 = hF(x_i + \frac{1}{2}h, y_i + \frac{1}{2}K_0),$$
$$K_2 = hF(x_i + \frac{1}{2}h, y_i + \frac{1}{2}K_1),$$
$$K_3 = hF(x_{i+1}, y_i + K_2),$$

h is the step size.

For the details, please refer to Greenspan (2006).

3. Numerical Tests and Discussion

3.1. Numerical test

In this paper, different FA models with Poisson arrival processes and different service processes are tested. Different inflow profiles and different step sizes are used in the tests to show the non-negativity of queue length of the FA of Vickrey's point-queue model. The same results can be obtained with different time intervals and the time interval in our paper is set to be 30s.

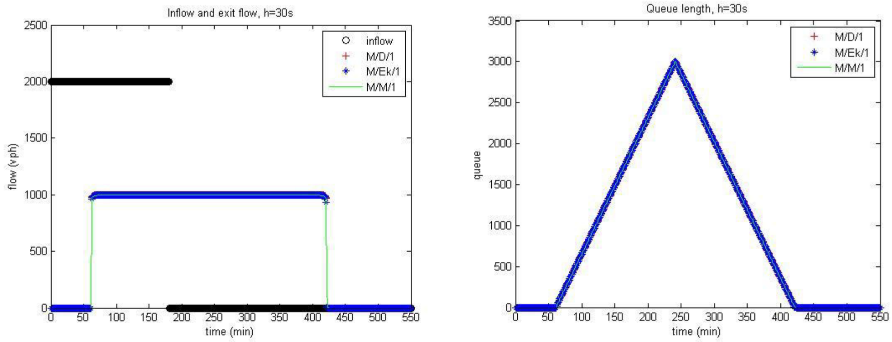
We test the three inflow profiles from time zero for 350 min with the free flow travel time t_0 = 60 min of the Nie and Zhang (2005) and Ban et al. (2012). The first profile is a constant inflow rate 2000 vph from time 0-180 min and then immediately drops to zero after that, thus is discontinuous at time t =180. The second one is a trapezoidal type of input with the demand peaks at 1200 vph from 60 to 120 min; it is (continuous) piece-wise linear but not differentiable at the times t =60, 120, 180. The third profile follows a sigmoidal curve from time zero to 180 min and then drops to 0, and is thus discontinuous at time t = 180 but smooth from 0 to t = 180 min. They represent, respectively, heavy traffic without much variation in terms of traffic volume, slowly varying inflow in moderately-congested traffic, and fast varying inflow in moderately-congested traffic. The mathematical forms of the three profiles are given below:

Inflow Profile #1: $p(t) = \begin{cases} 2000 & \text{for } 0 \le t < 180 \\ 0 & \text{otherwise.} \end{cases}$

Inflow Profile #2: $p(t) = \begin{cases} 20t & \text{for } 0 \le t < 60 \\ 1200 & \text{for } 60 \le t < 120 \\ 3600 - 20t & \text{for } 120 \le t < 180 \\ 0 & \text{otherwise.} \end{cases}$

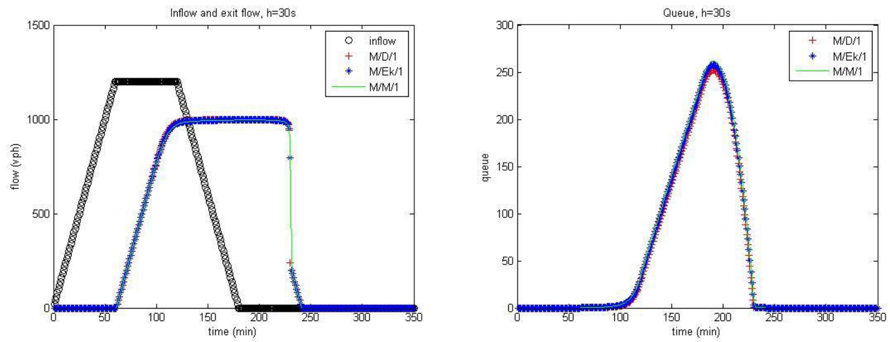
Inflow Profile #3: $p(t) = \begin{cases} 800 + 600\sin(6t) & \text{for } 0 \le t < 180 \\ 0 & \text{otherwise.} \end{cases}$

The time unit in our paper is in minutes and the capacity of the link is 1000 vph. The exit flows and queue lengths are depicted form Fig.3 to Fig. 5.



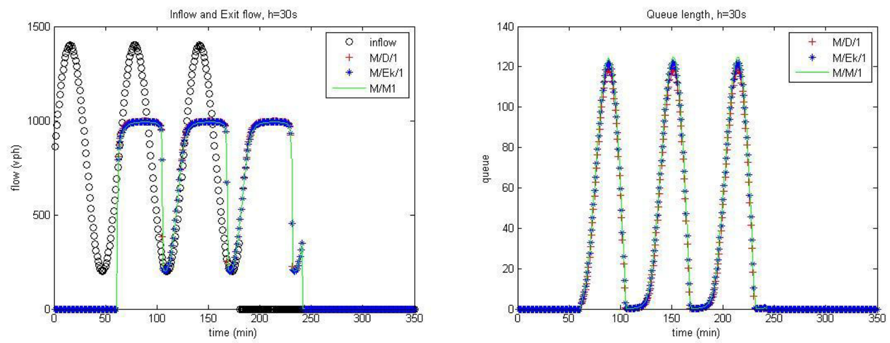
(a) (b)

Fig. 3. Exit flow and queue length under profile #1.



(a) (b)

Fig. 4. Exit flow and queue length under profile #2.



(a) (b)

Fig. 5. Exit flow and queue length under profile #3.

From Fig. 3 to Fig. 5, we can easily see that the queue and exit flow is non-negative. Moreover, the exit flow of M/D/1 is relatively larger than M/ E_k /1 and M/M/1 and the queue length of M/D/1 is relatively smaller than M/ E_k /1 and M/M/1. The reason of that is shown in Fig. 2. With the same queue length, the utilization ration of M/D/1 is the largest. Compared between the three results, we can see that the inflow rate, outflow rate and queue length is strongly inter-correlated, such as in Fig. 5, the periodicity of inflow rate leads to the periodicity of outflow rate and queue length. In a nutshell, the numerical tests show the PSFFA models can be used to approximate the original point-queue model and the non-negativity of the queue length can be deprived implicitly.

The advantage of model in this paper is that it indeed overcomes the negativity of the original point-queue model. However, the speed of calculation is relatively slow, which will restrict the use in the dynamic user equilibrium analysis. As in our discussion in the conclusion, the discrete form of FA model can be used to ensure the fast calculation.

3.2. Discussion

From the above tests, we can see the capability of our model in the approximation of the original point-queue model and the deprivation of the negativity of the queue length. If a detailed comparison is made between our study and Ban et al. (2012), we can see the following difference:

(1)With the same inflow profile, the queue length in our study is a little larger than that of Ban et al. (2012). When $q(t) \neq 0$, the exit flow rate is C in Ban et al. (2012), as the same in the Vickrey's model. However, in our study, when $q(t) \neq 0$, the exit flow rate is $\bar{C}\rho(t) = \bar{C}G^{-1}(q(t))$, which is state dependent. The state dependent exit flow rate in our paper is the same as Jain and Smith (1997).

(2)With the larger queue length and state dependent exit flow rate, the travel time in our paper is also a little larger, which can be clearly seen.

(3)With our model, the exit flow in our model is smoother than that in Ban et al. (2012), because vehicles in our model are treated as fluid in and out.

4. Conclusion

In this paper, the continuous-time point queue model is studied, which is the foundation of Dynamic Traffic Assignment (DTA) and Dynamic User Equilibrium (DUE) problem. The negativity of the queue-length of the original Vickrey's model is the focus during the recent research on this model, such as Nie and Zhang (2005), and Ban et al. (2012). Based on the observation that the point-queue model can be treated as a queuing model with a server and an infinite buffer, a fluid approximation (FA) model is used to study the point-queue model. The arrival rate is time-dependent and is assumed to be a Poisson distribution in this paper. The service rate of our FA model is the capacity of the link. With different state-dependent utilization ratio, the exit flow rate can be obtained. As a whole, the arrival rate and exit rate form the balance function of the FA model. With numerical testing, we can show that the capability of our model in approximating the point-queue model and the negativity of queue length can be deprived. With a detailed comparison between our model and the model in Ban et al. (2012), the difference of queue length, travel time and curve smoothness can be found, which is also discussed in our paper.

In any case, the point-queue model is still a pretty primitive approximation of real traffic flow dynamics, mainly because it cannot model the queue storage capacity of a network link. Important traffic dynamics such as queue spillback cannot be modelled using the point-queue model. From the perspective of finite queue, our following paper is under progress to extend the FA model to take the spillback under consideration in order to study its effect in the dynamic network loading problem.

Another interesting direction is the discrete form of fluid approximation. Dividing the time domain into different intervals, we can get the discrete form of fluid approximation as follows.

$$q_{t+1} = q_t + f_t^{in} - f_t^{out} \tag{7}$$

$$\rho_t = \frac{q_t + 1 - \sqrt{(q_t)^2 + 2\times(C_s)^2 \times q_t + 1}}{1-(C_s)^2} \tag{8}$$

Where (7) is the discrete form of (1) and (8) is the discrete form of $\rho_{(t)}$ in Eq. (2). Moreover, with the detailed comparison, Eq. (7) is the similar form of Merchant and Nemhauser (1978). Therefore, applying the discrete form of FA into dynamic system optimal assignment is worth doing in future.

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