Stability of Moving Invariant Sets and Uncertain Dynamic Systems on Time Scales

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Abstract—In this paper, we study the stability of moving invariant sets and uncertain dynamic systems on time scale. A control application is considered. © 1998 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Nonlinear differential equations with uncertain parameters may cause change of equilibrium states. To investigate such situations, Siljak, Ikeda and Ohata [1] have introduced the notion of parametric stability and discussed its study which is interesting in itself.

A fundamental feedback control problem is that of obtaining some desired behavior from the given system which has uncertain information. Leitmann and associates [2-4] have dealt with such a problem in a series of papers. They have investigated continuous and discrete uncertain systems by means of Lyapunov functions.

Recently, a theory known as dynamic systems on time scales has been built which incorporates both continuous and discrete times, namely, time as an arbitrary closed sets of reals, and permit us to handle both systems simultaneously [5,6]. This theory allows one to get some insight into and better understanding of the subtle differences between discrete and continuous systems.

To study uncertain systems, a different idea is employed in [7-9], which exhibits moving invariant sets as the parameter changes. By reducing the problem to a simpler comparison problem, the stability of moving invariant sets is discussed employing comparison method. The derivative of the Lyapunov function involved is estimated from opposite directions relative to suitable sets in phase space that depend on the moving parameter.

In this paper, utilizing the framework of the theory of dynamic systems on time scale, we shall investigate uncertain dynamic systems on time scale relative to stability of moving invariant sets. As an application of our results, we shall consider the control of uncertain dynamic system on time scales and obtain the desired stability behavior of moving invariant sets. For some preliminary work in this direction, see [10].
2. PRELIMINARIES

Let \( T \) be a time scale (any subset of \( \mathbb{R} \) with order and topological structure defined in a canonical way) with \( t_0 \geq 0 \) as a minimal element and no maximal element. Since a time scale \( T \) may or may not be connected, we need the following concept of jump operators.

**Definition 2.1.** The mappings \( \sigma, \rho : T \to T \) defined by
\[
\sigma(t) = \inf\{s \in T : s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in T : s < t\}
\]
are called the jump operators.

**Definition 2.2.** A nonmaximal element \( t \in T \) is called right-dense (rd) if \( \sigma(t) = t \), right-scattered (rs) if \( \sigma(t) > t \), left-dense (ld) if \( \rho(t) = t \), left-scattered (ls) if \( \rho(t) < t \). In the case \( T = R, \sigma(t) = t \), and \( T = hZ, \sigma(t) = t + h \).

**Definition 2.3.** The mapping \( \mu^* : T \to R_+ \) defined by \( \mu^*(t) = \sigma(t) - t \) is called graininess. If \( T = R, \mu^*(t) = 0 \), and when \( T = Z, \mu^*(t) = 1 \).

**Definition 2.4.** The mapping \( u : T \to X \), where \( X \) is a Banach space is called rd-continuous if at each right-dense \( t \in T \), it is continuous and at each left-dense \( t \), the left-sided limit \( u(t^-) \) exists.

Let \( C_{rd}[T, X] \) denote the set of rd-continuous mappings from \( T \) to \( X \). It is clear that a continuous mapping is rd-continuous. However, if \( T \) contains left-dense and right-scattered points, then rd-continuity does not imply continuity. But on a discrete time scale, the two notions coincide.

**Definition 2.5.** A mapping \( u : T \to X \) is said to be differentiable at \( t \in T \), if there exists an \( \alpha \in X \) such that for any \( \epsilon > 0 \) there exists a neighborhood \( N \) of \( t \) satisfying
\[
|u(\sigma(t)) - u(s) - \alpha(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s|, \quad \text{for all } s \in N.
\]

Let \( u^\Delta(t) \) denote the derivative of \( u \). Note, that if \( T = R, \alpha = u^\Delta = \frac{du(t)}{dt} \) and if \( T = Z, \alpha = u^\Delta = u(t + 1) - u(t) \). It is easy to see that if \( u \) is differentiable at \( t \), then it is continuous at \( t \), if \( u \) is continuous at \( t \) and \( t \) is right-scattered, then \( u \) is differentiable and
\[
u^\Delta(t) = \frac{u(\sigma(t)) - u(t)}{\mu^*(t)}.
\]

**Definition 2.6.** For each \( t \in T \), let \( N \) be a neighborhood of \( t \). Then, we define the generalized derivative (or Dini derivative), \( D^+u^\Delta(t) \), to mean that, given \( \epsilon > 0 \), there exists a right neighborhood \( N_e \subset N \) of \( t \) such that
\[
\frac{u(\sigma(t)) - u(s)}{\mu^*(t, s)} < D^+u^\Delta(t) + \epsilon, \quad \text{for } s \in N_e, \ s > t, \quad \text{where } \mu(t, s) = \sigma(t) - s.
\]

In case \( t \) is rs and \( u \) is continuous at \( t \), we have, as in the case of the derivative,
\[
D^+u^\Delta(t) = \frac{u(\sigma(t)) - u(t)}{\mu^*(t)}.
\]

**Definition 2.7.** Let \( h \) be a mapping from \( T \) to \( X \). The mapping \( g : T \to X \) is called the antiderivative of \( h \) on \( T \) if it is differentiable on \( T \) and satisfies \( g^\Delta(t) = h(t) \) for \( t \in T \).

The following known properties of the antiderivative are useful.

(a) If \( h : T \to X \) is rd-continuous, then \( h \) has the antiderivative \( g : t \to \int_s^t h(s)ds, s, t \in T \).
(b) If the sequence \( \{h_n\}_{n \in N} \) of rd-continuous functions \( T \to X \) converge uniformly on \([r, s]\) to rd-continuous function \( h \) then \( \left( \int_r^s h_n(t)dt \right)_{n \in N} \to \int_r^s h(t)dt \), in \( X \).
DEFINITION 2.8. The mapping \( f : T \times X \to X \) is rd-continuous if it is continuous at each \((t,x)\) with right-dense \(t\), \( \lim_{(s,y) \to (t-,x)} f(s,y) = f(t-,x) \) exists and at each \((t,x)\) with left dense \(t\), \( \lim_{y \to x} f(t,y) \) exists.

A basic tool employed in the proofs is the following induction principle, well suited for time scales.

THEOREM 2.1. Suppose that for any \( t \in T \), there is a statement \( A(t) \) such that the following conditions are verified:

(I) \( A(t_0) \) is true.

(II) If \( t \) right-scattered and \( A(t) \) is true, then \( A(\sigma(t)) \) is also true.

(III) For each right-dense \( t \), there exists a neighborhood \( N \) such that, whenever \( A(t) \) is true, \( A(s) \) is also true for all \( s \in N, s \geq t \).

(IV) For left-dense \( t \), \( A(s) \) is true for all \( s \in [t_0, t) \) implies \( A(t) \) is true.

Then, the statement \( A(t) \) is true for all \( t \in T \).

Following Definition 2.6, define, for \( V \in C_{rd}[T \times R^n, R^+] \), \( D^+V^\Delta(t, x(t)) \) to mean that, given \( \epsilon > 0 \), there exists a right neighborhood \( N_r \subset N \) of \( t \) such that

\[
\frac{1}{\mu(t, s)} \left[ V(\sigma(t), x(\sigma(t))) - V(s, x(\sigma(t))) - \mu(t, s)f(t, x(t)) \right] < D^+V^\Delta(t, x(t)) + \epsilon,
\]

for each \( s \in N_r, s > t \). As before, if \( t \) is rs and \( V(t, x(t)) \) is continuous at \( t \), this reduces to

\[
D^+V^\Delta(t, x(t)) = \frac{V(\sigma(t), x(\sigma(t))) - V(t, x(t))}{\mu(t, s)}.
\]

We need the following comparison results in terms of Lyapunov-like functions. See [10].

THEOREM 2.2. Let \( V \in C_{rd}[T \times R^n, R^+] \), \( V(t,x) \) be locally Lipschitzian in \( x \) for each \( t \in T \) which is rd, and let

\[
D^+V^\Delta(t, x(t)) \leq g(t, V(t, x(t)));
\]

where \( g \in C_{rd}[T \times R^+, R], g(t, u) \mu^*(t) + u \) is nondecreasing in \( u \) for each \( t \in T \) and \( r(t) = r(t, t_0, u_0) \) is the maximal solution of \( u^\Delta = g(t, u), u(t_0) = u_0 \geq 0 \), existing on \( T \). Then, \( V(t_0, x_0) \leq u_0 \) implies that \( V(t, x(t)) \leq r(t, t_0, u_0), t \in T, t \geq t_0 \).

A result giving the lower estimate is also true.

THEOREM 2.3. Let \( V \in C_{rd}[T \times R^n, R^+] \), \( V(t,x) \) be locally Lipschitzian in \( x \) for each \( t \in T \) which is rd, and let

\[
D^+V^\Delta(t, x(t)) \geq g(t, V(t, x(t)));
\]

where \( g \in C_{rd}[T \times R^+, R], g(t, u) \mu^*(t) + u \) is nondecreasing in \( u \) for each \( t \in T \) and \( p(t) = p(t, t_0, u_0) \) is the minimal solution of \( u^\Delta = g(t, u), u(t_0) = u_0 \geq 0 \), existing on \( T \). Then, \( V(t_0, x_0) \geq u_0 \) implies that \( V(t, x(t)) \geq p(t), t \in T, t \geq t_0 \).

We need both comparison results in our discussion below.

3. MAIN RESULTS

Consider the dynamic system on time scale

\[
x^\Delta = f(t, x, \lambda), \quad x(t_0) = x_0, \quad t_0 \in T,
\]

where \( f \in C_{rd}[T \times R^n \times R^d, R^n], \lambda \in R^d \) is an uncertain parameter and \( T \) is the time scale. Consider also the comparison dynamic equation

\[
u^\Delta = g(t, u, \mu), \quad u(t_0) = u_0 \geq 0,
\]

where \( g \in C_{rd}[T \times R^d, R] \) and \( \mu = \mu(\lambda) \geq 0 \) is a parameter depending on \( \lambda \).
Let \( \rho_0 \leq r_0 \leq r \leq \rho \) depending on \( \lambda \). Then, we shall say that the set \( B = \{ x \in \mathbb{R}^n : \rho_0 \leq |x| \leq \rho \} \) is conditionally invariant with respect to \( A = \{ x \in \mathbb{R}^n : r_0 \leq |x| \leq r \} \) and is uniformly asymptotically stable (UAS) relative to (3.1) if

(i) \( r_0 \leq |x_0| \leq r \) implies \( \rho_0 \leq |x(t)| \leq \rho, \ t \in T, \ t \geq t_0; \)

(ii) given \( \epsilon > 0 \) and \( t_0 \in T \),

then

(a) there exists a \( \delta = \delta(\epsilon) > 0 \) such that \( r_0 - \delta \leq |x_0| \leq r + \delta \) implies \( \rho_0 - \epsilon \leq |x(t)| \leq \rho + \epsilon, \ t \geq t_0, \ t \in T; \)

(b) there exists a \( \delta_0 > 0 \) and a \( T = T(\epsilon) > 0 \) such that \( r_0 - \delta_0 \leq |x_0| \leq r + \delta_0 \) implies \( \rho_0 - \epsilon \leq |x(t)| \leq \rho + \epsilon, \ t \geq t_0 + T, \ t \in T; \)

where \( x(t) = x(t, t_0, x_0) \) is any solution of (3.1).

Relative to the comparison equation (3.2), we shall say that \( Q = \{ u \in \mathbb{R}^n : \rho_0 \leq u \leq \rho \} \) is invariant and is UAS relative to (3.2) if

(i) \( \rho_0 \leq u_0 \leq R \) implies \( R_0 \leq u(t) \leq R, \ t \geq t_0, \ t \in T; \)

(ii) given \( \epsilon > 0 \) and \( t_0 \in T \),

then

(a) there exists a \( \delta = \delta(\epsilon) > 0 \) such that \( R_0 - \delta \leq u_0 \leq R + \delta \) implies \( R_0 - \epsilon \leq u(t) \leq R + \epsilon, \ t \geq t_0, \ t \in T; \)

(b) there exists a \( \delta_0 > 0 \) and a \( T = T(\epsilon) > 0 \) such that \( \rho_0 - \delta_0 \leq |u_0| \leq \rho_0 + \delta_0 \) implies \( \rho_0 - \epsilon \leq |u(t)| \leq \rho_0 + \epsilon, \ t \geq t_0 + T, \ t \in T; \)

where \( u(t) = u(t, t_0, u_0) \) is any solution of (3.2).

We can now prove the following result on UAS of the conditionally invariant set \( B \) with respect to \( A \), relative to the system (2.1). Let us define the sets \( \Omega_r, \Omega_{r_0} \) by \( \Omega_r = \{ x \in \mathbb{R}^n : x \in A^{\mu^*(t)} \} \) and \( |x| \geq r \}, \Omega_{r_0} = \{ x \in \mathbb{R}^n : x \in A^{\mu^*(t)} \) and \( |x| \leq r_0 \} \).

**Theorem 3.1.** Assume that

(A0) for each \( \lambda \in \mathbb{R}^d \), there exist \( r = r(\lambda), r_0 = r_0(\lambda), r_0 \leq r \) satisfying \( r \to 0 \) as \( |\lambda| \to 0 \) and \( r_0 \to \infty \) as \( |\lambda| \to \infty \);

(A1) there exists \( V \in C_{rd}[T \times \mathbb{R}^n, \mathbb{R}] \), \( V(t, x) \) be locally Lipschitzian in \( x \) for each right-dense \( t \in T \) and for \( a_i, b_i \in K, i = 1, 2 \),

\[
\begin{align*}
&b_1(|x|) \leq V(t, x) \leq a_1(|x|), \quad \text{if} \ x \in \Omega_r, \\
&b_2(|x|) \leq V(t, x) \leq a_2(|x|), \quad \text{if} \ x \in \Omega_{r_0};
\end{align*}
\]

(A2) if \( x \in \Omega_r, D^+V^\Delta(t, x) \leq g(t, V(t, x), r) \), and if \( x \in \Omega_{r_0}, D^+V^\Delta(t, x) \geq g(t, V(t, x), r_0) \),

where \( g \in C_{rd}[T \times \mathbb{R}^n, \mathbb{R}], g(t, u, \mu^*(t) + u \) is nondecreasing in \( u \) for each \( (t, u) \);

(A3) for each \( r_0 \leq r \), there exists \( R_0 \leq R \) such that \( R = a_1(r) = b_1(\rho) \) and \( R_0 = b_2(\rho_0) = a_2(\rho_0) \), where \( \rho_0 \leq r_0 \leq r \leq \rho \) and \( R \to \infty \) as \( r \to 0 \), \( R_0 \to \infty \) as \( r_0 \to \infty \);

(A4) the set \( \Omega \) is invariant is UAS with respect to (3.2).

Then, the set \( B \) is conditionally invariant with respect to \( A \) and is UAS relative to the system (3.1).

**Proof.** We shall first prove that \( B \) is conditionally invariant with respect to \( A \) and (3.1). If not, there would exist a solution \( x(t) = x(t, t_0, x_0) \) of (3.1) with \( r_0 \leq |x_0| \leq r \) and \( t_0 < t_2 \) such that either

(i) \( |x(t_2)| > \rho \) and \( r_0 \leq |x(t)|, \ t \in [t_0, t_2] \cap T; \)

(ii) \( |x(t_2)| < \rho_0 \) and \( |x(t)| \leq r, \ t \in [t_0, t_2] \cap T. \)
Because of (A2), using comparison Theorems 2.2, 2.3, we get either
\[ V(t, x(t)) \leq \rho(t, t_0, V(t_0, x_0)), \]
or
\[ V(t, x(t)) \leq \rho(t, t_0, V(t_0, x_0)), \]
for \( t \in [t_0, t_2] \cap \mathbb{T} \), where \( \rho(t, t_0, u_0), \rho(t, t_0, u_0) \) are the maximal and minimal solutions of (3.2). Hence, using (A3) and (A4) in Case (i), we have
\[ b_1(\rho) < b_1(|x(t_2)|) \leq V(t_2, x(t_2)) \leq \rho(t_2, t_0, V(t_0, x_0)) \leq r(t_2, t_0, a_1(|x_0|)) \leq r(t_2, t_0, a_1(\rho)) \leq a_1(\rho) = b_1(\rho), \]
or, in Case (ii), we get
\[ a_2(\rho_0) > a_2(|x(t_2)|) \geq V(t_2, x(t_2)) \geq \rho(t_2, t_0, V(t_0, x_0)) \geq \rho(t_2, t_0, b_2(|x_0|)) \geq b_2(r_0) \geq a_2(\rho_0). \]
Thus, we have a contradiction in both cases and hence, \( B \) is conditionally invariant with respect to \( A \) and (3.1).

Let \( 0 < \epsilon < \rho_0 \) and \( t_0 \in \mathbb{T} \) be given. Since (A4) holds and
\[ a_1(\rho) = b_1(\rho) = R, \quad R_0 = b_2(r_0) = a_2(\rho_0), \quad \text{given } a_2(\rho_0 - \epsilon), \ b_1(\rho + \epsilon), \]
there exist \( \epsilon_1, \delta_1, \delta > 0 \) such that
\[ R + \delta = a_1(\rho + \delta) < b_1(\rho + \epsilon) = R + \epsilon_1, \]
and
\[ R_0 - \epsilon = a_2(\rho_0 - \epsilon) < b_2(r_0 - \delta) = R_0 - \delta_1, \]
satisfying
\[ R_0 - \delta_1 < u_0 < R + \delta_1, \quad \text{implies } R_0 - \epsilon_1 < u(t) < R + \epsilon_1, \quad t \geq t_0, \ t \in \mathbb{T}, \]
where \( u(t) - u(t_0, u_0) \) is any solution of (3.2). We claim that with this \( \delta > 0 \), the set \( B \) is US relative to \( A \), that is,
\[ r_0 - \delta < |x_0| < r + \delta, \quad \text{implies } u(t_0, u_0) < |x(t)| < u(t), \quad t \geq t_0, \ t \in \mathbb{T}. \]
If this is not true, there would exist a solution \( x(t) \) of (3.1) with \( r_0 - \delta < |x_0| < r + \delta \) and a \( t_2 > t_0 \) such that either
\[ \begin{align*}
(\text{a}) & \quad |x(t_2)| \geq \rho + \epsilon \quad \text{and} \quad |x(t)| \geq r_0, [t_0, t_2] \cap \mathbb{T}, \quad \text{or} \\
(\text{b}) & \quad |x(t_2)| \leq \rho_0 - \epsilon \quad \text{and} \quad |x(t)| \leq r, [t_0, t_2] \cap \mathbb{T}.
\end{align*} \]
Consider (a). As before, we obtain
\[ V(t, x(t)) \leq r(t, t_0, V(t_0, x_0)), \quad [t_0, t_2] \cap \mathbb{T}, \]
and therefore, we arrive at the contradiction
\[ b_1(\rho + \epsilon) \leq b_1(|x(t_2)|) \leq V(t_2, x(t_2)) \leq r(t_2, t_0, V(t_0, x_0)) \leq r(t_2, t_0, a_1(|x_0|)) \leq r(t_2, t_0, a_1(\rho + \delta)) < b_1(\rho + \epsilon). \]
Similarly, in Case (b), we first get,
\[ V(t, x(t)) \geq \rho(t, t_0, V(t_0, x_0)), \quad [t_0, t_2] \cap \mathbb{T}, \]
and then it follows that
\[ a_2(\rho_0 - \epsilon) > a_2(|x(t_2)|) \geq V(t_2, x(t_2)) \geq \rho(t_2, t_0, V(t_0, x_0)) \geq \rho(t_2, t_0, b_2(|x_0|)) \geq b_2(r_0 - \delta) \geq b_2(r_0) = a_2(\rho - \epsilon), \]
which is again a contradiction. Hence, the set \( B \) is US relative to \( A \).
To prove UAS of the set $B$ relative to $A$, let us fix $\epsilon = \rho_0$ and designate by $\delta_0 = \delta(\rho_0)$ so, that we have

$$r_0 - \delta_0 < |x_0| < r + \delta_0, \quad \text{implies} \quad 0 < |x(t)| < \rho + \rho_0, \quad t \geq t_0, \quad t \in T.$$ 

Let $0 < \epsilon < \rho_0$ and $t_0 \in T$. Since $\Omega$ is UAS, given $a_2(\rho_0 - \epsilon)$, $b_1(\rho + \epsilon)$, there exists a $T = T(\epsilon) > 0$, with $t_0 + T \in T$ such that

$$b_2(r_0 - \delta_0) < u_0 < a_1(r + \delta_0), \quad \text{implies} \quad a_2(\rho_0 - \epsilon) < u(t) < b_1(\rho + \epsilon), \quad t \geq t_0 + T.$$ 

We claim that whenever $r_0 - \delta_0 < |x_0| < r + \delta_0$, we have

$$\rho_0 - \epsilon < |x(t)| < \rho + \epsilon, \quad t \geq t_0 + T, \quad t \in T.$$ 

If this is not true, there would exist a solution $x(t)$ of (3.1) such that

(a) $|x(t_2)| \geq \rho + \epsilon, \quad t_2 \geq t_0 + T, \quad t_2 \in T,$
(b) $|x(t_2)| \leq \rho_0 - \epsilon, \quad t_2 \geq t_0 + T, \quad t_2 \in T,$

where $r_0 - \delta_0 < |x_0| < r + \delta_0$. As before, using (A2) and (A3), we get successively

$$b_1(\rho + \epsilon) \leq V(t_2, x(t_2)) \leq r(t_2, t_0, a_1(r + \delta_0)) < b_1(\rho + \epsilon),$$

and

$$a_2(\rho_0 - \epsilon) \geq V(t_2, x(t_2)) \geq \rho(t_2, t_0, b_2(r_0 - \delta_0)) > a_2(\rho_0 - \epsilon),$$

which are contradictions. Hence, we have $B$ is UAS with respect to $A$ relative to system (3.1) and the proof is complete.

**REMARKS.** If $T = R$, then (3.1),(3.2) reduce to the continuous differential systems. Since, in this case, $\mu^*(t) = 0$, the results of Theorem 3.1 reduce to those in [7]. Note, that the conditions (A1) and (A2) are than weaker which are sufficient to prove UAS. If, on the other hand, $T = Z$, so that $\mu^*(t) = 1$, (3.1) and (3.2) reduce to difference equations, and consequently, one needs stronger conditions (A1), (A2). Theorem 3.1 offers new result in this special case.

As an application of Theorem 3.1, we shall consider the control of uncertain dynamic system on time scales of the form

$$x^\Delta = f_0(t, x, \lambda) + B(t, x, \lambda)F(t, x, 2, \lambda), \quad x(t_0) = x_0, \quad t_0 \in T, \quad (3.3)$$

under the following assumptions.

(B0) $f_0 \in C_{rd}[T \times R^n \times \Omega_0, R^n], \quad B \in C_{rd}[T \times R^n \times \Omega_0, R^{n \times m}], \quad F \in C_{rd}[T \times R^n \times R^m \times \Omega, R^n], \quad \Omega_0 \subset R^d$ is a nonempty set and $u \subset R^m$ is the control function.

(B1) There exist $r_0 = r_0(\lambda) \leq r = r(\lambda)$ and $V \in C_{rd}[T \times R^n, R_{+}]$ such that

$$V_{f_0}^\Delta(t, x) \leq -c_1(V(t, x)), \quad \text{if} \quad x \in \Omega_r,$$

$$V_{f_0}^\Delta(t, x) \geq -c_2(V(t, x)), \quad \text{if} \quad x \in \Omega_{r_0},$$

where $V_{f_0}^\Delta(t, x) = V^\Delta(t, x) + V_z(\sigma(t), x)f_0(t, x, \lambda), \quad c_i \in K, \quad i = 1, 2,$ and $c_i(u)\mu^*(t) + u$ is nondecreasing in $u$.

(B2) $b_1(|x|) \leq V(t, x) \leq a_1(|x|)$ if $x \in \Omega_r$,

$$b_2(|x|) \leq V(t, x) \leq a_2(|x|) \text{ if } x \in \Omega_{r_0} \text{ where } a_i, b_i \in K, \quad i = 1, 2.$$

(B3) For $x \in \Omega_r$, $u^T F(t, x, u, \lambda) \geq -b_1(t, x)|u| + b_2(t, x)|u|^2,$ where $b_1, b_2 \in C_{rd}[T \times R^n, R_{+}], \quad b_1 < b_2, \quad b_2 < k, \quad k \in C_{rd}[T \times R^n, R_{+}].$

(B4) For $x \in \Omega_{r_0}$, $u^T F(t, x, u, \lambda) \leq -r_1(t, x)|u| + r_2(t, x)|u|^2,$ where $r_1, r_2 \in C_{rd}[T \times R^n, R_{+}], \quad r_1 \geq r_2, \quad r_2 > k.$

(B5) $P = [p_\mu \in C_{rd}[T \times R^n, R^n]$ for $\mu > 0$ is the stabilizing family of controllers satisfying $|\alpha|p_\mu = -|\alpha|p_\mu, \quad \alpha = B^TV^\Delta \text{ and } \eta = k \alpha,$ and if $|\eta| > 0, \quad x \in \Omega_r, \quad |p_\mu| \geq \tilde{\rho}(1 - r/|\eta|)$, if $|\eta| > 0, \quad x \in \Omega_{r_0}, \quad |p_\mu| \leq \tilde{\rho}(1 - r_0/|\eta|)).$

We are now in a position to prove the following result.
THEOREM 3.2. Assume that Conditions (B1) to (B5) hold. Suppose further that $c_2^{-1}(u) \leq c_1^{-1}(u)$. Then, the set $B$ is conditionally invariant with respect to $A$ and is UAS relative to the system (3.3).

PROOF. Let us first consider the case $x \in \Omega_r$. Then, we have

$$V^\Delta_{\Omega}(t,x) \leq -c_1(V(t,x))$$

and

$$\alpha(t,x,\lambda)F(t,x,p,\lambda) = -\frac{\vert\alpha(t,x,\lambda)\vert}{\vert\mu(t,x)\vert}F(t,x,p,\lambda)p \leq |\alpha(t,x,\lambda)||r_1(t,x) - r_2(t,x)|p(t,x)|$$

$$\leq |\alpha(t,x,\lambda)||r_1(t,x) - r_2(t,x)|\beta(t,x)
\left(1 - \frac{r_0}{|\eta|}\right)$$

$$\leq |\alpha(t,x,\lambda)||r_1(t,x)|\frac{r}{|\eta|} \leq r,$$

and therefore, we obtain

$$V^\Delta_{(2.3)}(t,x) \leq -c_1(V(t,x)) + r, \quad \text{if } x \in \Omega_r.$$

Similarly, if $x \in \Omega_{r_0}$, we get

$$V^\Delta_{\Omega_0}(t,x) \geq -c_2(V(t,x)),$$

and

$$\alpha(t,x,\lambda)F(t,x,p,\lambda) = -\frac{\vert\alpha(t,x,\lambda)\vert}{\vert\mu(t,x)\vert}F(t,x,p,\lambda)p \geq |\alpha(t,x,\lambda)||r_1(t,x) - r_2(t,x)|p(t,x)|$$

$$\geq |\alpha(t,x,\lambda)||r_1(t,x) - r_2(t,x)|\beta(t,x)
\left(1 - \frac{r_0}{|\eta|}\right)$$

$$\geq |\alpha(t,x,\lambda)||r_1(t,x)|\frac{r_0}{|\eta|} \geq r_0.$$

Thus, we have

$$V^\Delta_{(2.3)}(t,x) \geq -c_2(V(t,x)) + r_0, \quad \text{if } x \in \Omega_{r_0}.$$

This implies that

$$g(t,u,r) = -c_1(u) + r, \quad g(t,u,r_0) = -c_2(u) + r_0,$$

and therefore, $u = c_1^{-1}(u) = R, \quad u = c_2^{-1}(u) = R_0$. Hence, in view of the properties of $a_i, b_i, c_i, i = 1,2$, we can find, for each $\lambda, \rho_0 \leq r_0 \leq r \leq \rho$ such that $R = a_1(\rho) = b_1(\rho), R_0 = b_2(\rho_0) = a_2(\rho_0)$, and $R_0 \leq R$.

To apply Theorem 3.1, we need to show that the set $\Omega = \{u \in R_+ : \rho_0 \leq u \leq R\}$ is invariant and is UAS relative to the comparison dynamic equation (3.2). Because of the specific nature of $g$, it is not difficult to prove it following the proof of Theorem 3.1 We omit the details to avoid monotony. The proof of Theorem 3.3 is, therefore, complete.

REFERENCES


