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On Drazin inverse of operator matrices [☆]

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ABSTRACT

In this short paper, we offer (another) formula for the Drazin inverse of an operator matrix for which certain products of the entries vanish. We also give formula for the Drazin inverse of the sum of two operators under special conditions.

1. Introduction

Let X be, in general, infinite dimensional Banach vector space, and let $\mathcal{L}(X)$ denote set of all bounded linear operators defined on X. We recall that the Drazin inverse of the operator $A \in \mathcal{L}(X)$ is the unique operator $A^D \in \mathcal{L}(X)$, provided it exists, satisfying the following conditions

$$A^D A = A A^D, \qquad A^D A A^D = A^D, \qquad A^{\nu+1} A^D = A^\nu.$$

The smallest natural number ν satisfying the previous system of equations is known as the index of the operator A and is denoted by $\mathrm{ind}(A)$. It is well known that the Drazin inverse of the operator $A \in \mathcal{L}(X)$ exists if and only if $0 \notin \overline{\sigma(A) \setminus \{0\}}$ and the point zero, provided $0 \in \sigma(A)$, is a pole of the resolvent $R(\lambda, A) = (\lambda - A)^{-1}$. Here $\sigma(A)$ denotes spectrum of the operator A, and for $W \subset \mathbb{C}$ symbol $\overline{W}$ denotes closure of W.

Assume that Banach space X has the splitting $X = Y \oplus Z$, where $Y, Z \subset X$ are closed subspaces of X. Denote by P_Y, $P_Z = 1 - P_Y$, projections on Y, Z, along Z, Y, respectively. Let $M \in \mathcal{L}(X)$, where $X = Y \oplus Z$, then we can express M in the form of the operator matrix

$$M = \begin{Vmatrix} A & B \\ C & D \end{Vmatrix},$$

where $A = P_Y(M\restriction_Y) \in \mathcal{L}(Y)$, $B = P_Y(M\restriction_Z) \in \mathcal{L}(Z, Y)$, $C = P_Z(M\restriction_Y) \in \mathcal{L}(Y, Z)$ and $D = P_Z(M\restriction_Z) \in \mathcal{L}(Z)$. Here $\mathcal{L}(Y, Z)$ is the set of bounded linear operators mapping Y to Z, and $M\restriction_Z$ is the operator M restricted to the subspace Z. We can express the resolvent of the operator matrix M in the following form

$$(\lambda - M)^{-1} = \begin{Vmatrix} Q^{-1} & Q^{-1}B(\lambda - D)^{-1} \\ (\lambda - D)^{-1}CQ^{-1} & (\lambda - D)^{-1} + (\lambda - D)^{-1}CQ^{-1}B(\lambda - D)^{-1} \end{Vmatrix}, \tag{1}$$

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where $Q = \lambda - A - B(\lambda - D)^{-1}C$. Expression is valid, provided Q is invertible, at least in the neighborhood of complex infinity, since all operators A, B, C and D are bounded (see [12,23]). In the sequel we are going to provide exact conditions under which we are going to have invertibility of Q.

In an earlier paper Castro-González et al. (see [6]) gave an expression for the Drazin inverse M^D subject to the condition $BCA = 0$. In the present paper we assume instead $ABC = 0$. The earlier result [6, Theorem 4.2] is derived from a result [6, Theorem 2.3] about sums; here we proceed directly to the matrix result, deriving some result for the sum as a consequence. The presented results for the operator matrices are proven using factorization of the operator Q. It is shown that such simple approach can be used to derive much of the results about the Drazin inverse of operator matrices, including results derived in [6].

If the Drazin inverse A^D of the operator A exists, we have

$$(\lambda - A)^{-1} = \sum_{k=1}^{\nu} \lambda^{-k} A^{k-1} A^{\pi} - \sum_{k=0}^{+\infty} \lambda^k \left(A^D\right)^{k+1}, \quad A^{\pi} = 1 - AA^D,\ \nu = \operatorname{ind}(A). \tag{2}$$

From this identity it is clear that we can identify the Drazin inverse of an operator by reading the coefficient with λ^0 in the Laurent expansion of the resolvent function in the punctured neighborhood of zero, i.e.,

$$A^D = -\frac{1}{2\pi \mathrm{i}} \oint_{|\lambda|=\varepsilon} \frac{1}{\lambda} (\lambda - A)^{-1}\, \mathrm{d}\lambda,$$

where ε is such that $\{\lambda \mid |\lambda| \leqslant \varepsilon\} \cap \sigma(A) = \{0\}$. A list of references about Drazin inverses include [1–26].

2. Results

The results we present are in the series form like already presented in [10,26].

Theorem 1. *Assume A, BC and D are Drazin invertible with $\nu_1 = \operatorname{ind}(A)$, $\nu_2 = \operatorname{ind}(BC)$, $\nu_3 = \operatorname{ind}(D)$. If $ABC = 0$, $BD = 0$ and $DC = 0$ then*

$$M^D = \begin{Vmatrix} \Psi A & \Psi B \\ C\Psi & D^D + C(\Psi A^D + (BC)^D(\Psi A - A^D))B \end{Vmatrix},$$

where

$$\Psi = \sum_{k=1}^{\nu_2} (BC)^{k-1}(BC)^{\pi} \left(A^D\right)^{2k} + \sum_{k=1}^{[(\nu_1+1)/2]} \left((BC)^D\right)^k A^{2k-2} A^{\pi}.$$

Proof. Under given assumptions we have $B(\lambda - D)^{-1} = \lambda^{-1}B$, $(\lambda - D)^{-1}C = \lambda^{-1}C$, $Q = \lambda - A - \lambda^{-1}BC$. The point is the following factorization, namely,

$$\lambda - A - \lambda^{-1}BC = (\lambda - A)\left(1 - \lambda^{-2}BC\right),$$

which is valid under the condition $ABC = 0$. If we assume that $BCA = 0$, we would have an inverse order of factors and we end-up in the results presented in [6]. Under this factorization of Q, we can state that Q^{-1} has the Laurent expansion in the punctured neighborhood of zero, since Q^{-1} is a product of two analytic functions at the punctured neighborhood of the point zero. Similarly, all other entries at the resolvent matrix $(\lambda - M)^{-1}$ have the Laurent expansion at the punctured neighborhood of the point zero. We note that being a product of meromorphic functions in the neighborhood of zero, all elements of the matrix $(\lambda - M)^{-1}$ are meromorphic as well. Accordingly, all elements in the operator matrix have Drazin inverses. Finally, M has the Drazin inverse since

$$\begin{aligned}(\lambda - M)^{-1} &= Q^{-1}P_Y + Q^{-1}B(\lambda - D)^{-1}(1 - P_Y) + (\lambda - D)^{-1}CQ^{-1}P_Y \\ &\quad + \left((\lambda - D)^{-1} + (\lambda - D)^{-1}CQ^{-1}B(\lambda - D)^{-1}\right)(1 - P_Y),\end{aligned}$$

is the sum of meromorphic functions at the neighborhood of the point zero. Using this representation of $(\lambda - M)^{-1}$ and according to linearity of integration, in order to determine the Drazin inverse of M we only need to determine coefficients with λ^0 in the Laurent series of the elements of the operator matrix $(\lambda - M)^{-1}$.

The result presented in this theorem is obtained in this way. For example, at the position $(1,1)$ the term with λ^0 in the operator matrix, we read from

$$\begin{aligned} Q^{-1} &= \lambda^2\left(\lambda^2 - BC\right)^{-1}(\lambda - A)^{-1} \\ &= \lambda^2 \left(\sum_{k=1}^{\nu_2} \lambda^{-2k}(BC)^{k-1}(BC)^{\pi} - \sum_{k=0}^{+\infty} \lambda^{2k}\left((BC)^D\right)^{k+1} \right) \left(\sum_{k=1}^{\nu_1} \lambda^{-k}A^{k-1}A^{\pi} - \sum_{k=0}^{+\infty} \lambda^k \left(A^D\right)^{k+1} \right). \end{aligned}$$

Similarly, we proceed for all other entries of the operator matrix. □

We mention special cases of this theorem already presented in the literature. In [26], results are given under consideration $A = D = 0$. In which case $Q = \lambda - \frac{1}{\lambda}BC$, expression for resolvent becomes

$$(\lambda - M)^{-1} = \begin{Vmatrix} (\lambda - \frac{1}{\lambda}BC)^{-1} & \frac{1}{\lambda}(\lambda - \frac{1}{\lambda}BC)^{-1}B \\ \frac{1}{\lambda}(\lambda - \frac{1}{\lambda}BC)^{-1} & \frac{1}{\lambda}(1 + \frac{1}{\lambda}C(\lambda - \frac{1}{\lambda}BC)^{-1}B) \end{Vmatrix}.$$

We note that Laurent expansion for $(\lambda^2 - BC)^{-1}$ is needed.

In [13], or [24], authors considered case $B = 0$. In such case $Q = \lambda - A$, so that results are generated directly, with resolvent given by

$$(\lambda - M)^{-1} = \begin{Vmatrix} (\lambda - A)^{-1} & 0 \\ (\lambda - D)^{-1}C(\lambda - A)^{-1} & (\lambda - D)^{-1} \end{Vmatrix}.$$

In this case we need Laurent expansions for $(\lambda - A)^{-1}$ and $(\lambda - D)^{-1}$, as well as product of these series.

Theorem 2. *Assume A, D and BC are Drazin invertible with $\nu_1 = \operatorname{ind}(A)$, $\nu_2 = \operatorname{ind}(BC)$, $\nu_3 = \operatorname{ind}(D)$. If $ABC = 0$, $DC = 0$ and BC be nilpotent, then*

$$M^D = \begin{Vmatrix} \Psi A & \Psi\Phi + \Theta \\ C\Psi & D^D + C(\Psi A^D \Phi + \Sigma) \end{Vmatrix},$$

where

$$\Phi = \sum_{m=1}^{\nu_3} \left(A^D\right)^{m-1} BD^{m-1}D^\pi, \qquad N_{k+1} = \left(N_k - \left(A^D\right)^k B\right)D^D,$$

$$N_2 = \left(\Delta D^D - A^D B\right)D^D, \qquad \Delta = \sum_{k=1}^{\nu_1} A^{k-1}A^\pi B\left(D^D\right)^{k-1},$$

$$\Theta = \sum_{k=1}^{\nu_2} (BC)^{k-1} N_{2k}, \qquad \Sigma = \sum_{k=1}^{\nu_2} (BC)^{k-1} N_{2k+1}.$$

Proof. Now we have $(\lambda - D)^{-1}C = \lambda^{-1}C$, $Q = (\lambda - A)(1 - \lambda^{-2}BC)$ and $(BC)^D = 0$, since BC is nilpotent. This gives

$$Q^{-1} = \lambda^2\left(\lambda^2 - BC\right)^{-1}(\lambda - A)^{-1} = \lambda^2\left(\sum_{k=1}^{\nu_2} \lambda^{-2k}(BC)^{k-1}\right)(\lambda - A)^{-1}.$$

The rest of the proof is the same as in the previous theorem. □

In the case $BC = 0$ and $BD = 0$ we have $Q = \lambda - A$, and resolvent matrix is given by

$$(\lambda - M)^{-1} = \begin{Vmatrix} (\lambda - A)^{-1} & (\lambda - A)^{-1}\frac{1}{\lambda}B \\ (\lambda - D)^{-1}C(\lambda - A)^{-1} & (\lambda - D)^{-1} + (\lambda - D)^{-1}C(\lambda - A)^{-1}\frac{1}{\lambda}B \end{Vmatrix}.$$

Expression for the Drazin inverse can be generated using method presented in previous theorem. Special results, in which case resolvent matrix is even more reduced, are presented in [13] and [17].

Finally, we state the last result.

Theorem 3. *Assume A and BC are Drazin invertible with $\nu_1 = \operatorname{ind}(A)$, $\nu_2 = \operatorname{ind}(BC)$ and $\nu_3 = \operatorname{ind}(D)$. If $ABC = 0$, $DC = 0$ and D be nilpotent, then*

$$M^D = \begin{Vmatrix} \Psi A & \Psi AA^D\Phi + F(\Sigma) - \Theta D \\ C\Psi & C(\Psi A^D \Phi + (BC)^D F(\Omega) - \Theta) \end{Vmatrix},$$

where

$$\Sigma = \Psi A^\pi(B + ABD), \qquad \Omega = \Psi A^\pi(AB + BD), \qquad \Theta = \sum_{m=1}^{\nu_3} N_{m+1}BD^{m-1},$$

$$N_{2m+1} = N_{2m}A^D, \qquad N_{2m+2} = \left(N_{2m+1} + \left((BC)^D\right)^{m+1}\right)A^D, \qquad N_1 = 0,$$

$$F(X) = \sum_{m=1}^{[(\nu_3+1)/2]} \left((BC)^D\right)^{m-1} XD^{2m-2}.$$

Proof. In this case $(\lambda - D)^{-1}C = \lambda^{-1}C$, $Q = (\lambda - A)(1 - \lambda^{-2}BC)$ and $D^D = 0$, since D is nilpotent. $\square$

We give an illustration using matrices. Let

$$A = \begin{Vmatrix} 1 & -1 & 0 & 0 & 1 & 2 \\ 1 & -1 & -1 & 0 & 3 & 4 \\ 1 & -1 & -1 & 0 & 5 & 6 \\ 1 & -1 & -1 & 0 & 7 & 8 \\ 1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{Vmatrix}, \qquad B = \begin{Vmatrix} 1 & -1 & 2 & 3 & 4 \\ 1 & -1 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 & 10 \\ 0 & 0 & 11 & 12 & 13 \\ 0 & 0 & 14 & 15 & 16 \\ 0 & 0 & 17 & 18 & 19 \end{Vmatrix},$$

$$C = \begin{Vmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{Vmatrix}, \qquad D = \begin{Vmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{Vmatrix}.$$

Then, we easily verify $ABC = 0$, $DC = 0$ and clearly D is nilpotent. Applying Theorem 3, we obtain

$$M^D = \begin{Vmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{9}{4} & -\frac{7}{4} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{55}{2} & -30 & \frac{637}{16} \\ -1 & 1 & 1 & -1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{57}{4} & -\frac{63}{4} & \frac{97}{4} \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & \frac{57}{4} & \frac{63}{4} & -\frac{65}{4} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 1 & 0 & 0 & -6 & -6 & -\frac{57}{8} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{7}{4} & \frac{5}{4} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 19 & \frac{39}{2} & \frac{149}{16} \\ \frac{1}{2} & -\frac{1}{2} & \frac{5}{4} & -\frac{3}{4} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{45}{4} & -\frac{45}{4} & -\frac{171}{16} \\ \frac{3}{2} & -\frac{1}{2} & -\frac{29}{4} & \frac{11}{2} & \frac{1}{4} & -2 & 0 & 0 & \frac{1285}{16} & \frac{1359}{16} & -\frac{425}{8} \\ \frac{3}{2} & -\frac{3}{2} & -\frac{21}{4} & \frac{9}{2} & \frac{1}{4} & -2 & 0 & 0 & \frac{1029}{16} & \frac{1095}{16} & -\frac{175}{8} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{Vmatrix}.$$

We want to illustrate an application of derived results to the computation of the Drazin inverse of the sum of two operators. Classical approach uses the fact that if $F, G \in \mathcal{L}(X)$ then $Q = \begin{Vmatrix} F & 1 \end{Vmatrix} \in \mathcal{L}(X \oplus X, X)$ and $R = \begin{Vmatrix} 1 \\ G \end{Vmatrix} \in \mathcal{L}(X, X \oplus X)$, then obviously $QR = F + G$ and $RQ = \begin{Vmatrix} F & 1 \\ GF & G \end{Vmatrix} \in \mathcal{L}(X \oplus X)$. Expressions for QR and RQ actually give connection between Drazin inverses of operator matrices and Drazin inverses of the sum of two operators, using the following fact (see [6,16]), let $Q \in \mathcal{L}(X, Y)$, $R \in \mathcal{L}(Y, X)$ if QR is Drazin invertible then RQ is Drazin invertible, moreover,

$$(RQ)^D = Q\big((QR)^2\big)^D R = Q\big((QR)^D\big)^2 R.$$

Using this observation we give the following lemma.

Lemma 4. *Let $F, G \in \mathcal{L}(X)$ and assume that F, G and GF are Drazin invertible. If $FGF = 0$ and $G^2F = 0$ then $F + D$ is Drazin invertible and*

$$(F+G)^D = \big(1 + GF^D\big)F^D + F\big[F^D H + HG^D\big]G + GF^D(\Psi\Phi + N_3)G + \big(G^D + G\big(F^D\big)^2\Phi + GFN_3\big)G^D G,$$

where $H = \Psi\Phi + N_2$, Ψ is as in Theorem 1 and Φ, N_k, $k = 2, 3$, are as in Theorem 2.

Proof. Using described procedure we get that $F + G$ is Drazin invertible if and only if operator matrix $M = \begin{Vmatrix} F & 1 \\ GF & G \end{Vmatrix}$ is Drazin invertible. We easily check, assuming $A = F$, $B = 1$, $C = GF$ and $D = G$, that $ABC = FGF = 0$, $DC = G^2F = 0$ and $BC = GF$ is nilpotent, since $(GF)^2 = GFGF = 0$. Accordingly, we can apply Theorem 2. We have $\Psi = (1 + GF^D)(F^D)^2$, $\Phi = \sum_{k=1}^{\nu_3} (F^D)^{k-1} F^\pi (G^D)^{k-1}$, $\Delta = \sum_{k=1}^{\nu_1} F^{k-1} F^\pi (G^D)^{k-1}$, $\Theta = N_2 + GFN_4$, $\Sigma = N_3 + GFN_5$, $N_2 = \Delta(G^D)^2 - F^D G^D$, $N_3 = \Delta(G^D)^3 - (F^D)^2 G^D - F^D (G^D)^2$, $N_4 = \Delta(G^D)^4 - (F^D)^3 G^D - (F^D)^2 (G^D)^2 - F^D (G^D)^3$, $N_5 = \Delta(G^D)^5 - (F^D)^4 G^D - (F^D)^3 (G^D)^2 - (F^D)^2 (G^D)^3 - F^D (G^D)^3$, which gives

$$M^D = \begin{Vmatrix} (1 + GF^D)F^D & \Psi\Phi + N_2 + GFN_4 \\ GF^D & G^D + G(F^D)^2\Phi + GFN_3 \end{Vmatrix}.$$

Using previous formula, we obtain

$$
\begin{aligned}
(F+G)^D &= \left\| F \quad 1 \right\| \left(M^D\right)^2 \left\| \begin{matrix} 1 \\ G \end{matrix} \right\| \\
&= \left\| F \ \ 1 \right\| \left\| \begin{matrix} \Psi & (1+GF^D)F^D H + (H + GFN_4)G^D \\ G(F^D)^2 & GF^D H + (G^D + G(F^D)^2\Phi + GFN_3)G^D \end{matrix} \right\| \left\| \begin{matrix} 1 \\ G \end{matrix} \right\| \\
&= \left(1+GF^D\right)F^D + F\left[F^D H + HG^D\right]G + GF^D(\Psi\Phi + N_3)G + \left(G^D + G\left(F^D\right)^2\Phi + GFN_3\right)G^D G.
\end{aligned}
$$

In the derivation we used heavily identities $G^D F = G^D F^D = F^D GF = FGF^D = F^D GF^D = 0$, which can be derived easily from the conditions of the lemma. □

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