REALIZING STRONG SHAPE EQUIVALENCES

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We use an Artin-Mazur type strong shape functor to prove that the strong shape category of compact metric spaces (D.A. Edwards and the second-named author) is the category of fractions obtained by inverting strong shape equivalences. An example using the dyadic solenoid shows that this fails for (weak) shape theory.

1. Introduction

Shape theory studies the algebraic topology of arbitrary (usually compact metric or compact Hausdorff) spaces by approximating these spaces by inverse systems of “nice spaces”. See D.A. Edwards [15], or the forthcoming survey by Edwards and the second-named author [17] for a history of shape theory.


On the other hand, pro-Ho(Top) is inadequate as a target for shape functors in several respects. It is too weak: D. Christie’s [9] work on the dyadic solenoid (see also Section 6, below) and T. Chapman’s [8] proof that the Borsuk shape category is isomorphic to the weak proper homotopy category of the complements of nicely embedded compacta suggest a stronger possible theory.

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Finally D. Quillen [36] described an abstract structure for homotopy theory—a closed model categories in which cofibrations, fibrations, and weak equivalences are defined and satisfy the "usual" properties, and observed that pro-Ho(Top) is not the homotopy category of a closed model structure on pro-Top.

D.A. Edwards and the second-named author [16, especially Sections 3, 8] described a closed model structure on pro-Top, and used T. Porter's [35] Vietoris functor to describe a satisfactory, although complex, strong shape theory. This resolves the above difficulties. The second-named author [23] (see also Sections 2 and 3 below) gave an equivalent geometric formulation of strong shape theory by rigidifying the Mardešić–Segal functor. J. Dydak and J. Segal [13], and Y. Kodama and J. Ono [28] developed alternate equivalent versions of strong shape theory.

The remaining sections are organized as follows. In Sections 4 and 5 we prove that the strong shape category of compact metric spaces CM (developed in Sections 2 and 3) is precisely the localization \{strong shape equivalences\}^{-1} CM. In Section 6 we prove that W. Holtsztynski's "universal" shape category \{shape equivalences\}^{-1} CM is not equivalent to the (Mardešić–Segal) shape category. Section 7 contains some further interesting questions.

2. Background

We develop the background necessary to define precisely the geometric strong shape functor.

Let Top denote the category of topological spaces and continuous maps, let CH \subset Top be the full subcategory of compact Hausdorff spaces, and let CM \subset CH be the full subcategory of compact metric spaces. We shall also need a small category of finite polyhedra. Let Q be the Hilbert cube, and let PL be the category whose objects are polyhedral subspaces whose linear structure is induced from Q, and piecewise linear maps. Then every finite polyhedron is piecewise-linearly equivalent to an object of PL, and PL is a (non-full) small subcategory of CM. (We only use the embeddings of objects of PL in Q to make PL small, and to define a unique piecewise-linear structure on each object considered as a topological space.)

Following [23], we shall define a strong shape functor s-sh: Top \to pro-PL. We shall need a category of inverse systems in which cofinal subsystems are isomorphic. For any category C, Grothendieck defined such a category pro-C. We recall the description of pro-C in the appendix of Artin and Mazur [2]. Objects of pro-C are functors from small, filtering categories to C, denoted \( X: A \to C \), or simply \( A \to C \).

If \( \alpha_1 \to \alpha_2 \) is a morphism of \( A \), we call the induced morphism \( X(\alpha_1) \to X(\alpha_2) \) a bonding map of the inverse system. We shall sometimes simply write \( \{X_\alpha\}_{\alpha \in A} \) or \( \{X_\alpha\} \).
for \(X : A \to C\) when there is now danger of confusion. We now define morphisms in pro-\(C\). Let \(\{X_\alpha\}_{\alpha \in A}\) and \(\{Y_\beta\}_{\beta \in B}\) be objects in pro-\(C\). Then

\[
\text{pro-}C(\{X_\alpha\}, \{Y_\beta\}) = \lim_B \text{colim}_A \{C(X_\alpha, Y_\beta)\}.
\]

It is convenient to have explicit representatives of morphisms in pro-\(C\). A morphism \(f : \{X_\alpha\} \to \{Y_\beta\}\) may be represented by a function \(\theta : \text{obj } B \to \text{obj } A\) and a set of morphisms in \(C(f_\beta : X_{\theta(\beta)} \to Y_\beta)\), such that for any bonding map \(Y_\beta \to Y_{\beta'}\) of \(\{Y_\beta\}\), there exists a commutative diagram

\[
\begin{array}{ccc}
X_\alpha & \xrightarrow{\text{bond}} & X_{\theta(\beta)} \\
\downarrow & & \downarrow \\
X_{\theta(\beta')} & \xrightarrow{f_\beta} & Y_\beta \\
\end{array}
\]

A functor \(T : A \to B\) between filtering categories is called cofinal if for each object \(\beta \in B\), there is a morphism \(T(\alpha) \to \beta\) in \(B\), and for any two morphisms \(T(\alpha) \to \beta\) in \(B\), there is a morphism \(\alpha' \to \alpha\) in \(A\) such that the two composed morphisms \(T(\alpha') \to T(\alpha) \to \beta\) are equal. Let \(T : A \to B\) be a cofinal functor between small filtering categories and let \(X : B \to C\) be an object of pro-\(C\). Then induced map

\[
(X : B \to C) \to (X \circ T : A \to C)
\]

is an isomorphism of pro-\(C\). If \(T\) is an inclusion, we shall sometimes simply say that \(\{X_\alpha\}_{\alpha \in A}\) is a cofinal subsystem of \(\{X_\beta\}_{\beta \in B}\).

Finally, a small filtering category \(A\) is called a strongly directed set if for any objects \(\alpha_1, \alpha_2 \in A\), there is at most one morphism \(\alpha_1 \to \alpha_2\), denoted \(\alpha_2 \preceq \alpha_1\) (\(\alpha_2\) precedes \(\alpha_1\)), and if \(\alpha_1 \leq \alpha_2\) and \(\alpha_2 \leq \alpha_1\) imply \(\alpha_1 = \alpha_2\). If, in addition, each object of \(A\) has only finitely many predecessors, we call \(A\) cofinite. For any small filtering category \(A\), there is a functorial cofinite strongly directed set \(M(A)\), and a cofinal functor \(M(A) \to A\). This is the Mardešić construction as given in [16, Theorem (2.1.6)].

In Section 3 we shall need and develop a generalization of cofinality: cofinality up to coherent homotopy. This concept, and its use in a continuity theorem for strong shape, suggests the development of a Vogt-style (R. Vogt [43]) approach to coherent pro-homotopy. See the discussion in Section 7, and T. Porter’s recent paper [45].

We shall need one more construction, a special case of S. MacLane’s [30] comma categories, to define s-sh. Let \(C\) be a category, and let \(D\) be a subcategory of \(C\). For an object \(X \in C\), let

\[
X \downarrow D
\]

be the category whose objects are morphisms \(X \to Y\), with \(Y\) an object of \(D\), and whose morphisms are commutative triangles in \(C\).
with $Y \rightarrow Y'$ a morphism in $D$. This construction extends to a contravariant functor on $C$; if $f : X \rightarrow X'$ is a morphism of $C$, the corresponding morphism (= functor)

$$f \downarrow D : (X \downarrow D) \rightarrow (X \downarrow D)$$

is defined on objects by

$$(X \rightarrow Y) \mapsto (X \xrightarrow{f} X' \rightarrow Y).$$

2.2. Proposition. For any topological space $X$, the comma category $X \downarrow \mathbf{PL}$ is a small, filtering category.

Proof (from [23]). The category $X \downarrow \mathbf{PL}$ is small because $\mathbf{PL}$ is small. For any two objects $X \rightarrow K'$ and $X \rightarrow K''$ of $X \downarrow \mathbf{PL}$, the object $X \rightarrow K' \times K''$ maps to $X \rightarrow K'$ and $X \rightarrow K''$. Given any two morphisms $(X \rightarrow K') \Rightarrow (X \rightarrow K'')$ in $X \downarrow \mathbf{PL}$, let $K$ be equalizer in $\mathbf{PL}$ (and hence in $\mathbf{Top}$) of the induced maps $K' \Rightarrow K''$. There is an induced map $X \rightarrow K$, and this yields an equalizer diagram

$$(X \rightarrow K) \Rightarrow (X \rightarrow K') \Rightarrow (X \rightarrow K'')$$

in $X \downarrow \mathbf{PL}$. Thus $X \downarrow \mathbf{PL}$ is filtering.

We may therefore define a (covariant) geometric strong shape functor $s \text{-sh} : \mathbf{Top} \rightarrow \text{pro-PL}$ by the formula

$$(2.3) \quad s \text{-sh}(X) = \{(X \downarrow \mathbf{PL}) \rightarrow \mathbf{PL}\}$$

on objects. Morphisms are induced by the contravariant functor $(- \downarrow \mathbf{PL})$ on indexing categories. The inverse system $s \text{-sh}(X)$ is described explicitly as follows. Objects of $s \text{-sh}(X)$ are targets of maps $f : X \rightarrow P$, $P \in \mathbf{PL}$ indexed by the set of such maps $\{f\}$. Bonding maps of $s \text{-sh}(X)$ correspond to strictly commutative triangles

$$(2.4) \quad \begin{array}{c}
X \\
\downarrow^f \\
P \xrightarrow{\phi} P' \nend{array}$$

with $\phi$ a PL map. If $f : X \rightarrow Y$ is a map in $\mathbf{Top}$, the induced map $s \text{-sh}(X) \rightarrow s \text{-sh}(Y)$ is defined as follows. Associate to each map $g : Y \rightarrow P$ in the indexing category $Y \downarrow \mathbf{PL}$ for $s \text{-sh}(Y)$ the composite map $gf : X \rightarrow P$ in the indexing category $X \downarrow \mathbf{PL}$ of $s \text{-sh}(X)$, and the identity map

$$(s \text{-sh}(X) \ni P) \rightarrow P \in s \text{-sh}(Y).$$
This defines a map $s\text{-sh}(f) : s\text{-sh}(X) \to s\text{-sh}(Y)$ in pro-$\text{PL}$. We shall call $s\text{-sh}(X)$ the strong shape of $X$.

2.5. Remarks. If $X$ is a polyhedron $P$, there is a natural map $s\text{-sh}(P) \to P$ in pro-$\text{PL}$ (here $P$ denotes also the constant inverse system $\{P\}$). $P$ is clearly cofinal in $s\text{-sh}(P)$ in pro-$\text{Top}$. Note however, that $P$ is not cofinal in $s\text{-sh}(P)$ in pro-$\text{PL}$; such a result only holds up to contiguity.

We shall need an appropriate homotopy theory $\text{Ho}(\text{pro-PL})$ to describe strong shape theory. The inclusion $\text{PL} \to \text{Top}$ yields an inclusion pro-$\text{PL} \to \text{pro-Top}$. We shall describe the homotopy theory of pro-$\text{PL}$ by defining a closed model structure (D.G. Quillen [36]) on pro-$\text{Top}$.

Quillen introduced closed model categories as an abstraction and generalization of the categories $\text{Top}$ (equipped with weak homotopy equivalences, Serre fibrations, and the corresponding cofibrations) and $\text{SS}$ (simplicial sets, equipped with weak homotopy equivalences, Kan fibrations, and inclusions as the corresponding cofibrations). A closed model category $C$ consists of a category $C$ together with three classes of morphisms of $C$, called weak equivalences, fibrations, and cofibrations. These classes are required to satisfy (and be interrelated by) axioms motivated by the homotopy structures of $\text{Top}$ and $\text{SS}$ defined above. In particular, the lifting axiom states that for any commutative solid-arrow diagram

\[
\begin{array}{ccc}
A & \longrightarrow & Y \\
\downarrow i & & \downarrow \rho \\
X & \longrightarrow & B
\end{array}
\]

(2.6)

in which $i$ is a cofibration, $\rho$ is a fibration, and $i$ or $\rho$ is also a weak equivalence, there exists a filler $f$. The factorization axiom states that any morphism $f : X \to Y$ in $C$ can be factored as

\[
X \longrightarrow i \longrightarrow Z \longrightarrow p \longrightarrow Y,
\]

(2.7)

with $i$ a cofibration, $p$ a fibration, and in addition either map may be required to be a weak equivalence. The above model structure on $\text{Top}$ (developed by Quillen in [36, Section II.3]) is called the singular model structure on $\text{Top}$. Roughly, a map $A \to X$ in $\text{Top}$ is a cofibration in this singular model structure if $X$ is obtained from $A$ by adding cells. A. Strøm [41] proved that the "usual" homotopy structure on $\text{Top}$ (homotopy equivalences, Hurewicz fibrations, and the corresponding cofibrations) is a closed model category. We call this the usual closed model structure on $\text{Top}$.

2.8. Remarks. We shall use the singular closed model structure on $\text{Top}$ because we need the following result [36, Section II.3]. Any map $f : X \to Y$ in $\text{Top}$, with $X$
compact can be factored as in (2.7) with Z the colimit of a diagram of cofibrations and homotopy equivalences of compact spaces 

\[ \{X \to Z_a\}. \]

In [16, Section 3] D.A. Edwards and the second-named author showed that the usual closed model structure on Top induced a closed model structure (called the usual closed model structure) on pro-Top in which basic cofibrations and basic weak equivalences are defined levelwise. Fibrations are not defined levelwise.

The homotopy category [36, Section I.5]

\[(2.9) \quad \text{Ho}(\text{pro-Top}) = \{\text{levelwise homotopy equivalences}\}^{-1}(\text{pro-Top})\]

is obtained by formally inverting basic weak equivalences. There is an analogous singular structure on pro-Top. This also uses [24] and the fact that a map \( f \) in pro-Top is a levelwise weak homotopy equivalence if and only if its image \( \text{Sing}(f) \) in pro-SS is a levelwise weak equivalence. Here \( \text{Sing} \) denotes the singular functor and \( R \) denotes J. Milnor’s [33] geometric realization functor. As above, there is a homotopy category

\[(2.10) \quad \text{Ho}_{\text{sing}}(\text{pro-Top}) = \{\text{levelwise weak homotopy equivalences}\}^{-1}(\text{pro-Top}).\]

Consider the natural functors

\[(2.11) \quad \text{pro-PL} \xrightarrow{\pi} \text{Ho}(\text{pro-Top}) \xrightarrow{\pi} \text{Ho}_{\text{sing}}(\text{pro-Top}).\]

2.12. Proposition. The functor \( \pi \) induces an equivalence of the full subcategories generated by the images of \( \pi' \) and \( \pi'' \).

Proof. We need the following property of closed model categories [36, Section I.1.16]. If \( X \) is cofibrant and \( Y \) is fibrant in a closed model category \( C \), then there is a natural isomorphism

\[ [X, Y] \to \text{Ho}(C)(X, Y) \]

from homotopy classes of maps from \( X \) to \( Y \) to \( \text{Ho}(C)(X, Y) \). Let \( X, Y \in \text{pro-PL} \). Factor the map \( Y \to \ast \) in pro-Top as

\[(2.13) \quad Y \xrightarrow{i} Y' \xrightarrow{p} \ast \]

with \( i \) a cofibration and weak equivalence, and \( p \) a fibration in the usual closed model structure [16, Section 3], see above, on pro-Top. Also note that \( X \) is cofibrant (the natural map \( \phi \to X \) is a cofibration) and \( Y' \) is fibrant in both structures. This yields a commutative diagram
We now define strong shape categories and strong shape equivalences.

2.15. Definitions and notation. Let $C$ denote any of the following categories: Top, CH, or CM. The associated strong shape category $s\text{-}sh(C)$ has the same objects as $C$. Morphisms in $s\text{-}sh(C)$ are pulled back from $\text{Ho}_{\text{pro-Top}}$:

$$s\text{-}sh(C)(X, Y) = \text{Ho}_{\text{pro-Top}}(s\text{-}sh(X), s\text{-}sh(Y)).$$

The associated shape category $\text{sh}(C)$ is defined similarly. It has the same objects as $C$, and morphisms are pulled back from $\text{pro-Ho}_{\text{Top}}$ (or, equivalently because $s\text{-}sh$ takes values in pro-PL, pro-$\text{Ho}(\text{Top})$):

$$\text{sh}(C)(X, Y) = \text{pro-Ho}_{\text{Top}}(\text{sh}(X), \text{sh}(Y)).$$

Here $\text{sh}(X)$ denotes the image of $s\text{-}sh(X)$ under the canonical functor $\text{Ho}_{\text{sing}}(\text{pro-Top}) \to \text{pro-Ho}_{\text{sing}}(\text{Top})$.

The algebraic topology of $\text{pro-Ho}_{\text{sing}}(\text{Top})$ was extensively studied by M. Artin and B. Mazur [2]. The relationship between $\text{Ho}_{\text{sing}}(\text{pro-Top})$ and $\text{pro-Ho}_{\text{sing}}(\text{Top})$ is developed in J. Grossman [22] and [16, Section 5].

A map $s\text{-}sh(X) \to s\text{-}sh(Y)$ in $s\text{-}sh(C)$ is called a strong shape equivalence, denoted s.s.e., if it is invertible in $s\text{-}sh(C)$. We shall also call a (continuous) map $f: X \to Y$ in $C$ a strong shape equivalence if its image $[(s\text{-}sh(f))]$ is invertible in $s\text{-}sh(C)$. Shape equivalences, denoted s.c.e., are defined analogously using $\text{sh}$ and $\text{sh}(C)$.

2.16. Remarks. A map $X \to Y$ in CH is a shape equivalence if and only if the induced map $[Y, P] \to [X, P]$ is a bijection for all polyhedra $P$. For each object $X$ of CH, our $\text{sh}(X)$ is cofinal in the Mardešić–Segal [31] shape of $X$ (see [23]), hence our definition agrees with the usual shape category.

Unfortunately, $s\text{-}sh(\text{Top})$ does not behave well with respect to homotopy, see the first-named author and J. Siegel [5] and 3.12–3.14 below.

2.17. Proposition. The following approaches to strong shape theory are equivalent where defined:

(a) The Vietoris functor approach (T. Porter [35]) of D.A. Edwards and the second-named author [16, Sections 3, 8], defined on CH,
(b) The above geometric approach of [23], defined on CH,
(c) Definition 2.15 above, of s-sh(CH),
(d) J. Dydak and J. Segal’s natural transformation approach for CM [13], and
(e) Y. Kodama and Y. Ono’s fine shape theory on CH [28].

Proof. For (a) ~ (b), see [23]. Proposition 2.12 implies (b) ~ (c). Dydak and Segal prove (a) ~ (d); Kodama and Ono prove (a) ~ (e).

3. Strong shape theory

We establish the following main properties of the strong shape functor

\[ s-sh : \text{CH} \rightarrow \text{pro-PL} \hookrightarrow \text{pro-Top}. \]

(i) (Corollary 3.4) \( \lim_{\text{pro-PL}} \circ s-sh = \text{id}_{\text{CH}}. \)
(ii) Exactness (Proposition 3.5): for any pair \((X,A)\) of compact Hausdorff spaces, the sequence

\[ s-sh(A) \rightarrow s-sh(X) \rightarrow s-sh(X/A) \]

is a cofibration sequence in \( \text{pro-Top} \).

(iii) Continuity (Proposition 3.8): s-sh preserves inverse limits up to equivalence in \( \text{Ho}_{\text{sing}}(\text{pro-Top}) \).

(iv) Homotopy invariance (Proposition 3.10): s-sh maps homotopy equivalences to equivalences in \( \text{Ho}_{\text{sing}}(\text{pro-Top}) \).

These results were announced in [23]. We include a proof of 3.5 from [23] for completeness. The other proofs are new. We would like to study s-sh on all of \( \text{Top} \). However, see 3.12–3.14 below, s-sh need not preserve weak equivalences outside \( \text{CH} \).

For any topological space \( X \) there is a natural map

\[ X \rightarrow \{ (X \downarrow \text{PL}) \rightarrow \text{PL} \} = s-sh(X) \]

in \( \text{pro-Top} \). This induces a natural map

(3.1) \[ X \rightarrow \lim \circ s-sh(X). \]

3.2. Proposition. For any topological space \( X \), \( \lim \circ s-sh(X) \equiv \beta X \), the Stone-Čech compactification of \( X \), and the map (3.1) is the canonical map \( X \rightarrow \beta X \).

Proof. This is a restatement of a classical result of Stone and Čech (see, e.g. R.C. Walker [44, p. 25]) in modern language.

3.3. Corollary. For any compact Hausdorff space \( X \), \( \lim \circ s-sh(X) \equiv X \).

The inverse limit of any object of \( \text{pro-PL} \) is a compact Hausdorff space. This yields the following.
3.4. Corollary. $\lim_{\text{pro-PL}} s\text{-sh} = \text{id}_{\text{CH}}$.

3.5. Proposition (exactness). For any compact Hausdorff pair $(X, A)$, the sequence

$$s\text{-sh}(A) \to s\text{-sh}(X) \to s\text{-sh}(X/A)$$

is a cofibration sequence in the singular model structure on pro-$\text{Top}$.

**Proof.** Consider the sequence $A \to X \to X/A$. Let $C$ be the category whose objects are cofibration sequences in $\text{PL}$ under this sequence, that is, commutative diagrams

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow & & \downarrow \\
L & \longrightarrow & K
\end{array}
\quad
\begin{array}{ccc}
X & \longrightarrow & X/A \\
\downarrow & & \downarrow \\
K & \longrightarrow & K/L
\end{array}
$$

with $(K, L)$ a pair in $\text{PL}$. Morphisms of $C$ are defined by generalizing the comma category construction of $s\text{-sh}$. It suffices to show that the restriction functors $C \to A \uplus \text{PL}$, $C \to (X \uplus \text{PL})$ and $C \to (X/A \uplus \text{PL})$ are cofinal. In the first case, the Tietze extension theorem provides an extension $X \to CL$, the cone on $L$, for any map $A \to L$. This observation is due to G. Kozlowski. To check that the restriction functor $C \to A \uplus \text{PL}$ satisfies the equalizer condition, consider any solid-arrow commutative diagram

$$
\begin{array}{ccc}
L'' & \longrightarrow & K \\
\downarrow & & \downarrow \\
L & \longrightarrow & K
\end{array}
\quad
\begin{array}{ccc}
K & \longrightarrow & K/L
\end{array}
$$

Let $L$ be the equalizer of $L \Rightarrow L'$ in $\text{PL}$. Fill in the dotted arrows; this yields a cofibration sequence $L'' \to K \to K/L''$ in $\text{PL}$. Clearly this construction can be performed under $A \to X \to X/A$, and the restriction of the map $(L'' \to K \to K/L'') \to (L \to K \to K/L)$ is the required equalizer. Thus $C \to (A \uplus \text{PL})$ is a cofinal functor.

It is even easier to check that the restrictions $C \to (X \uplus \text{PL})$ and $C \to (X/A \uplus \text{PL})$ are cofinal. To define the required sequences under $A \to X \to X/A$, in the first case, given a map $X \to K$, let $L = K$. In the second case, given a map $X/A \to P$, let $K = P$, and let $L$ be the image of $[A]$ in $P$. The equalizer conditions follows easily.
The following general result will yield a continuity theorem for s-sh.

3.6. Proposition. Let \( \{P_a\}_{a \in A} \) be an object of \( \text{pro-PL} \). Let \( X = \lim \{P_a\} \) in \( \text{CH} \). Then there is a natural map \( X \to \lim \{P_a\} \) in \( \text{pro-Top} \) which becomes an equivalence in \( \text{Ho}_{\text{sing}}(\text{pro-Top}) \).

The proof is deferred until 3.15.

In order to state the continuity theorem we need the following notation. Let \( \{X_a\}_{a \in A} \) be an inverse system over \( \text{CH} \). Applying s-sh levelwise yields an inverse system

\[
\{\text{s-sh}(X_a)\}_{a \in A}
\]

over \( \text{pro-PL} \), that is, an object of \( \text{pro(pro-PL)} \). In the language of comma categories it is just \( \{(X_a)_{/\text{PL}}\to \text{PL}\} \). However \( \{\text{s-sh}(X_a)\}_{a \in A} \) is easily interpreted as an object, in fact its own inverse limit (see [2, Appendix]), in \( \text{pro-PL} \).

More explicitly, let \( B \) be the category whose objects are continuous functions

\[
X_a \to P, \quad a \in A, \quad P \in \text{PL},
\]

and whose morphisms are commutative diagrams

\[
\begin{array}{ccc}
X_{a_1} & \xrightarrow{\text{bond}} & X_{a_2} \\
\downarrow & & \downarrow \\
P_1 & \xrightarrow{\phi} & P_2,
\end{array}
\]

where "bond" denotes a bonding map of \( X_a \), and \( \phi \in \text{PL} \). It is easy to show that \( B \) is filtering by imitating the proof of Proposition 2.2. Then

\[
(P_\beta)_{\beta \in B} = \{\text{codom} : B \to \text{PL}\} \\
\equiv \{\text{s-sh}(X_a)\}_{a \in A}.
\]

3.8. Proposition (continuity). Let \( A \) be a small filtering category, and let \( \{X_a\}_{a \in A} \) be an inverse system over \( \text{CH} \) with limit \( X \). Then there is a natural map in \( \text{pro-Top} \)

\[
\text{s-sh}(X) \to \{P_\beta\}_{\beta \in B} (\equiv \{\text{s-sh}(X_a)\}_{a \in A})
\]

which becomes an equivalence in \( \text{Ho}_{\text{sing}}(\text{pro-Top}) \).

Proof. The natural maps \( X \to \{X_a\} \) and \( X_a \to \text{s-sh}(X_a) \) in \( \text{pro-Top} \) induce a natural map

\[
X \to \{X_a\} \to \{\text{s-sh}(X_a)\} \equiv \{P_\beta\}.
\]

This map induces an isomorphism \( X \to \lim \{P_\beta\} \) because \( X \equiv \lim \{X_a\} \) and \( X_a \equiv \lim \circ \text{s-sh}(X_a) \). Now apply Proposition 3.6 to \( \{P_\beta\} \).
3.9. Remarks. Consider the functors

\[ \text{pro-PL} \xrightarrow{\text{lim}} \text{s-sh} \xrightarrow{\text{CH}} \]

Proposition 3.2 and the first half of Proposition 3.6 imply that s-sh is coadjoint to lim. The isomorphism \( \text{id}_{\text{CH}} \xrightarrow{\text{lim}} \text{s-sh} \) and the natural weak equivalence \( \text{s-sh} \circ \text{lim} \xrightarrow{\text{id}_{\text{pro-PL}}} \) suggest a possible analogy with the geometric realization and singular functors (J. Milnor [33], see also J.P. May [32]). Unfortunately, this fails because \( \text{lim} \) does not preserve weak equivalences. Homotopy limits (A.K. Bousfield and D.M. Kan [3], R. Vogt [43], D.A. Edwards and the second named author [16, Section 4]) were introduced to rectify this kind of difficulty.

3.10. Proposition (homotopy invariance). Let \( f, g : X \rightarrow Y \) be homotopic maps in \( \text{CH} \). Then \( \text{s-sh}(f) \) and \( \text{s-sh}(g) \) become equivalent in \( \text{Ho}_{\text{sing}}(\text{pro-Top}) \).

Proof. We give a simpler and more direct proof than [23]. Let \( H : X \times I \rightarrow Y \) be a homotopy from \( f \) to \( g \). Form the commutative diagram in \( \text{pro-Top} \).

We will show that \( \phi_0 \) and \( \phi_1 \) are equivalences in \( \text{Ho}_{\text{sing}}(\text{pro-Top}) \). Let \( \{ P_\alpha \} = \text{s-sh}(X) \). We have a commutative diagram in which \( \phi_0 \) is a "restriction"

\[ \text{s-sh}(X \times 0) \xrightarrow{\phi_0} \text{s-sh}(X \times I) \]

Similarly for \( \phi_1 \). Clearly, \( \phi_0 \) is a levelwise homotopy equivalence. By the continuity theorem (3.8), \( \phi_0 \) is an equivalence in \( \text{Ho}_{\text{sing}}(\text{pro-Top}) \). The conclusion follows.

3.11. Corollary. The functor \( \text{s-sh} : \text{CH} \rightarrow \text{pro-Top} \) induces a functor, also denoted \( \text{s-sh} : \text{Ho}(\text{CH}) \rightarrow \text{Ho}_{\text{sing}}(\text{pro-Top}) \).

Before giving the proof of Proposition 3.6 we consider the homotopy properties of \( \text{s-sh} \) on \( \text{Top} \). The first-named author and J. Siegel proved the following.
3.12. Proposition [5, proof of Theorem 1.14]. The Stone–Čech compactification \( \beta \) induces natural isomorphisms in pro-\( \text{Top} \)

\[
\beta : s\text{-sh}(X) \to s\text{-sh}(\beta X).
\]

3.13. Corollary. The functor \( s\text{-sh} : \text{Top} \to \text{pro-Top} \) does not induce a functor on homotopy categories.

Proof. \( \beta R \) is not contractible (see [5]).

On the other hand, uniformly homotopic maps of spaces \( f, g : X \to Y \) induce homotopic maps on the Stone–Čech compactifications \( \beta f, \beta g : \beta X \to \beta Y \). This yields the following positive result.

3.14. Proposition. The functor \( s\text{-sh} \) does induce a functor from the uniform homotopy category of spaces to \( \text{Ho}_{\text{sing}}(\text{pro-Top}) \).

3.15. Proof of Proposition 3.6. An indirect proof using Porter's Vietoris functor [35] was given in [23] and [16, Section 8]. Here is a geometric proof. Roughly, we shall show that \( \{P_a\} \) is cofinal in \( s\text{-sh}(P) \) up to contiguity, hence up to coherent homotopy, and argue that this yields an equivalence in \( \text{Ho}_{\text{sing}}(\text{pro-Top}) \). To illustrate the relationship between contiguity and higher homotopies, consider three pairwise contiguous maps of polyhedra \( f, g, h : K \to L \). There are natural linear homotopies \( H_1 : f \sim g, H_2 : g \sim h, \) and \( H_3 : f \sim h \). Also, the composed homotopy \( H_1 \ast H_2 \) is linearly homotopic to \( H_3 \), relative to the endpoints. Because of this higher homotopy, \( H_1, H_2, \) and \( H_3 \) are called coherent. See R. Vogt [43].

3.16. Preliminaries. Since \( X = \lim_a \{P_a\}_{a \in A} \), there is a natural functor

\[
J : A \to (X \downarrow \text{PL}), \quad \text{with}
\]

\[
J(\alpha) = (X \to P_\alpha),
\]

and

\[
J(\alpha \to \beta) = \begin{array}{c}
X \\
\rightarrow \\
\downarrow \\
X
\end{array}
\]

\[
P_\alpha \\
\rightarrow \\
\downarrow \\
P_\beta.
\]

Because \( J \) is injective on objects and morphisms, we may identify \( A \) with its image \( J(A) \), a non-full subcategory of \( X \downarrow \text{PL} \). Also, \( J \) induces the required natural map (restriction)

\[
f : s\text{-sh}(X) = \{(X \downarrow \text{PL}) \to \text{PL}\} \to \{P_a\}_{a \in A}
\]

in pro-\( \text{PL} \). We shall show that \( f \) becomes an equivalence in \( \text{Ho}_{\text{sing}}(\text{pro-Top}) \) by (roughly) showing that \( \{P_a\} \) is cofinal in \( s\text{-sh}(X) \) up to coherent homotopy.
Apply a suitable functorial version of the Mardešić construction [16, 2.1.6] to $s$-$\text{sh}(X)$ and the subsystem $\{P_\alpha\}_{\alpha \in A}$. This yields a commutative diagram

$$
\begin{array}{c}
\{P_\beta\}_{\beta \in B} \\
\uparrow \\
\text{s-sh}(X)
\end{array} \xrightarrow{g} \begin{array}{c}
\{P_\gamma\}_{\gamma \in C} \\
\downarrow \\
\{P_\alpha\}_{\alpha \in A}
\end{array}
$$

(3.17)

in which $B$ and $C$ are cofinite strongly directed sets, $C$ is a subset of $B$, $g$ is the corresponding restriction functor, and the vertical arrows are cofinal functors, hence isomorphisms in pro-PL. Because $B$ is a cofinite (each element has finitely many predecessors) strongly directed set (for any pair of objects $\beta$ and $\beta'$ there is at most one morphism $\beta \to \beta'$, denoted $\beta' \leq \beta$, and $\beta = \beta'$ whenever $\beta \leq \beta'$ and $\beta' \leq \beta$) we may do induction over $B$. Also, the equalizer condition for cofinality holds trivially.

Now inductively triangulate the polyhedra $P_\beta$, $\beta \in B$, so that under any bonding map $P_\beta \to P_{\beta'}$, the image of each simplex of $P_\beta$ lies within a single simplex of $P_{\beta'}$.

3.18. Factorization up to contiguity. Consider any map $X \to P_{\beta_1}$ "in" $\{P_\beta\}$. We seek a factorization

$$
\begin{array}{c}
X \\
\downarrow \\
P_{\beta_1}
\end{array} \xrightarrow{\theta} \begin{array}{c}
P_{\gamma_1} \\
\downarrow \\
P_{\beta_1}
\end{array}
$$

(3.19)

up to contiguity for some $\gamma_1$ in $C$. There are two cases.

First, let $D \subseteq B$ be the full subcategory with

$$\text{obj } D = \{ \beta \in B \mid \text{for some } \gamma \text{ in } C, \beta \leq \gamma \}.$$ 

For such $\beta$, any bonding maps $P_\gamma \to P_\beta$ provides such a factorization. Because $D$ is also cofinite and strongly directed, and $C \subseteq D$, we may use the identity factorization for all $\gamma$ in $C$, and otherwise inductively choose $\gamma$'s so that the function

$$\theta : \text{obj } D \to \text{obj } C$$

is non-decreasing.

Otherwise, consider the following construction. Let $U_{\beta_1}$ be the open covering of $P_{\beta_1}$ by open stars of vertices, and let $U$ be the pullback of $U_{\beta_1}$ to $X$. Because $U$ is finite, $X = \lim \{P_\gamma\}$, and $C = \{ \gamma \}$ is filtering, $U$ is the pullback to $C$ of an open covering $V$ of some $P_\gamma$. Let $\tilde{X}$ denote the image of $X$ in $P_\gamma$. There is an isomorphism of Čech nerves

$$N(U) (\equiv P_\beta) \xrightarrow{\theta} N(V | \tilde{X})$$
By Kuratowski [29, p. 122], for some neighborhood $G$ of $\bar{X}$ in $P_y$ and refinement $V'$ of $V|G$

\[ N(V'|\bar{X}) \leftarrow \xrightarrow{\alpha} N(V'|X) \xrightarrow{\alpha} N(V'|G). \]

We may now choose $\gamma_1 > \gamma'$ in $C$ so that the image $P_{\gamma_1}$ of $P_{\gamma}$ in $P_y$ lies in $G$, using compactness of the polyhedra $\{P_{\gamma}\}$. This yields maps

\[ X \xrightarrow{\alpha} P_{\gamma_1} \xrightarrow{\alpha} N(V'|P_{\gamma}) \xrightarrow{\alpha} N(V'|G) \xrightarrow{\alpha} P_{\beta}, \]

which yield the required factorization (3.19).

3.21. Induction over $B \setminus D$. Perform the above construction inductively over all $\beta$ in \text{obj} $B \setminus \text{obj} D$, always choosing $\gamma$'s so that the function \text{obj} $B \rightarrow \text{obj} C$, extending $\theta$ above, is non-decreasing. This is possible because $B$ is cofinite and $C$ is filtering. Note that our factorization for $\beta$ in \text{obj} $D$ is a special case of our factorization for $\beta$ in \text{obj} $D \setminus \text{obj} B$.

3.22. Claim. Each of the diagrams

\[
\begin{array}{c}
X \\
\downarrow \\
P_{\gamma_1} \xrightarrow{\alpha} P_{\gamma_2} \\
\downarrow \\
P_{\beta_1} \xrightarrow{\alpha} P_{\beta_2}
\end{array}
\]

commutes up to contiguity.

Verification. Let $U_{\beta}$ be the open covering of $P_{\beta_1}$ by open stars of vertices, similarly for $U_{\beta_2}$ and $P_{\beta_2}$. Then the pullback of $U_{\beta_2}$ to $P_{\beta_1}$ refines $U_{\beta}$ by construction (see 3.21 above). Claim 3.22 now follows from construction (3.20).

3.23. We have therefore constructed a pro-map up to contiguity $\{P_{\gamma}\} \rightarrow \{P_{\beta}\}$, which will yield the required isomorphism. Because any two contiguities are contiguous, etc., all of the associated homotopies in the map $\{P_{\gamma}\} \rightarrow \{P_{\beta}\}$ are coherent.

Now replace $\{P_{\beta}\}$ by an equivalent (in $\text{Ho}_{\text{sing(pro-Top)}}$) cofibrant-fibrant object $\{P'_{\beta}\}$. We recall from [16, Section 3.1] that an inverse system $\{X_\alpha\}$, indexed by a cofinite strongly directed set is cofibrant if each $X_\alpha$ is cofibrant; and is fibrant if each map $X_\alpha \rightarrow \lim_{\beta < \alpha} \{X_\beta\}$ is a (Serre) fibration. (We do not need here the definition that retracts of cofibrant or fibrant objects are respectively cofibrant or fibrant.) The $\{P'_{\beta}\}$ are constructed inductively, using [36, Section II.3]. Let $\{P'_{\gamma}\} \subset \{P'_{\beta}\}$ be the subsystem corresponding to $\{P_{\gamma}\} \subset \{P_{\beta}\}$. Then $\{P'_{\gamma}\}$ is also
cofibrant and fibrant; the latter because each predecessor in $P'_\beta$ of any $P'_\gamma$ lies in 
{$P'_\gamma$}.

Consider the resulting map $h: \{P'_\gamma\} \rightarrow \{P'_\beta\}$. We can use the coherence data to
inductively rigidify this map (using the lifting techniques of [16, Section 3, especially
Proposition 3.3.9]) to obtain a commutative diagram in pro-$\text{Top}$

\[
\begin{array}{c}
\{P'_\gamma\} \\
\downarrow \\
\{P'_\beta\} \\
\downarrow \\
\{P'_\gamma\}.
\end{array}
\]

\[(3.24)\]

We illustrate this process by describing three cases explicitly. Coherence is used only
in the third case. Here is a preliminary definition. Let $\beta' < \beta$ in $B$. Call $\beta'$ a maximal
predecessor if $\beta'' < \beta'$ whenever $\beta'' < \beta$ in $B$.

First suppose that $\beta$ has a unique maximal predecessor $\beta_1$, and that we have
already rigidified (the restriction of) $h$ into $P'_\beta$. Consider the resulting homotopy
commutative diagram

\[
P'_\gamma \xrightarrow{h_\beta} P'_\beta
\]

\[
h_{\beta_1}, \text{ bond}
\]

\[
P'_\beta
\]

\[(3.25)\]

The required homotopy $H_1: P'_\gamma \times I \rightarrow P'_\beta$ is obtained from the "factorization up to
contiguity". Because \{$P'_\beta$\} is fibrant, the bonding map $P'_\beta \rightarrow P'_\beta_1$ is a Serre fibration.
Because $P'_\gamma$ is also cofibrant, we may lift $H_1$ to $P'_\gamma$, and thus define a map $h'_\beta: P'_\gamma \rightarrow
P'_\beta$ which makes diagram (3.25) strictly commute. This is the required rigidification.

Next, suppose that $\beta$ has two maximal predecessors $\beta_1$ and $\beta_2$. If $\beta_1$ and $\beta_2$ have
no common predecessors, we may regidify $h$ into $P'_\beta$ by following the above
construction. Our assumptions imply that

$$P' = \lim \{P'_\beta| b' < \beta\} = P'_\beta_1 \times P'_\beta_2,$$

and yield a homotopy commutative diagram

\[
P'_\gamma \xrightarrow{h_\beta} P'_\beta
\]

\[
P'_\beta_1
\]

\[
P'_\beta_2
\]

\[
(3.26)
\]
The required homotopies

\[ H_i : P'_\gamma \times I \to P'_\beta \]

are obtained from the "factorization up to contiguity". The homotopies \( H_i \) yield a homotopy

\[ H = (H_1, H_2) : P'_\gamma \times I \to P = P'_\beta \times P'_\beta. \]

We rigidify \( h_\beta \) by lifting \( H \) to \( P'_\beta \) as in (3.25) above.

Finally, consider the general case in which \( \beta \) has two maximal predecessors \( \beta_1 \) and \( \beta_2 \). Let

\[ Q' = \lim\{ P'_\beta \mid \beta' < \beta_1 \text{ and } \beta' < \beta_2 \} \]

be the limit of the common predecessors of \( P'_\beta \) and \( P'_\beta \). Because \( \{ P'_\beta \} \) is fibrant, one can check that the induced maps

\[ P'_\beta \to Q' \]

are Serre fibrations. Now consider the homotopy commutative diagram

\[
\begin{array}{ccc}
P'_\gamma & \xrightarrow{h_\beta} & P'_\beta \\
\downarrow{h_{\beta_1}} & & \downarrow{h_{\beta_2}} \\
Q' & \xleftarrow{h_\beta} & P' \\
\end{array}
\]

in which \( P' \) is a pullback. The composite mappings \( P'_\gamma \to P'_\beta \to Q' \) are equal by construction. The other required homotopies \( H_i : P'_\gamma \times I \to P'_\beta \) are obtained from the "factorization up to contiguity." Serre fibrations are labeled with "fib"; \( P'_\beta \to P' \) is a Serre fibration because \( \{ P'_\beta \} \) is fibrant. By construction the composed homotopies

\[ P'_\gamma \times I \xrightarrow{H_1} \xrightarrow{H_2} Q' \]

are contiguous, hence homotopic relative to their endpoints. Because the map \( P' \to P'_\beta \) is a Serre fibration and \( P'_\gamma \) is cofibrant we may deform \( H_1 \), relative to its endpoints, to a homotopy \( H'_1 \) which makes diagram (3.29) commute. This yields a homotopy suitable homotopy

\[ H : P'_\gamma \times I \to P' \]

which we lift to \( P'_\beta \) to obtain the required rigidification \( h'_\beta : P'_\gamma \to P'_\beta \).

Now define an enlarged inverse system \( \{ P'_\beta \}_{\beta \in B} \), by including all maps in
diagram (3.23) as bonding maps. Because $C \subset B$, obj $B = \text{obj} B'$. By diagram (3.23), \( \{P'_\beta\}_{\beta \in B} \) is cofinal in \( \{P'_\beta\}_{\beta \in B'} \). Hence so is \( \{P'_\beta\}_{\beta \in B} \); for the equalizer condition any \( P'_\beta \) has a predecessor \( P'_\gamma \), thus cofinality of \( \{P'_\gamma\} \) (\( \subset \{P'_\beta\} \)) yields the required weak equalizer. This yields the commutative diagram

\[
\begin{array}{ccc}
\{P'_\beta\}_{\beta \in B} & \xrightarrow{s} & \{P'_\gamma\} \\
\downarrow & & \downarrow \\
\{P'_\beta\}_{\beta \in B} & \xleftarrow{s} & \{P'_\gamma\}
\end{array}
\]

and thus the required equivalence \( \{P'_\beta\}_{\beta \in B} = \{P'_\gamma\} \). This completes the proof of Proposition 3.6.

4. The realization theorem

Let $C$ be a category and let $\Sigma$ be a class of morphisms in $C$. The localization of $C$ with respect to $\Sigma$, denoted $\Sigma^{-1}C$, is the category obtained by formally inverting morphisms in $\Sigma$. Any functor $C \to D$, which sends morphisms in $\Sigma$ to isomorphisms in $D$ factors uniquely through $\Sigma^{-1}C$. We shall prove the following.

4.1. Theorem. The strong shape category of compact metric spaces, $s\text{-sh}(CM)$, is the localization of the homotopy category $\text{Ho}(CM)$ with respect to strong shape equivalences.

Because every homotopy equivalence is a strong shape equivalence, and the homotopy category $\text{Ho}(CM)$ is a localization, the natural functor $CM \to s\text{-sh}(CM)$ factors through $\text{Ho}(CM)$. Let \( \{\text{s.s.e.}\}^{-1}\text{Ho}(CM) \) denote the localization of $\text{Ho}(CM)$ with respect to strong shape equivalences. This yields a commutative diagram of categories and functors

\[
\begin{array}{ccc}
CM & \xrightarrow{\text{Ho}(CM)} & s\text{-sh}(CM) \\
\downarrow & & \downarrow F \\
\{\text{s.s.e.}\}^{-1}\text{Ho}(CM)
\end{array}
\]

We shall prove that $F$ is an isomorphism. Because $F$ is an isomorphism on objects by definition, we need only show that $F$ is full (Proposition 4.2) and faithful (Proposition 4.8).
4.2. Proposition. Let $X, Y \in \text{CM}$, and let $f : \text{s-sh}(X) \to \text{s-sh}(Y)$ be a map in $\text{Ho}_{\text{sing}}(\text{pro-Top})$. Then there is a diagram

\[
\begin{array}{cc}
X & \rightarrow & Y \\
\downarrow{\phi} & & \downarrow{\gamma} \\
Y' & \rightarrow & \text{in CM with } \gamma \text{ an inclusion and strong shape equivalence which realizes } f, \text{ that is,}
\end{array}
\]

\[\{f\} = [\text{s-sh}(\gamma)]^{-1}[\text{s-sh}(\phi)]\]

in $\text{Ho}_{\text{sing}}(\text{pro-Top})$.

Proof. We first represent $f$ as a map of towers (countable inverse systems) in $\text{Ho}_{\text{sing}}(\text{pro-Top})$. Because $X$ and $Y$ are compact metric spaces, we may choose countable subsystems $\{X_m\} \subseteq \text{s-sh}(X)$ and $\{Y_n\} \subseteq \text{s-sh}(Y)$ with limits $X$ and $Y$, respectively. The continuity theorem (Proposition 3.8) yields the vertical equivalences in the diagram

\[
\begin{array}{ccc}
\text{s-sh}(X) & \longrightarrow & \text{s-sh}(Y) \\
\downarrow & & \downarrow \\
\{X_m\} & \longrightarrow & \{Y_n\}
\end{array}
\]

in $\text{pro-Top}$. We define a filler $f'$ in $\text{Ho}_{\text{sing}}(\text{pro-Top})$ by inverting the equivalence $\text{s-sh}(X) \to \{X_m\}$.

We shall now use the closed model structure on $\text{Ho}_{\text{sing}}(\text{pro-Top})$ [16, Section 3] to define a diagram of towers in $\text{pro-Top}$ analogous to (4.3). The central idea is that maps from cofibrant objects (in particular, towers of polyhedra) into fibrant objects (for example, towers of Serre fibrations) in $\text{Ho}_{\text{sing}}(\text{pro-Top})$ can be realized by maps in $\text{pro-Top}$. We therefore first replace $\{Y_n\}$ by an equivalent fibrant object $\{Y'_n\}$. Unfortunately, the spaces $Y'_n$ need not be compact. A geometric argument is used to rectify this difficulty and replace $\{Y'_n\}$ by a compact tower (tower of compact spaces) $\{Y''_n\}$. The choice of $\{Y''_n\}$ depends upon the map $f$. Finally, applying limits to our diagram in $\text{pro-Top}$ will yield the required diagram (4.3).

We inductively construct a tower of Serre fibrations $\{Y''_n\}$ and a levelwise trivial cofibration (cofibration and weak equivalence) $\{y''_n : Y_n \to Y''_n\}$. See, for example, Bousfield and Kan [3, p. 299], or [16, Section 3].

Let $Y'_1 = Y_1$, and let $\gamma'_1$ be the identity map. Now assume inductively that $Y'_n$ and $\gamma'_n$ have been suitably defined for $n \leq m$. To define $Y''_{m+1}$ and $\gamma''_{m+1}$, first form the solid-arrow diagram
Define $Y'_{m+1}$ and $\gamma'_{m+1}$ by factoring the composite map $Y_{m+1} \to Y'_m$ as a trivial cofibration $\gamma'_{m+1}$ followed by a Serre fibration $p_{m+1}$ using Quillen [36, Section 11.3]. Continue inductively to define the required tower $\{Y'_n\}$, whose bonding maps $p_m$ are Serre fibrations, and the required levelwise trivial cofibration

$\gamma' = \{\gamma'_n\} : \{Y_n\} \to \{Y'_n\}.$

Quillen's description of the homotopy category of a model category [36, Section 1.5] yields a diagram in pro-$\text{Top}$.

$$\begin{align*}
\{ X_m \} & \xrightarrow{\phi'} \{ Y_m \} \\
\{ Y'_m \} & \xrightarrow{\gamma} \{ Y''_m \}
\end{align*}$$

which realizes $f$, that is, $f = [\gamma']^{-1}[\phi']$. By reindexing $\{X_m\}$ if necessary, we may assume that $\phi'$ is also a levelwise map $\{\phi'_n : X_n \to Y''_n\}$.

We shall now choose a suitable tower $\{Y'_n\}$ of compact spaces and a global homotopy $H = \{H_n : X_n \times I \to Y'_n\}$ from $\{\phi'_n\}$ to $\{\phi_n\}$ so that

(i) $\{Y_n\} \xrightarrow{\psi} \{Y'_n\} \xrightarrow{\gamma'} \{Y''_n\}$, levelwise, and

(ii) $\{Y'_n\}$ contains the image of $\{\phi_n\}$.

In general $\{Y'_n\}$ will not be fibrant.

Again we proceed inductively. Let $Y'_1 = Y_1 (= Y''_1)$ which is compact, let $\phi'_1 = \phi_1$, and let $H_1$ be the constant homotopy. Now assume inductively that the diagram

$$\begin{align*}
\{ X_n \} & \xrightarrow{\phi_n} \{ Y'_n \} \\
\{ Y_n \} & \xrightarrow{\gamma'_n} \{ Y''_n \}
\end{align*}$$

and the homotopy $\{H_n\}$ have been defined for $n \leq m$, so that the left-hand triangle commutes up to $\{H_n\}$, and the right-hand triangle strictly commutes. We also assume that $(H_m)_t$ is constant for $t \geq 1 - 1/m$. We shall now define level $m+1$ of the diagram and homotopy.

The map $i_m : Y'_m \to Y''_m$ is a trivial cofibration (cofibration and weak equivalence), and all objects in $\text{Top}$ are fibrant. Hence, there is a retraction $r : Y''_m \to Y'_m$ and a
homotopy $\Gamma: Y'_m \times I \rightarrow Y''_m$ relative to $Y'_m$, from the identity to $i_m r$. We may assume that $\Gamma_t=\text{id}$ for $t \leq 1 - 1/m$ and $\Gamma_t = i_m r$ for $t \geq 1 - 1/(m+1)$.

Recall the construction of $Y''_{m+1}$ and the associated maps
$$Y_{m+1} \xrightarrow{r_{m+1}} Y''_{m+1} \xrightarrow{p_m} Y'_m,$$
a trivial cofibration followed by a Serre fibration, using [36, Section II.3]. Define $Y_{m+1}^*$ and maps
$$Y_{m+1} \xrightarrow{j} Y_{m+1}^* \xrightarrow{q} Y'_m$$
by similarly factoring the composite mapping
$$Y_{m+1} \xrightarrow{\text{bond}} Y_m \xrightarrow{r_m} Y'_m$$
Because $Y'_m \subseteq Y''_m$, the second construction is a restriction of the first. This yields a cofibration $Y_{m+1}^* \rightarrow Y''_{m+1}$ which is also a weak equivalence because $Y'_m \approx Y''_m$. We have thus defined a solid arrow commutative diagram

![Diagram](image)

Define the filler $s$ by lifting the composite homotopy
$$Y_{m+1}^* \times I \xrightarrow{p_{m+1} \times \text{id}} Y''_m \times I \xrightarrow{r} Y'_m$$
to a homotopy
$$\Gamma': Y'' \times I \rightarrow Y''_{m+1}$$
with $\Gamma'_t=\text{id}$ for $t \leq 1 - 1/m$ and $\Gamma'_t$ stationary for $t \geq 1 - 1/(m+1)$. By construction, $\text{Im}(\Gamma'_t) \subset Y''_m$. Let $s=\Gamma'_1$.

(For Proposition 4.8 below, observe that the inclusion $Y_{m+1} \subseteq Y''_{m+1}$ is a cofibration and $\Gamma$ is stationary on $Y_m=p_{m+1}(Y_{m+1})$. This allows us to choose a homotopy $\Gamma$ which is stationary on $Y_{m+1}$.)

Also lift the composite homotopy
$$X_{m+1} \times I \xrightarrow{\text{bond} \times \text{id}} X_m \times I \xrightarrow{h_m} Y''_m$$
defined by our inductive assumption to a homotopy $\theta: X_{m+1} \times I \rightarrow Y''_{m+1}$, with $\theta_0=\phi'_{m+1}$, and $\theta$ stationary for $t \geq 1 - 1/m$. Let $\phi''=\phi_1$. 
Finally, by construction, $Y^*_{m+1}$ is a union of finite CW complexes $Y_a$, each of which contains $Y_{m+1}$ as a subcomplex and is weakly equivalent to $Y_{m+1}$. These complexes are defined by solving finitely many of the extension problems used to define $Y^*_{m+1}$. Because $X$ is compact, we can choose a suitable $Y_a$, which we call $Y^*_{m+1}$, containing $\text{Im}(\phi^*)$. Let $i_{m+1}$ be the composite inclusion $Y^*_{m+1} \hookrightarrow Y^*_{m+1} \hookrightarrow Y^*_{m+1}$.

Define

$$H_{m+1}: X_{m+1} \times I \to Y^*_{m+1}$$

by the formula

$$H_{m+1}(x, t) = \Gamma'(\theta(x, t), t).$$

Then $p_{m+1}H_{m+1}(x, t) = \Gamma[H_m(x', t), t]$ where $x'$ is the image of $x$ under the bonding map. But $\Gamma_t = \text{id}$ for $t \leq 1 - 1/m$, and $H_m(x', t) \in Y_m$, on which $\Gamma_t$ is also stationary, for $t \geq 1 - 1/m$. Thus $H_{m+1}$ covers $H_m$. By construction $H_{m+1}$ is stationary for $t \geq 1 - 1/(m+1)$.

Continue inductively to obtain diagram (4.5) for all $n$, and the required global homotopy $H = \{H_n\}$. Define the required diagram (4.3) by applying inverse limits to diagram (4.5).

We shall need an easy lemma to prove that $F$ is faithful.

4.7. Lemma. Let $i: A \to X$ be an inclusion and strong shape equivalence in $\text{CH}$. For any continuous map $f: A \to Y$ the induced map (inclusion) $X \to Z = Y \cup_A X$ is a strong shape equivalence.

Proof. Form the pushout diagram

$$\begin{array}{ccc}
\text{s-sh}(A) & \xrightarrow{\text{s-sh}(i)} & \text{s-sh}(X) \\
\downarrow \text{s-sh}(f) & & \downarrow \\
\text{s-sh}(Y) & \longrightarrow & \{Z_a\}
\end{array}$$

in pro-$\text{Top}$. Then $\{Z_a\}$ is an inverse system of compact polyhedra with inverse limit $Z$. The continuity theorem (Proposition 3.8) implies that $\text{s-sh}(Z) = \{Z_a\}$. By hypothesis and exactness of $\text{s-sh}$ (Proposition 3.5), the map $\text{s-sh}(i)$ is a weak equivalence and cofibration. Hence the induced map $\text{s-sh}(Y) \to \{Z_a\}$ is also a weak equivalence (and cofibration) by a model category axiom [36, Section I.1], verified for pro-$\text{Top}$ in [16, Section 3], as required.

4.8. Proposition. Suppose the diagrams

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y' \\
\downarrow \gamma & & \downarrow \\
X & \xrightarrow{f'} & Y'' \\
\end{array}$$

and

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\end{array}$$
in CM, in which \( \gamma' \) and \( \gamma'' \) are strong shape equivalences and inclusions represent equivalent maps in \( s\text{-}sh(CM) \). Then there is a homotopy commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f^*} & Y^* \\
\downarrow{f} & & \downarrow{\gamma^*} \\
Y' & \xrightarrow{\gamma} & Y
\end{array}
\]

(4.9)

in CM, in which \( \gamma^* \) is a strong shape equivalence.

Proof. Let \( P \) be the pushout of the square

\[
\begin{array}{ccc}
Y & \xrightarrow{r} & Y' \\
\downarrow{\gamma} & & \downarrow{r^*} \\
Y'' & \xrightarrow{\gamma} & P
\end{array}
\]

By Lemma 4.7, the inclusion \( Y \rightarrow P \) is a strong shape equivalence. Let \( \{P_n\} \) be a tower of finite polyhedra and PL maps with \( P = \lim\{P_n\} \), thus \( s\text{-}sh(P) = \{P_n\} \), naturally, by the continuity theorem (Proposition 3.8). As in the proof of Proposition 4.2, let \( \{P'_n\} \) be an equivalent fibrant tower (tower of Serre fibrations).

There is a solid-arrow commutative diagram in \( \text{pro-Top} \)

\[
\begin{array}{ccc}
\text{s-sh}(X) = \{X_\alpha\} & \xrightarrow{\text{s-sh}(f)} & \text{s-sh}(Y') \\
\downarrow{î_0} & & \downarrow{î} \\
\text{s-sh}(X \times I) & \xrightarrow{î} & \text{s-sh}(Y'') \\
\downarrow{î_1} & & \downarrow{î} \\
\text{s-sh}(X) = \{X_\alpha\} & \xrightarrow{\text{s-sh}(f)} & \text{s-sh}(Y) \\
\end{array}
\]

(4.10)

We shall use model-category [36, Section I.1] properties to find a filler (homotopy) \( H \). By construction, \( \{X_\alpha \times I\} \), together with the inclusions \( i_0, i_1: \{X_\alpha\} \rightarrow \{X_\alpha \times I\} \) is a cylinder-object for \( \{X_\alpha\} \), and \( \{P'_n\} \) is fibrant. Therefore there is a homotopy \( H \) between the two equivalent (by hypothesis) composite maps \( \{X_\alpha\} \Rightarrow \{P'_n\} \).

Now apply the proof of Proposition 4.2 simultaneously to the upper half and the lower half of diagram (4.10). In each case the homotopy \( P'_n \times I \rightarrow P'_n \) corresponding to \( r^* \) are stationary on \( P'_n \). We obtain a tower of finite CW complexes \( \{Y'_n\} \), a levelwise trivial cofibration \( \gamma: \{P_n\} \rightarrow Y'_n \), and a commutative diagram
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Proposition 4.2 implied that the functor \( F: \{s.s.e.\} \to \text{Ho}(\text{CM}) \) is full. Proposition 4.8 implies that \( F \) is faithful. This completes the proof of Theorem 4.1.

4.12. Remarks. The proofs of Propositions 4.2 and 4.8 require that the codomain \( Y \) be a compact metric space so that its strong shape \( s\text{-sh}(Y) \) is equivalent to a tower in \( \text{Ho}_{\text{sing}}(\text{pro}-\text{Top}) \); similarly for \( Y', Y'' \), and \( P \). Given an uncountable inverse system \( \{Y_a\} \) in \( \text{pro}-\text{PL} \), we do not know how to factor the canonical inclusion \( \{Y_a\} \to \{Y'_a\} \) into an equivalent fibrant system through a suitable inverse system of compact spaces. It may not even be possible!

S. Ferry [46] recently generalized Propositions 4.2 and 4.8 for mappings between suitable compacta. For any strong-shape-simple-connected compact metric space \( Y \), Ferry obtained a universal target \( Y' \) for mappings out of finite-dimensional compacta.

5. Calculus of fractions

We prove that the strong shape category of compact metric spaces admits a calculus of left fractions as in P. Gabriel and M. Zisman [20, p. 12].

5.1. Definition. Let \( \Sigma \) be the class of maps in \( \text{CM} \) which are inclusions and strong shape equivalences. Let \( \Sigma' \) be the image of \( \Sigma \) in \( \text{Ho}(\text{CM}) \).

5.2. Theorem. The class \( \Sigma' \) in \( \text{mor Ho}(\text{CM}) \) satisfies the Gabriel–Zisman axioms for a calculus of left fractions. Also, \( s\text{-sh}(\text{CM}) \) is isomorphic to the localization \( \Sigma'\to^{1}\text{Ho}(\text{CM}) \).

Proof. The first two axioms, that \( \Sigma' \) contains identity maps and is closed under composition, are immediate consequences of 5.1. Lemma 4.7 is a strong form of the third axiom. Proposition 4.8 implies the last axiom. Finally, Propositions 4.2 and 4.8 imply that \( s\text{-sh}(\text{CM}) \cong \Sigma'\to^{1}\text{Ho}(\text{CM}) \).
5.3. Corollary
(a) If $Y$ is a compact metric space, then

$$s\text{-}sh(CH)(X, Y) = \operatorname{colim}\{ [X, Y'] | Y' \in CH, Y \subset Y' \text{ and the inclusion is a strong shape equivalence} \}.$$  

for $X$ in $CH$.

(b) If $Y$ is a fibered compactum (the inverse limit of a tower of compacta of the homotopy type of compact ANR's, or equivalently, finite polyhedra), then

$$s\text{-}sh(CH)(X, Y) = [X, Y],$$  

for $X$ in $CH$.

5.4. Remarks. R. Geoghegan [21] introduced fibered compacta as a convenient class of spaces on which shape theory and homotopy theory "agree", but see also the proof of Proposition 6.3 below. Geoghegan proved that there are no strange fibered compacta in homotopy theory; i.e., a finitely dominated fibered compactum has finite homotopy type. S. Ferry [18] constructed a beautiful, strange (non-fibered) compactum.

6. Holstzynski's universal shape category

W. Holtzynski [25] observed that continuous functors on $CH$, for example, Čech homology $H_*$, factor through the localization of $CH$ at shape equivalences, $\{\text{s.e.}\}^{-1}CH$. He therefore proposed $\{\text{s.e.}\}^{-1}CH$ as a universal shape category. There is an obvious functor $\{\text{s.e.}\}^{-1}CH \to \text{sh}(CH)$, the usual shape category. (It is unknown whether $\{\text{s.e.}\}^{-1}CH$ admits a calculus of left fractions.) We shall prove that Steenrod homology $^S\text{H}_*$ ([40], see also [33] and [16, Section 8]) of compact metric spaces factors through $\{\text{s.e.}\}^{-1}CM$. This distinguishes $\{\text{s.e.}\}^{-1}CM$ from $\text{sh}(CM)$, hence $\{\text{s.e.}\}^{-1}CH \not\cong s\text{-}sh(CH)$.

For any object $\{X_a\}$ of pro-$\text{Top}$ the homology pro-groups are defined by

$$\text{pro } H_\ast \{X_a\} = \{H_\ast(X_a)\} \subset \text{pro-}(\text{abelian groups}).$$  

The pro-homotopy of a compact Hausdorff space $X$ is given by

$$\text{pro-}H_\ast(X) \cong \text{pro-}H_\ast(s\text{-}sh(X)).$$

Čech homology is the inverse limit of pro-homology. The pro-homology of a compact metric space $X$ is pro-isomorphic to the tower $\{H_\ast(X_n)\}$ where $\{X_n\}$ is any tower of polyhedra whose limit is $X$. See, for example, Proposition 3.6.

6.1. Proposition. Shape equivalences in $CM$ induce isomorphisms on Steenrod homology.
Proof. Let \( f: X \rightarrow Y \) be a shape equivalence in CM. Consider the induced map on Milnor [33, 34] short exact sequences

\[
0 \rightarrow \lim^1 \text{pro-}H_{\ast+1}(X) \rightarrow H_{\ast}(X) \rightarrow H_{\ast}(X) \rightarrow 0
\]

(6.2)

\[
0 \rightarrow \lim^1 \text{pro-}H_{\ast+1}(Y) \rightarrow H_{\ast}(Y) \rightarrow H_{\ast}(Y) \rightarrow 0
\]

By hypothesis \( f_1 \) and \( f_2 \) are isomorphisms. By the five lemma, \( f_2 \) is an isomorphism.

It is well known (and essentially contained in Christie's [9] thesis) that

\[
[*]S_2 = \text{sh}(S_2) = Z_2^2,
\]

where \( S_2 \) is the dyadic solenoid and \( Z_2^2 \) is the 2-adic integers. However, there is a unique shape map \( * \rightarrow S_2 \). This implies the following.

6.4. Proposition. \{s.e.\} \( ^{-1}\text{CM} \) is not the usual shape category \( \text{sh}(\text{CM}) \).

7. Discussion

We briefly summarize the relationships among shape categories and discuss several open questions. The following diagram of categories and functors summarizes the main results of Sections 4 and 6.

\[
\begin{array}{ccc}
\{\text{s.s.e.\} } ^{-1}\text{CM} \rightarrow \{\text{s.e.\} } ^{-1}\text{CM} \\
\downarrow \cong \downarrow \cong \\
\text{s-sh(CM)} \rightarrow \text{s-sh(CM)}
\end{array}
\]

The distinction between shape and strong shape is developed in [16, Section 8], using Christie's [9] example.

A natural interesting question is whether every shape equivalence in CM is also a strong shape equivalence. This would trivially imply an isomorphism of localizations \{s.e.\} \( ^{-1}\text{CM} \rightarrow \{\text{s.e.\} } ^{-1}\text{CM} \), and thus combine the simple definition of \{s.e.\} \( ^{-1}\text{CM} \) with the rich homotopy theory of s-sh(CM). Here are some partial positive results. For finite dimensional pointed connected compacta, L. Siebenmann's \( \pi_{\infty} \)-criterion [39] (see also J. Grossman [22], and [16, Theorem 5.5.6] implies that every pointed shape equivalence is a pointed strong shape equivalence. Every shape equivalence in CM is equivalent to a strong shape equivalence [16, Section 5]. Every hereditary shape equivalence in CM is a strong shape equivalence (J. Dydak and J. Segal [14, Corollary 10.4.3], see also R.B. Sher [38]). Every shape
equivalence in \(\text{CM}\) satisfying the analogue of the pushout lemma 4.7 is a strong shape equivalence (Dydak and Segal [13]).

This question has an obvious analogue in pro-homotopy — does the natural functor \(\text{Ho}(\text{pro-Top}) \to \text{pro-Ho(Top)}\) reflect isomorphisms of towers? An affirmative answer would easily imply the splitting of homotopy idempotents. However, J. Dydak and P. Minc [11], and P. Freyd and A. Heller [19] independently found an \textit{unsplit} idempotent in unpointed homotopy theory, yielding a negative answer. Their example involved a complex of infinite homological dimension. Dydak and the second-named author [12] proved that unpointed homotopy idempotents split on two-dimensional complexes. The general splitting question on finite or finite-dimensional complexes remains open.

Several results (exactness of \(\text{s-sh}(3.5)\) and a pushout lemma (4.7)) suggest that all inclusions in \(\text{CH}\) are \textit{cofibrations} in strong shape theory. Further evidence is provided by the following "homotopy extension theorem".

\textbf{7.1. Proposition.} Let \(j: A \to X\) be an inclusion in \(\text{CH}\) and a strong shape equivalence. Let \(Y\) be the inverse limit of a tower of fibrations of polyhedra \(\{Y_n\}\). Then any solid-arrow diagram of the form

\[
\begin{array}{ccc}
A & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y
\end{array}
\]

admits a filler.

\textbf{Proof.} Because \(\text{s-sh}(Y) \equiv \{Y_n\}\), diagram (7.2) yields a diagram

\[
\begin{array}{ccc}
\text{s-sh}(A) & \longrightarrow & \text{s-sh}(Y) & \longrightarrow & \{Y_n\} \\
\downarrow & & \downarrow & & \downarrow \\
\text{s-sh}(X) & & & & \ast
\end{array}
\]

in which \(\text{s-sh}(j)\) is a cofibration (Proposition 3.2) and a weak equivalence (hypothesis), and \(p\) is a fibration. The lifting axiom for model categories [36, Section I.1] yields a filler \(F\). Let \(F = \lim F'\).

This suggests the problem of deciding how much of a closed model structure corresponding to strong shape theory can be defined on \(\text{CH}\)?

The proof of Proposition 3.6 suggests further development of coherent pro-homotopy, following R. Vogt [43], and T. Porter’s recent paper [45]. In particular, homotopy limits (see [3], [16, Section 4], and [43]) should be easily extended to this
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setting. In fact, homotopy limits may exist more generally. Consider for example, the generalized pro-categories gpro-\text{C} of A. Deleanu and P. Hilton [10], and J.L. Aronson and D.A. Edwards [1]. These authors removed the requirement that indexing categories be filtering, but retained the familiar formula for morphisms

\[ \text{gpro-}\text{C}(\{X_\alpha\}, \{Y_\beta\}) = \lim_\beta \colim_\alpha \{C(X_\alpha, Y_\beta)\}. \]

Colim is well defined on the subcategory of gpro-\text{Top}_0 (pointed, connected spaces) for which the Bousfield–Kan spectral sequence [3] converges completely. This suffices for the descriptions in [16, Sections 6, 8] of completions à la Sullivan [43], and Quillen’s +-construction [36].

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We thank Alex Heller for helpful discussions. F. Cathey [47] recently obtained similar results with different methods.

Note added in proof

A. Heller and the second-named author recently proved that (unpointed) homotopy idempotents on \textit{finite-dimensional} complexes split.

References


[38] R.B. Sher, Realizing cell-like maps in Euclidean spaces, General Topology Appl. 2 (1972) 75–89.


