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Coexistence of pion condensation and color superconductivity in two flavor quark matter

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Abstract

We show that the superconducting 2SC phase at high density and normal chiraly broken quark phase at low density is separated by the mixed non-uniform phase along the baryon density line.

1. Introduction

In QCD with two flavors at high baryon density the superconducting phase is the lowest energy state [1,2]. In this phase u, d quarks with two different colors (say “red”, “green”) create Cooper pairs whereas the “blue” quarks remain free. The chiral symmetry $SU(2) \times SU(2)$ is restored and only gauge color group is broken. On the other hand at lower density one expects the state with broken chiral symmetry. This can be a hadronic phase but the free quark plasma is not ruled out. In such a situation the sequence of the phase transitions along the density line (at zero or very low temperature) follows the chain: hadronic phase/quark phase/superconducting phase. All the phase transitions are expected to be first order. In this Letter we present an explicit model describing the transition between the normal quark and the superconducting phases. It occurs that the transition actually goes through the mixed state which consists of the non-uniform chiral phase and superconducting phase.

The non-uniform chiral phase had been introduced in the context of the non-relativistic nuclear physics [3] and then was extended for the systems of high density plasma of quarks [4,5]. In this new phase the chiral fields condense in both scalar and pseudoscalar channels creating a static chiral wave along one direction in space with the wavelength $2\pi/|\vec{q}|$ where $\vec{q}$ is a wave vector. It was shown that in some temperature—density range this configuration has lower energy then the uniform quark phase. On the other hand we have learned from the studies [1,2] that at higher densities one should compare the non-uniform quark phase with the superconducting phase rather then with the uniform quark phase. As a result one can show that the phase diagram of strongly interacting matter is more complicated and contains a region of the mixed state.

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The high density matter exists in the neutron stars. If the density is high enough then the superconducting phase resides in their cores. In this situation one can also expect the mixed phase to be present there—the mixed state which separates uniform chiraly broken phase from the superconducting phase. We cannot prove that this mixed state is the same as the one considered in this Letter but we provide an explicit realization of such a scenario.

In the next section we introduce a thermodynamic potential describing all the interesting phases. In the third section we present the numerical analysis and finally we discuss the results in the conclusions.

2. Model of chiral and 2SC phases

Let us consider the NJL model [6]:

$$H = \int d^3x \left\{\bar{\psi}(-i\vec{\gamma}\cdot\vec{\nabla} - \mu\gamma_0)\psi - G\left[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\vec{\tau}\psi)^2\right] - G'\left(\psi\tau_2 t^A C\gamma_5\psi\right)^\dagger\left(\psi\tau_2 t^A C\gamma_5\psi\right)\right\}, \tag{1}$$

where ψ is the quark field, μ the quark chemical potential and the color, flavor and spinor indices are suppressed. The vector $\vec{\tau}$ is the isospin vector of Pauli matrices and t^A are three color antisymmetric group generators. The coupling constant G describes the interaction in the isospin singlet, Lorentz scalar and the isospin triplet, Lorentz pseudoscalar, quark–antiquark channels whereas G' describes the interaction in the color $\bar{\mathbf{3}}$, flavor singlet, Lorentz scalar diquark channel. Both couplings are related through the Fierz transformation but we treat them as the independent parameters. There is an additional parameter (or function) which regularize infinities in the model. In our case we use the simplest momentum cut-off Λ. Our main results are independent of this choice. We analyze the four-fermion point interactions in the mean field approximation using the general anstaz:

$$\langle\bar{\psi}\psi\rangle = -\frac{M}{2G}\cos\vec{q}\cdot\vec{x}, \qquad \left\langle\bar{\psi}i\gamma_5\tau^a\psi\right\rangle = -\frac{M}{2G}\delta_{a3}\sin\vec{q}\cdot\vec{x}, \qquad \left\langle\psi\tau_2 t^A C\gamma_5\psi\right\rangle = \frac{\Delta}{2G'}\delta_{A1}, \tag{2}$$

where the directions of symmetry breaking in the isospin and color spaces are chosen arbitrarily. The wave vector $\vec{q}$ describes the chiral wave along some arbitrary direction in space. In the limit of vanishing $\vec{q}$ we recover usual uniform chiral phase. The gap parameter Δ describes the superconducting phase. Let us notice that (2) is not the analog of the LOFF phase in the superconductivity [7,8] which arises due to the difference in the chemical potentials of the pairing species. In our case it is rather the dynamical interaction between the spin and isospin degrees of freedom which effectively lower the energy [4,5]. Using formulae (1), (2) in the mean-field approximation we get:

$$H_{\mathrm{MF}} = \int d^3x \left\{\bar{\psi}(-i\vec{\gamma}\cdot\vec{\nabla} + MU - \mu\gamma_0)\psi - \frac{\Delta}{2}\left(\psi\tau_2 t^1 C\gamma_5\psi\right)^\dagger - \frac{\Delta^*}{2}\left(\psi\tau_2 t^1 C\gamma_5\psi\right) + \frac{M^2}{4G} + \frac{|\Delta|^2}{4G'}\right\}, \tag{3}$$

where the Dirac operator is now space dependent $U = \cos\vec{q}\cdot\vec{x} + i\gamma_5\tau_3\sin\vec{q}\cdot\vec{x}$. One can eliminate this dependence by the unitary transformation to the new Fermi fields $\tilde{\psi} = U^{1/2}\psi$ which gives the free energy in the form:

$$\begin{aligned}\widetilde{H}_{\mathrm{MF}} = \int d^3x \Bigg\{&\tilde{\psi}^\dagger\left(-i\vec{\alpha}\cdot\vec{\nabla} - \frac{1}{2}\vec{\Sigma}\cdot\vec{q}\tau_3 + M\gamma_0 - \mu\right)\tilde{\psi} - \frac{\Delta}{2}\left(\tilde{\psi}\tau_2 t^1 C\gamma_5\tilde{\psi}\right)^\dagger \\ &- \frac{\Delta^*}{2}\left(\tilde{\psi}\tau_2 t^1 C\gamma_5\tilde{\psi}\right) + \frac{M^2}{4G} + \frac{|\Delta|^2}{4G'}\Bigg\},\end{aligned} \tag{4}$$

where $\vec{\Sigma}$ is the vector of Dirac spin matrices. The Hamiltonian (4) describes two kinds of interactions: the "chiral" part and the "superconducting part". The chiral part mixes particles and antiparticles leaving color and isospin structure untouched. On the other hand the "superconducting" part mixes colors and isospins of particles. To be more precise, one can introduce the plane wave basis

$$\tilde{\psi}^j_\alpha(t,\vec{x}) = \sum_{s=1,2}\int\frac{d^3k}{(2\pi)^3\sqrt{2E(\vec{k})}}\left\{u_s(\vec{k})a^j_{\alpha,s}(\vec{k})\exp(-ikx) + v_s(\vec{k})b^{j\dagger}_{\alpha,s}(\vec{k})\exp(ikx)\right\}, \tag{5}$$

where $E(\vec{k}) = \sqrt{\vec{k}^2 + M^2}$, u_s, v_s are Dirac bispinors, $a^i_{\alpha,s}(\vec{k})$ ($b^i_{\alpha,s}(\vec{k})$) is an annihilation operator of quark (antiquark) of color α, flavor j, spin s, and momentum $\vec{k}$, satisfying usual anticommutation relations. In this basis Hamiltonian (4) creates the 24×24 matrix which decays into three independent diagonal 8×8 blocks. Two of them are related to pairing between "red", "green" u, d quarks and antiquarks whereas the third one describes the pairing between "blue" quarks and antiquarks (which do not create Cooper pairs). The Hamiltonian (4) can be diagonalized with the positive energy eigenvalues:

$$\lambda_{1,\pm}(\vec{k}) = \sqrt{(E_\pm(\vec{k}) - \mu)^2 + |\Delta|^2}, \qquad \lambda_{2,\pm}(\vec{k}) = \sqrt{(E_\pm(\vec{k}) + \mu)^2 + |\Delta|^2},$$
$$E_\pm(\vec{k}) = \sqrt{\vec{k}^2 + M^2 + \frac{q^2}{4} \pm \sqrt{q^2 M^2 + \vec{k}\cdot\vec{q}}}, \tag{6}$$

for u, d, "red", "green" quarks and $\lambda_{3,\pm}(\vec{k}) = E_\pm(\vec{k})$ for "blue" u, d quarks. The negative eigenvalues has opposite signs. After diagonalization the Hamiltonian (4) takes the general form $\widetilde{H}_{\rm MF} = \widehat{H} + H_0$. The first term is a diagonal operator and the second term defines the energy of the ground state as a function of chemical potential. The vacuum expectation value $\langle \widetilde{H}_{\rm MF} \rangle = H_0$ and

$$H_0 = \frac{|\Delta|^2}{4G'} + \frac{M^2}{4G} - 2\sum_{s=\pm} \int^{\Lambda} \frac{d^3k}{(2\pi)^3}(\lambda_{1,s} + \lambda_{2,s}) - 2\sum_{s=\pm} \int^{\Lambda} \frac{d^3k}{(2\pi)^3}\lambda_{3,s} + 2\sum_{s=\pm} \int_{E_s<\mu} \frac{d^3k}{(2\pi)^3}(\lambda_{3,s} - \mu). \tag{7}$$

The first integral describes the contribution from the quarks which create the chiral condensate as well as Cooper pairs. The factor 2 is the number of colors minus one ("red", "green"). The parameter Λ reminds of the momentum cut-off of the divergent integral. The last two terms are connected to "blue" quarks which build only chiral condensate. The factor of 2, this time, describes the number of flavors. Let us rewrite (7) in the form:

$$H_0 = \frac{|\Delta|^2}{4G'} + \frac{M^2}{4G} - 6\sum_{s=\pm} \int^{\Lambda} \frac{d^3k}{(2\pi)^3}\lambda_{3,s}$$
$$- 2\sum_{s=\pm}\left\{\int \frac{d^3k}{(2\pi)^3}(\lambda_{1,s} + \lambda_{2,s} - 2\lambda_{3,s}) + \sum_{s=\pm}\int_{E_s<\mu} \frac{d^3k}{(2\pi)^3}(\lambda_{3,s} - \mu)\right\} \tag{8}$$

which distinguishes explicitly between the vacuum contribution—the first integral, and the finite density contributions—the last two integrals. The vacuum contribution can be expanded in the power of the wave vector $\vec{q}$ (nothing more but the gradient expansion in mesonic fields) and we arrive at the final result:

$$H_0^{njl} = \frac{|\Delta|^2}{4G'} + \frac{M^2}{4G} + \frac{M^2 F_\pi^2 \vec{q}^{\,2}}{2M_0^2} - 12\int^{\Lambda} \frac{d^3k}{(2\pi)^3} E(\vec{k})$$
$$- 2\sum_{s=\pm}\left\{\int \frac{d^3k}{(2\pi)^3}(\lambda_{1,s} + \lambda_{2,s} - 2\lambda_{3,s}) - \int_{E_s<\mu} \frac{d^3k}{(2\pi)^3}(\lambda_{3,s} - \mu)\right\}. \tag{9}$$

The coefficient at $\vec{q}^{\,2}$ is model independent and it is related to the pion decay constant with its vacuum value $F_\pi = 93$ MeV. M_0 is the constituent quark mass at zero density. Let us notice that in the limit of vanishing $|\Delta|$ we recover the chiral Hamiltonian [5] whereas in the limit of vanishing $\vec{q}$ we can find the Hamiltonian describing Cooper pairing [6]. Let us also mention that from the formula (9) one can easily guess the linear sigma model version of the Hamiltonian. Indeed the NJL and linear sigma models at the mean-field level are different only with respect to the description of the vacuum contribution to the energy. Thus instead of the formula (9) one can write in the linear sigma model:

$$H_0^{l\sigma} = \frac{|\Delta|^2}{4G'} + \frac{M^2}{4G} + \frac{M^2 F_\pi^2 \vec{q}^2}{2M_0^2} + \frac{m_\sigma^2}{8F_\pi^2}\left(\frac{m^2}{g^2} - F_\pi^2\right)^2 - 2\sum_{s=\pm}\left\{\int \frac{d^3k}{(2\pi)^3}(\lambda_{1,s} + \lambda_{2,s} - 2\lambda_{3,s}) - \int_{E_s<\mu} \frac{d^3k}{(2\pi)^3}(\lambda_{3,s} - \mu)\right\}, \tag{10}$$

where g is dimensionless coupling constant and m_σ is a mass of sigma meson. However in the numerical calculation we only use the formula (9).

3. Chiral/2SC phase transition

The model parameters can be fitted to the values of the chiral condensate and the pion decay constant which give $G = 5.01\ \text{GeV}^{-2}$ and the cut-off $\Lambda = 0.65$ GeV [9]. For this set of parameters the chiral phase transition at $\vec{q} = 0$ and $|\Delta| = 0$ takes place at the chemical potential $\mu = 0.316$ GeV and the value of the constituent quark mass at zero density is $M_0 = 0.301$ MeV. The coupling constant G' we treat as the additional model parameter. The main results which are robust against the change in G' are:

- there is the first order phase transition from the homogeneous chiraly broken phase $M = M_0$, $\vec{q} = 0$, $|\Delta| = 0$ to the mixed phase $M \neq M_0 \neq 0$, $\vec{q} \neq 0$, $|\Delta| \neq 0$;
- by increasing chemical potential one decreases constituent mass M and increases $\vec{q}$ and $|\Delta|$;
- there is additional first order phase transition from the mixed phase to the 2SC phase ($M = 0$, $\vec{q} = 0$, $|\Delta| \neq 0$).

From the above points one can see that the 2SC phase is separated from the chiraly broken phase by the non-uniform mixed state. This conclusion is still valid for non-zero temperatures. We discuss this feature in the last section. On the other hand the numerical details depend on the choice of G'. In particular for the range $G/2 \leqslant G' \leqslant G$ we can find:

- the transition to the mixed phase appears in the band of 10 MeV around $\mu = 0.29$ MeV (smaller G' larger critical chemical potential);
- the values of the gap parameter $|\Delta|$ can vary by a factor of two (smaller G' larger $|\Delta|$);
- the values of the wave vector $|\vec{q}|$ can change by 40–50 per cent (smaller G' larger $|\vec{q}|$);
- the range of the mixed phase along μ-axis can change from 10 to 100 MeV (smaller G' longer the range).

In Fig. 1 we present the values of M, $|\vec{q}|$, $|\Delta|$ for $G' = G/2$ as a function of chemical potential μ. The first order phase transition takes place at $\mu = 0.301$ GeV. The constituent mass drops to around half of its vacuum value

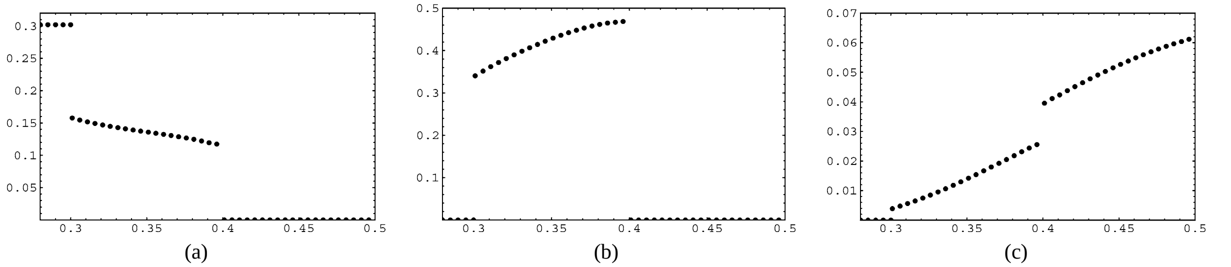

Fig. 1. Dependence of M, $|\vec{q}|$ and Δ in GeV on the quark chemical potential μ in GeV.

(Fig. 1(a)), the wave vector $|\vec{q}| = 0.34$ GeV (Fig. 1(b)) and $|\Delta| = 0.004$ GeV (Fig. 1(c)). The mixed phase exists up to the second critical point $\mu = 0.396$ GeV where another first order phase transition appears. The values of M, $\vec{q}$ drop to zero and the value of the gap $|\Delta| = 0.039$ GeV. For larger μ we enter the homogeneous 2SC phase.

4. Conclusions

In this Letter we showed that the high density superconducting phase 2SC is separated from the low density chiraly broken phase by the region of the mixed non-homogeneous phase. This mixed state consists of the chiral waves with non-zero expectation values in the scalar and pseudoscalar channels $\langle\bar{\psi}\psi\rangle$, $\langle\bar{\psi}i\gamma_5\tau_a\psi\rangle$ described by the wave vector $\vec{q}$ which points at arbitrary direction is space. In addition there is a non-zero value of the superconducting gap parameter $|\Delta|$. We cannot prove that this mixed state is the lowest energy state however we showed that its energy is lower then that of the homogeneous phases. Similar questions was already addressed in the context of the Overhauser effect [10] versus BCS pairing. It was shown that both types of pairing can be competitive [11,12] with each other. However it was also shown that within the NJL-type of models BCS state is favoured over the Overhauser state [12] which suggests that the non-homogeneous phase (2) is lower then BCS and Overhauser phases in the considered density range. Nevertheless one has to keep in mind that other approaches (instanton models) favours Overhauser effect over BCS pairing [11,12] thus one cannot drew the final conclusion and more work is required. We can then conclude that the superconducting phase and the normal phase are separated by some kind of crystal state. Let us stress that the crystal structure (2) is dynamical in origin, not "kinematical" in a sense of the LOFF state [7,8].

Our results are robust against the reasonable changes in the model parameters. What is more the same results one can derive not only in the NJL model but also in the linear sigma model. Thus also in that sense the results are model independent. The numerical details, like the location of the critical points, the values of the constituent mass, wave vector and superconducting gap, depends on the choice of the parameters. Although we perform our calculations for massless quarks the introduction of the small current quark mass does not change the main conclusions. It merely distorts the shape of the chiral wave as was shown in [13] for one-dimensional system. One can also expect that the mixed phase exists for some range of temperatures (below $T \sim 50$ MeV where superconductor melts) and densities. However the detailed picture of the phase diagram remain to be done.

The real QCD consists of two light flavors and one heavier, strange quark. In this situation one has to consider the three flavors NJL model. However we expect that our main points remain the same. If one considers the superconducting phase inside the neutron stars then the question of the color and charge neutrality rises [14]. It was shown by the explicit calculation [15] that at non-zero lepton chemical potential the 2SC phase borders with the normal quark phase at lower densities and only at higher densities it changes into the CFL phase. Thus even in the case of the three flavors QCD we can still have quark matter/2SC boundary in the neutron stars. This boundary is not just a line in the phase diagram but the mixed state region as we have shown. What kind of the crystal or non-uniformity is realized in the three flavors QCD has not been discoverd so far.

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