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Notes

Note on a Family of BIBD's and Sets of Mutually Orthogonal Latin Squares

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ABSTRACT

The connection between two families of balanced incomplete block designs (BIBD's) and sets of mutually orthogonal Latin squares (MOLS) is noted. Known sets of MOLS then allow construction of corresponding BIBD's in these families.

I. INTRODUCTION

In several recent papers [3, 4, 6, 8], methods have been given for the construction of families of balanced incomplete block designs with parameters

$$v, b = v(v-1), \qquad r = k(v-1), \qquad k, \lambda = k(k-1) \tag{1}$$

and

$$v, b = v(v-1)/2, \qquad r = k(v-1)/2, \qquad k, \lambda = k(k-1)/2. \tag{2}$$

(Of course, a design with parameters (2) will also yield a design with parameters (1).) The purpose of this note is to point out that the designs constructed in the above papers are easily derivable from known sets of mutually orthogonal latin squares (MOLS); and in some cases known sets of MOLS will yield designs that cannot be found by the other methods. The connection between the designs and sets of mutually orthogonal latin squares is simple, but does not seem to have been stated explicitly before.

We first give three lemmas which connect designs with parameters (1) with sets of MOLS:

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LEMMA 1. *Given a set of t MOLS of order v, we can construct a BIBD with parameters*

$$v, b = v(v-1), \qquad r = t(v-1), \qquad k = t, \qquad b = t(t-1).$$

Proof. We can permute the numbers in each of the Latin squares so that the first row in each square is $1, 2, \ldots, v$, without affecting orthogonality. We then form $v(v-1)$ blocks B_{ij} ($i = 2, \ldots, v; j = 1, \ldots, v$) by placing in B_{ij} the t numbers which lie in the (i, j)-th cell of the squares. From the properties of orthogonal Latin squares it follows easily that the blocks form a BIBD with the given parameters.

LEMMA 2. *Given a set of t MOLS fo order v, we can construct a BIBD with parameters*

$$v, b = v(v-1), \qquad r = (t+1)(v-1), \qquad k = t+1, \qquad \lambda = t(t+1).$$

Proof. As in Lemma 1, we can assume that the first row of each square is $1, 2, \ldots, v$. We now form $v(v-1)$ blocks B_{ij} ($i = 2, \ldots, v; j = 1, \ldots, v$) by placing in B_{ij} the t numbers which lie in the (i, j)-th cell of the square, along with the number j. Again, this is easily seen to yield a BIBD.

Before giving the third lemma, we make the following:

DEFINITION. A *transversal* of a Latin square of order v is a set of v cells, no two in the same row or column, such that each of the numbers $1, 2, \ldots, v$ appears in exactly one cell.

A set of Latin squares has a *common transversal* if there exists some set of v cells having the above property for all squares in the set.

We now give

LEMMA 3. *Given a set of t MOLS with a common transversal, we can construct a BIBD with parameters*

$$v, b = v(v-1), \quad r = (t+2)(v-1), \quad k = t+2, \quad \lambda = (t+1)(t+2).$$

Proof. Consider the "superposed square" formed by superposing the t MOLS. The elements in the various squares can be permuted (without destroying orthogonality) so that the ordered t-tuples appearing on the common transversal are $(1, \ldots, 1), \ldots, (v, \ldots, v)$. Further, the rows and columns of the superposed square can now be permuted so that the transversal is the main diagonal, and elements can be relabeled so that

(1,..., 1) appears in row 1 and column 1, (2,..., 2) in row 2 and column 2, and so on.

We now form $v(v-1)$ blocks B_{ij} ($i \neq j$; $i, j = 1,..., v$) by placing in B_{ij} the t elements in cell (i, j) of the superposed square, along with the numbers i and j. Again, it is easily checked that this yields the desired BIBD.

II. Applications of The Lemmas

If v can be factored into prime powers as $v = p_1^{e_1} \cdots p_n^{e_n}$, then it is well known (see, for example, [5]) that $N = \min(p_i^{e_i} - 1)$ MOLS of order v can be constructed. For $k \leqslant N + 1$, choosing any $k - 1$ of these squares and applying Lemma 2 yields

Theorem 1. *If $v = p_1^{e_1} \cdots p_n^{e_n}$, and if $k \leqslant \min(p_i^{e_i})$, then there exists a BIBD with parameters*

$$v, b = v(v-1), \qquad r = k(v-1), \qquad k, \lambda = k(k-1).$$

These designs have also been constructed in [4], [6], and [8].

Now consider v, an odd integer with prime factorization $p_1^{e_1} \cdots p_n^{e_n}$, and let $N = \min(p_i^{e_i} - 1)$. Consider the finite fields $\mathrm{GF}(p_1^{e_1}),...,\mathrm{GF}(p_n^{e_n})$, and consider the ring consisting of v elements $(a_1, ..., a_n)$, where $a_i \in \mathrm{GF}(p_i^{e_i})$, and where addition and multiplication of ring elements are given by

$$(a_1, ..., a_n) + (b_1, ..., b_n) = (a_1 + b_1, ..., a_n + b_n),$$
$$(a_1, ..., a_n) \times (b_1, ..., b_n) = (a_1 b_1, ..., a_n b_n),$$

according to addition and multiplication in the n finite fields. Then N MOLS of order v are formed as follows (see, for example, [5]):

> Choose N ring elements $\gamma_1, ..., \gamma_N$, such that, if $\gamma_i = (\gamma_{i1}, ..., \gamma_{in})$, then $\gamma_{ij} \neq 0$ and $\gamma_{ij} \neq \gamma_{i'j}$ ($i \neq i'$). Then form N MOLS by using as the entry in cell (x, y) of the j-th square
>
> $$L_{xy}^j = \gamma_j \gamma_x + \gamma_y,$$
> $$j = 1,..., N; \; x, y = 0, 1,..., v-1.$$

We now prove

Lemma 4. *The ring elements $\gamma_0 = (0,..., 0)$, $\gamma_1, ..., \gamma_{v-1}$ above can be ordered so that, with respect to any two Latin squares in the set, we find that,* *if the ordered pair (a, b), (where $a \neq b$), appears in the last $(v-1)/2$ rows of the superposed square, the pair (b, a) does not.*

Proof. Order the ring elements so that we have

$$\gamma_0, \gamma_1, \ldots, \gamma_{(v-1)/2}, \qquad -\gamma_1, \ldots, -\gamma_{(v-1)/2}.$$

Consider two squares L^j and $L^{j'}$; suppose (a, b) appears in cell (x, y) and, (b, a) in cell (x', y'). Then

$$\gamma_j\gamma_x + \gamma_y = a, \qquad \gamma_{j'}\gamma_x + \gamma_y = b \tag{3}$$

and

$$\gamma_j\gamma_{x'} + \gamma_{y'} = b, \qquad \gamma_{j'}\gamma_{x'} + \gamma_{y'} = a. \tag{4}$$

Hence

$$\gamma_x(\gamma_j - \gamma_{j'}) = a - b = \gamma_{x'}(\gamma_{j'} - \gamma_j),$$

and so

$$(\gamma_x + \gamma_{x'})(\gamma_j - \gamma_{j'}) = (0, \ldots, 0).$$

But, by construction, $(\gamma_j - \gamma_{j'})$ has no zero components, and hence $\gamma_x + \gamma_{x'} = 0$, that is, $\gamma_x = -\gamma_{x'}$. This proves the lemma.

Noting that the first row of each square constructed above is γ_0, $\gamma_1, \ldots, \gamma_{v-1}$, we can, in virtue of Lemma 4, form $v(v-1)/2$ blocks by using the procedure of Lemma 2 on rows $2, \ldots, (v-1)/2$ of the superposed square, to yield BIBD's with parameters (2). We have thus proved (see [4], [6]. [8]):

THEOREM 2. *If v is odd, and has prime factorization $v = p_1^{e_1} \cdots p_n^{e_n}$, and if $k \leqslant \min(p_i^{e_i})$, then there exists a BIBD with parameters*

$$v, b = v(v-1)/2, \qquad r = k(v-1)/2, \qquad k, \lambda = k(k-1)/2.$$

Theorems 1 and 2 yield designs with parameters (1) and (2) in all the cases in which designs were given in [3], [4], [6], and [8]. However, we can construct designs (1) for values of k not included in the above theorems, in cases in which more than $N = \min(p_i^{e_i} - 1)$ MOLS of order v are known to exist.

For example, Johnson, Dulmage, and Mendelsohn [2] have constructed sets of 5 MOLS of order 12. Applying Lemma 1, say, then yields a BIBD with parameters (12, 132, 55, 5, 20). (A design with the same parameters has been found by a completely different method in [7].)

As another example, consider sets of MOLS constructed by Bose, Shrikhande, and Parker [1]. Many of these sets have common transversals, and so we can apply any one of Lemmas 1, 2, or 3 above. For example, since their methods yield 4 MOLS of order 57, we can apply Lemma 2 to obtain a BIBD with parameters (57, 2862, 265, 5, 20).

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