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Universal Bethe ansatz solution for the Temperley–Lieb spin chain

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Abstract

We consider the Temperley–Lieb (TL) open quantum spin chain with "free" boundary conditions associated with the spin-s representation of quantum-deformed $sl(2)$. We construct the transfer matrix, and determine its eigenvalues and the corresponding Bethe equations using analytical Bethe ansatz. We show that the transfer matrix has quantum group symmetry, and we propose explicit formulas for the number of solutions of the Bethe equations and the degeneracies of the transfer-matrix eigenvalues. We propose an algebraic Bethe ansatz construction of the off-shell Bethe states, and we conjecture that the on-shell Bethe states are highest-weight states of the quantum group. We also propose a determinant formula for the scalar product between an off-shell Bethe state and its on-shell dual, as well as for the square of the norm. We find that all of these results, except for the degeneracies and a constant factor in the scalar product, are universal in the sense that they do not depend on the value of the spin. In an appendix, we briefly consider the closed TL spin chain with periodic boundary conditions, and show how a previously-proposed solution can be improved so as to obtain the complete (albeit non-universal) spectrum.

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1. Introduction

The generators $\{X_{(1)}, \ldots, X_{(N-1)}\}$ of the unital Temperley–Lieb (TL) algebra TL_N [1],

$$\begin{aligned} X_{(i)}^2 &= cX_{(i)}\,, \\ X_{(i)}X_{(i\pm1)}X_{(i)} &= X_{(i)}\,, \\ X_{(i)}X_{(j)} &= X_{(i)}X_{(j)}\,, \qquad |i-j|>1\,, \end{aligned} \tag{1.1}$$

can be used to define the Hamiltonian of an open quantum spin chain of length N with "free" boundary conditions

$$H = \sum_{i=1}^{N-1} X_{(i)}\,. \tag{1.2}$$

This type of model has been the subject of many investigations. For simplicity, we focus here on the models associated with $U_Q(A_1) = U_Q sl(2)$. The generators $X_{(i)}$ have been constructed for any value of spin s [2,3]. The $s = 1/2$ case is the well-known quantum-group-invariant spin-1/2 XXZ chain [4]. The $s = 1$ case is the quantum deformation [2] of the pure biquadratic spin-1 chain [5–7]. These models are integrable; and closed-chain versions with periodic boundary conditions have been investigated for $s > 1/2$ using inversion relations [8,9], numerically [10], by coordinate Bethe ansatz [5,6,11], and by analytical Bethe ansatz [12]. Additional results can be found in [13–17] for the open chain, and in [18–20] for the closed chain. TL models associated with higher-rank algebras have also been investigated [21–23].

Despite these and further efforts, a number of fundamental problems related to these models, such as the formulation of an algebraic Bethe ansatz solution, have remained unsolved. Moreover, the analytical Bethe ansatz solution proposed in [12] does not give the complete spectrum.

The goal of this paper is to address some of these problems. We construct the transfer matrix corresponding to the Hamiltonian (1.2), and we determine its eigenvalues using analytical Bethe ansatz. We prove that the transfer matrix has quantum group symmetry, which accounts for the degeneracies of the spectrum. We propose an algebraic Bethe ansatz construction of the Bethe states, which (when on-shell) we conjecture are highest-weight states of the quantum group. The scalar product between an off-shell Bethe state and an on-shell Bethe state is also considered, and we conjecture that it can be given in terms of a determinant formula; the square of the norm, i.e., the scalar product between on-shell Bethe states, follows as a limit.[1] We find that all of these results, except for the degeneracies and a constant factor in the scalar product, are universal in the sense that they do not depend on the value of the spin.

Although most of this paper concerns the open TL chain, we briefly consider the closed TL chain with periodic boundary conditions in an appendix. There we revisit the analytical Bethe ansatz computation in [12], and show how the proposed solution can be improved so as to obtain the complete spectrum. In contrast with the case of the open chain, the solutions of the closed-chain Bethe equations are not universal, as the Bethe roots depend on the value of the spin.

The outline of this paper is as follows. In section 2 we describe the construction of the Hamiltonian (1.2) and the corresponding transfer matrix. In section 3 we use analytical Bethe ansatz to determine the eigenvalues of the transfer matrix and the corresponding Bethe equations. In section 4 we show that the transfer matrix has quantum group symmetry, and we propose explicit

[1] Such formulas are generally known as Slavnov [24] and Gaudin–Korepin [25–27] formulas, respectively.

formulas for the number of solutions of the Bethe equations and the degeneracies of the transfer-matrix eigenvalues. In section 5 we present our proposals for the algebraic Bethe ansatz solution and scalar products. We briefly discuss these results and remaining problems in section 6. We treat the closed TL chain in Appendix A.

2. Transfer matrix

We begin this section by describing in more detail the construction of the Hamiltonian (1.2). We then construct the corresponding transfer matrix, which is the generating function of the Hamiltonian and the higher local conserved commuting quantities, and we review some of its important properties.

We consider the TL open quantum spin chain corresponding to the spin-s representation of $U_Q sl(2)$. The $X_{(i)}$ appearing in the Hamiltonian (1.2) are operators on $\left(\mathbb{C}^{2s+1}\right)^{\otimes N}$ defined by

$$X_{(i)} = X_{i,i+1}\,, \tag{2.1}$$

where X is a $(2s+1)^2$ by $(2s+1)^2$ matrix (an endomorphism of $\mathbb{C}^{2s+1} \otimes \mathbb{C}^{2s+1}$) with the following matrix elements [2]

$$\langle m_1, m_2|X|m_1', m_2'\rangle = (-1)^{m_1-m_1'} Q^{m_1+m_1'} \delta_{m_1+m_2,0}\delta_{m_1'+m_2',0}\,, \tag{2.2}$$

where $m_1, m_2, m_1', m_2' = -s, -s+1, \ldots, s$, and $s = \frac{1}{2}, 1, \frac{3}{2}, \ldots$. In other words, $X_{(i)}$ is an operator on N copies of $\mathbb{C}^{2s+1}$, which acts as X on copies i and $i+1$, and otherwise as the identity operator,

$$X_{(i)} = \mathbb{I}^{\otimes(i-1)} \otimes X \otimes \mathbb{I}^{\otimes(N-i-1)}\,, \tag{2.3}$$

where $\mathbb{I}$ is the identity operator on $\mathbb{C}^{2s+1}$. These operators satisfy the TL algebra (1.1), where c is given by

$$c = [2s+1]_Q = \frac{Q^{2s+1} - Q^{-2s-1}}{Q - Q^{-1}} = \sum_{k=-s}^{s} Q^{2k} \equiv -\left(q + \frac{1}{q}\right). \tag{2.4}$$

We assume throughout this paper that Q has a generic value.

The Hamiltonian (1.2) is integrable for any value of spin s. In the notation of [12], the corresponding R-matrix is given by [28]

$$R(u) = \left(uq - \frac{1}{uq}\right)\mathcal{P} + \left(u - \frac{1}{u}\right)\mathcal{P}X\,, \tag{2.5}$$

where $\mathcal{P}$ is the permutation matrix on $\mathbb{C}^{2s+1} \otimes \mathbb{C}^{2s+1}$. Indeed, the Yang–Baxter equation

$$R_{12}(u_1/u_2)\, R_{13}(u_1/u_3)\, R_{23}(u_2/u_3) = R_{23}(u_2/u_3)\, R_{13}(u_1/u_3)\, R_{12}(u_1/u_2) \tag{2.6}$$

is satisfied. This R-matrix has the unitarity property

$$R_{12}(u) R_{21}(u^{-1}) = \zeta(u)\,\mathbb{I}^{\otimes 2}\,, \qquad \zeta(u) = \omega(uq^{-1})\,\omega(u^{-1}q^{-1})\,, \tag{2.7}$$

where $R_{21} = \mathcal{P}_{12}\, R_{12}\, \mathcal{P}_{12} = R_{12}^{t_1 t_2}$, and $\omega(u)$ is defined as

$$\omega(u) = u - \frac{1}{u}\,. \tag{2.8}$$

This R-matrix also has crossing symmetry

$$R_{12}(u) = V_1 R_{12}^{t_2}(-u^{-1}q^{-1}) V_1 \,, \tag{2.9}$$

where V is an anti-diagonal matrix with elements

$$V_{jk} = (-1)^j Q^{s+1-j} \delta_{j+k,2s+2} \,. \tag{2.10}$$

The model (1.2) is an *open* spin chain. For an integrable open spin chain, the transfer matrix is given by [29]

$$t(u) = \mathrm{tr}_0 \, K_0^+(u) \, T_0(u) \, K_0^-(u) \, \hat{T}_0(u) \,, \tag{2.11}$$

where $T_0(u)$ and $\hat{T}_0(u)$ are the monodromy matrices

$$T_0(u) = R_{0N}(u) \cdots R_{01}(u) \,, \qquad \hat{T}_0(u) = R_{10}(u) \cdots R_{N0}(u) \,. \tag{2.12}$$

Moreover, the K-matrices (endomorphisms of $\mathbb{C}^{2s+1}$) satisfy the boundary Yang–Baxter equations

$$R_{12}(u/v) \, K_1^-(u) \, R_{21}(uv) \, K_2^-(v) = K_2^-(v) \, R_{12}(uv) \, K_1^-(u) \, R_{21}(u/v) \tag{2.13}$$

and [30]

$$\begin{aligned} &R_{12}(v/u) \, K_1^{+\,t_1}(u) \, M_1^{-1} \, R_{21}(u^{-1}v^{-1}q^{-2}) \, M_1 \, K_2^{+\,t_2}(v) \\ &\quad = K_2^{+\,t_2}(v) \, M_1 \, R_{12}(u^{-1}v^{-1}q^{-2}) \, M_1^{-1} \, K_1^{+\,t_1}(u) \, R_{21}(v/u) \,, \end{aligned} \tag{2.14}$$

where M is the diagonal matrix given by

$$M = V^t \, V = \mathrm{diag}(Q^{-2s} \,, Q^{-2(s-1)} \,, \ldots \,, Q^{2s}) \,. \tag{2.15}$$

The Hamiltonian (1.2) corresponds to the special case with quantum-group invariance [30]

$$K^- = \mathbb{I} \,, \qquad K^+ = M \,, \tag{2.16}$$

and therefore the transfer matrix (2.11) takes the simpler form

$$t(u) = \mathrm{tr}_0 \, M_0 \, T_0(u) \, \hat{T}_0(u) \,. \tag{2.17}$$

Indeed, the Hamiltonian is related to the transfer matrix as follows

$$H = \alpha \frac{d}{du} t(u) \Big|_{u=1} + \beta \, \mathbb{I}^{\otimes N} \,, \tag{2.18}$$

where

$$\alpha = -\left[4\omega(q^2) \, \omega(q)^{2N-2} \right]^{-1} \,, \qquad \beta = \frac{\omega(q)}{\omega(q^2)} - \frac{N}{2} \frac{\omega(q^2)}{\omega(q)} \,. \tag{2.19}$$

The higher conserved quantities can be obtained by taking higher derivatives of the transfer matrix. These quantities commute with each other by virtue of the commutativity property [29]

$$[t(u) \,, t(v)] = 0 \,. \tag{2.20}$$

The transfer matrix also has crossing symmetry [31]

$$t(u) = t(-u^{-1}q^{-1}) \,. \tag{2.21}$$

3. Analytical Bethe ansatz

We now proceed to determine the eigenvalues of the transfer matrix (2.17) by analytical Bethe ansatz [31–34]. To this end, it is convenient to introduce inhomogeneities $\{\theta_j\}$, i.e. to consider instead the inhomogeneous transfer matrix

$$t(u;\{\theta_j\}) = \mathrm{tr}_0\, M_0\, T_0(u;\{\theta_j\})\, \hat{T}_0(u;\{\theta_j\})\,, \tag{3.1}$$

where

$$T_0(u;\{\theta_j\}) = R_{0N}(u/\theta_N)\cdots R_{01}(u/\theta_1)\,, \qquad \hat{T}_0(u;\{\theta_j\}) = R_{10}(u\theta_1)\cdots R_{N0}(u\theta_N)\,. \tag{3.2}$$

As noted in [12], the R-matrix (2.5) degenerates into a one-dimensional projector at $u=q^{-1}$,

$$R(q^{-1}) = (q^{-1}-q)(2s+1)(-1)^{2s}\, P^-\,, \qquad P^- = \frac{(-1)^{2s}}{2s+1}\mathcal{P}X\,, \qquad (P^-)^2 = P^-\,. \tag{3.3}$$

Hence, we can use the fusion procedure [35,36], as generalized to the case of boundaries in [37], to obtain the fusion formula

$$t(u;\{\theta_j\})\, t(uq;\{\theta_j\}) = \frac{1}{\zeta(u^2q^2)}\left[\tilde{t}(u;\{\theta_j\}) + f(u)\,\mathbb{I}^{\otimes N}\right], \tag{3.4}$$

where $\tilde{t}(u;\{\theta_j\})$ is a fused transfer matrix, and the scalar function $f(u)$ is given by a product of quantum determinants

$$\begin{aligned} f(u) &= \Delta(K^+)\,\Delta(K^-)\,\delta(T(u))\,\delta(\hat{T}(u)) \\ &= g(u^{-2}q^{-3})\, g(u^2q) \prod_{i=1}^{N}\left[\zeta(uq/\theta_i)\,\zeta(uq\theta_i)\right], \end{aligned} \tag{3.5}$$

where $g(u)$ is given by

$$g(u) = \mathrm{tr}_{12}\, R_{12}(u) V_1 V_2 P_{12}^- = (-1)^{2s+1}\omega(uq^{-1})\,. \tag{3.6}$$

Using the fact that the fused transfer matrix vanishes when evaluated at $q^{-1}\theta_i$, i.e.

$$\tilde{t}(q^{-1}\theta_i;\{\theta_j\}) = 0\,, \qquad i=1,\ldots,N\,, \tag{3.7}$$

it follows from (3.4) that the fundamental transfer matrix (3.1) satisfies a set of exact functional relations

$$t(q^{-1}\theta_i;\{\theta_j\})\, t(\theta_i;\{\theta_j\}) = F(q^{-1}\theta_i)\,\mathbb{I}^{\otimes N}\,, \qquad i=1,\ldots,N\,, \tag{3.8}$$

where

$$F(u) = \frac{f(u)}{\zeta(u^2q^2)} = -\frac{\omega(u^2)\,\omega(u^2q^4)}{\omega(u^2q)\,\omega(u^{-2}q^{-3})} \prod_{i=1}^{N}\left[\omega(u/\theta_i)\,\omega(uq^2/\theta_i)\,\omega(u\theta_i)\,\omega(uq^2\theta_i)\right]. \tag{3.9}$$

Let us denote the eigenvalues of $t(u;\{\theta_j\})$ by $\Lambda(u;\{\theta_j\})$. From (2.21) it follows that the eigenvalues have crossing symmetry

$$\Lambda(u;\{\theta_j\}) = \Lambda(-u^{-1}q^{-1};\{\theta_j\})\,; \tag{3.10}$$

and from (3.8) it follows that the eigenvalues obey the functional relations

$$\Lambda(q^{-1}\theta_i;\{\theta_j\})\,\Lambda(\theta_i;\{\theta_j\}) = F(q^{-1}\theta_i)\,, \qquad i = 1,\dots,N\,. \tag{3.11}$$

To solve these equations, we introduce the functions

$$\begin{aligned} a(u;\{\theta_j\}) &= -\frac{\omega(u^2q^2)}{\omega(u^2q)}\prod_{i=1}^{N}\left[\omega(uq/\theta_i)\,\omega(uq\theta_i)\right]\,, \\ d(u;\{\theta_j\}) &= -\frac{\omega(u^2)}{\omega(u^2q)}\prod_{i=1}^{N}\left[\omega(u/\theta_i)\,\omega(u\theta_i)\right] = a(-u^{-1}q^{-1};\{\theta_j\})\,, \end{aligned} \tag{3.12}$$

which have the properties

$$a(q^{-1}\theta_i;\{\theta_j\}) = 0 = d(\theta_i;\{\theta_j\})\,, \qquad a(\theta_i;\{\theta_j\})\,d(q^{-1}\theta_i;\{\theta_j\}) = F(q^{-1}\theta_i)\,. \tag{3.13}$$

It is easy to see that equations (3.10) and (3.11) are satisfied by

$$\Lambda(u;\{\theta_j\}) = a(u;\{\theta_j\})\,\frac{\mathcal{Q}(uq^{-1})}{\mathcal{Q}(u)} + d(u;\{\theta_j\})\,\frac{\mathcal{Q}(uq)}{\mathcal{Q}(u)}\,, \tag{3.14}$$

where $\mathcal{Q}(u)$ is any crossing-invariant function

$$\mathcal{Q}(u) = \mathcal{Q}(-u^{-1}q^{-1})\,. \tag{3.15}$$

From the form (2.5) of the R-matrix and the commutativity property (2.20), it follows that $\Lambda(u;\{\theta_j\})$ must be a Laurent polynomial in u (with a finite number of terms). We assume that $\mathcal{Q}(u)$ is also a Laurent polynomial, and is given by

$$\mathcal{Q}(u) = \prod_{k=1}^{M}\omega(u/u_k)\,\omega(uqu_k)\,, \tag{3.16}$$

where the so-called Bethe roots $\{u_1,\dots,u_M\}$ are still to be determined, which is consistent with (3.15). Obviously $\mathcal{Q}(u_k) = 0$, which means that both terms in the expression (3.14) for $\Lambda(u;\{\theta_j\})$ have a simple pole at $u = u_k$. (We assume that the Bethe roots are distinct, and are not equal to 0 or ∞.) The corresponding residues must cancel (since $\Lambda(u;\{\theta_j\})$ must be finite for u not equal to 0 or ∞), which implies the so-called Bethe equations

$$\frac{a(u_k;\{\theta_j\})}{d(u_k;\{\theta_j\})} = -\frac{\mathcal{Q}(u_kq)}{\mathcal{Q}(u_kq^{-1})}\,, \qquad k = 1,\dots,M\,. \tag{3.17}$$

Since we no longer need the inhomogeneities, we now set them to unity $\theta_j = 1$.

To summarize, we have argued that the eigenvalues $\Lambda(u)$ of the transfer matrix $t(u)$ (2.17) are given by

$$\begin{aligned} \Lambda(u) = -\frac{1}{\omega(u^2q)}\Bigg[&\omega(u^2q^2)\,\omega(uq)^{2N}\prod_{j=1}^{M}\frac{\omega(uq^{-1}/u_j)\,\omega(uu_j)}{\omega(u/u_j)\,\omega(uqu_j)} \\ &+ \omega(u^2)\,\omega(u)^{2N}\prod_{j=1}^{M}\frac{\omega(uq/u_j)\,\omega(uq^2u_j)}{\omega(u/u_j)\,\omega(uqu_j)}\Bigg], \end{aligned} \tag{3.18}$$

Table 1
Solutions $\{u_k\}$ of the Bethe equations (3.19) and degeneracies of the corresponding eigenvalues (3.18) for $N=2$ and $s=\frac{1}{2}, 1, \frac{3}{2}$ with $q=0.5$.

M	$\{u_k\}$	Degeneracies		
		$s=\frac{1}{2}$	$s=1$	$s=\frac{3}{2}$
0	–	3	8	15
1	$1.34164+0.447214i$	1	1	1
	total:	4	9	16

Table 2
Solutions $\{u_k\}$ of the Bethe equations (3.19) and degeneracies of the corresponding eigenvalues (3.18) for $N=3$ and $s=\frac{1}{2}, 1, \frac{3}{2}$ with $q=0.5$.

M	$\{u_k\}$	Degeneracies		
		$s=\frac{1}{2}$	$s=1$	$s=\frac{3}{2}$
0	–	4	21	56
1	$1.22474+0.707107i$	2	3	4
1	$1.38873+0.267261i$	2	3	4
	total:	8	27	64

and the Bethe roots are given by the Bethe equations

$$\left[\frac{\omega(u_k q)}{\omega(u_k)}\right]^{2N} = \prod_{\substack{j\neq k\\ j=1}}^{M} \frac{\omega(u_k u_j^{-1} q)\,\omega(u_k u_j q^2)}{\omega(u_k u_j^{-1} q^{-1})\,\omega(u_k u_j)}\,. \tag{3.19}$$

These equations take a more symmetric form in terms of the rescaled Bethe roots $\tilde{u}_k \equiv u_k q^{1/2}$:

$$\left[\frac{\omega(\tilde{u}_k q^{1/2})}{\omega(\tilde{u}_k q^{-1/2})}\right]^{2N} = \prod_{\substack{j\neq k\\ j=1}}^{M} \frac{\omega(\tilde{u}_k \tilde{u}_j^{-1} q)\,\omega(\tilde{u}_k \tilde{u}_j q)}{\omega(\tilde{u}_k \tilde{u}_j^{-1} q^{-1})\,\omega(\tilde{u}_k \tilde{u}_j q^{-1})}\,. \tag{3.20}$$

Note that the results (3.18)–(3.20) do not depend on the value of s; in particular, they coincide with the well-known results for the case $s=1/2$ [4,31]. This is consistent with the TL equivalence (see e.g. [1,7,38–40]), which suggests that the spectrum (but not the degeneracies) of the TL Hamiltonian (1.2) is independent of the representation.

We have verified numerically for small values of N and s that every distinct eigenvalue of the transfer matrix can be expressed in the form (3.18). See Tables 1–3, and note that all $(2s+1)^N$ eigenvalues are accounted for.

In closing this section, we note that the eigenvalues of the Hamiltonian (1.2) are given by

$$E = \alpha \frac{d}{du}\Lambda(u)\Big|_{u=1} + \beta = \frac{1}{2}\omega(q) \sum_{j=1}^{M} \left[\frac{\omega(u_j^2)}{\omega(u_j)^2} - \frac{\omega(u_j^2 q^2)}{\omega(u_j q)^2}\right], \tag{3.21}$$

as follows from (2.18), (2.19) and (3.18).

Table 3
Solutions $\{u_k\}$ of the Bethe equations (3.19) and degeneracies of the corresponding eigenvalues (3.18) for $N = 4$ and $s = \frac{1}{2}, 1, \frac{3}{2}$ with $q = 0.5$.

M	$\{u_k\}$	Degeneracies		
		$s = \frac{1}{2}$	$s = 1$	$s = \frac{3}{2}$
0	–	5	55	209
1	$1.10176 + 0.886631i$	3	8	15
1	$1.34164 + 0.447214i$	3	8	15
1	$1.40092 + 0.193427i$	3	8	15
2	$1.81555 - 0.854196i$, $1.81555 + 0.854196i$	1	1	1
2	$1.28401 + 0.592723i$, $1.3969 + 0.220635i$	1	1	1
	total:	16	81	256

4. Quantum group symmetry

In this section we demonstrate the quantum-group invariance of the transfer matrix, and we discuss the implications of this symmetry for the Bethe ansatz solution.

4.1. Symmetry of the transfer matrix

Let us denote by $R^{\pm}$ the asymptotic limits of the R-matrix $R(u)$ (2.5)

$$R^{+} = \lim_{u\to\infty} \frac{1}{u} R(u) = \mathcal{P}(q + X)\,,$$
$$R^{-} = \lim_{u\to 0} -u\, R(u) = \mathcal{P}(q^{-1} + X)\,, \tag{4.1}$$

and let us similarly denote by $T_0^{\pm}$ the asymptotic limits of the monodromy matrix $T_0(u)$ (2.12)

$$T_0^{\pm} = R_{0N}^{\pm} \cdots R_{01}^{\pm}\,. \tag{4.2}$$

Regarding $T_0^{\pm}$ as a $(2s+1) \times (2s+1)$ matrix in the auxiliary space, its matrix elements $T_{ij}^{\pm}$ (which are operators on the quantum space $\left(\mathbb{C}^{2s+1}\right)^{\otimes N}$) define a quantum group, which has been identified in [41] as $U_q(2s+1)$.[2] We shall demonstrate that each of these matrix elements commutes with the transfer matrix

$$\left[T_{ij}^{\pm}\,, t(u)\right] = 0\,, \qquad i, j = 1, 2, \ldots, 2s+1\,. \tag{4.3}$$

The proof, which is similar to the one in [42] (see also [43]), requires two lemmas:

Lemma 1.

$$\left[R_{12}^{\pm}\, T_1^{\pm}\,, T_2(u)\, \hat{T}_2(u)\right] = 0\,. \tag{4.4}$$

Proof. We recall the fundamental relation

$$R_{12}(u_1/u_2)\, T_1(u_1)\, T_2(u_2) = T_2(u_2)\, T_1(u_1)\, R_{12}(u_1/u_2)\,. \tag{4.5}$$

[2] For $s > 1/2$, this symmetry is larger than the $U_Q sl(2)$ symmetry [2], which is also present.

Taking asymptotic limits of u_1 yields

$$R_{12}^{\pm}\, T_1^{\pm}\, T_2(u) = T_2(u)\, T_1^{\pm}\, R_{12}^{\pm}\,, \tag{4.6}$$

which further implies

$$T_2^{-1}(u)\, R_{12}^{\pm}\, T_1^{\pm} = T_1^{\pm}\, R_{12}^{\pm}\, T_2^{-1}(u)\,. \tag{4.7}$$

Therefore,

$$\begin{aligned} R_{12}^{\pm}\, T_1^{\pm}\, T_2(u)\, T_2^{-1}(u^{-1}) &= T_2(u)\, T_1^{\pm}\, R_{12}^{\pm}\, T_2^{-1}(u^{-1}) \\ &= T_2(u)\, T_2^{-1}(u^{-1})\, R_{12}^{\pm}\, T_1^{\pm}\,, \end{aligned} \tag{4.8}$$

where the first equality follows from (4.6), and the second equality follows from (4.7). We have therefore shown the commutativity property

$$\left[R_{12}^{\pm}\, T_1^{\pm}\,,\, T_2(u)\, T_2^{-1}(u^{-1})\right] = 0\,. \tag{4.9}$$

Furthermore,

$$\begin{aligned} T_0^{-1}(u) &= R_{01}^{-1}(u) \cdots R_{0N}^{-1}(u) \\ &\propto R_{10}(u^{-1}) \cdots R_{N0}(u^{-1}) = \hat{T}_0(u^{-1})\,, \end{aligned} \tag{4.10}$$

where the second line follows from unitarity (2.7). Substituting into (4.9) we obtain the desired result (4.4). □

Lemma 2.

$$M_1^{-1}\left((R_{12}^{\pm})^{-1}\right)^{t_2} M_1\, R_{12}^{\pm\, t_2} = \mathbb{I}^{\otimes 2}\,. \tag{4.11}$$

Proof. We write the unitarity condition (2.7) as

$$R_{12}(u)\, R_{12}^{t_1 t_2}(u^{-1}) = \zeta(u)\, \mathbb{I}^{\otimes 2}\,, \tag{4.12}$$

and then use crossing symmetry (2.9) to obtain

$$V_1\, R_{12}^{t_2}(-u^{-1}q^{-1})\, V_1\, V_1^{t_1}\, R_{12}^{t_1}(-uq^{-1})\, V_1^{t_1} = \zeta(u)\, \mathbb{I}^{\otimes 2}\,. \tag{4.13}$$

By taking asymptotic limits and noting that $V^2 = (-1)^{2s}\, \mathbb{I}$, we obtain

$$R_{12}^{\pm\, t_2}\, M_1^{-1}\, R_{12}^{\mp\, t_1}\, M_1 = \mathbb{I}^{\otimes 2}\,. \tag{4.14}$$

Moreover, from (4.12) we obtain $R_{12}^{\pm}\, R_{12}^{\mp\, t_1 t_2} = \mathbb{I}$, which implies that

$$R_{12}^{\mp\, t_1 t_2} = (R_{12}^{\pm})^{-1}\,, \qquad \text{or} \qquad R_{12}^{\mp\, t_1} = \left((R_{12}^{\pm})^{-1}\right)^{t_2}\,. \tag{4.15}$$

Substituting into (4.14), we obtain

$$R_{12}^{\pm\, t_2}\, M_1^{-1}\left((R_{12}^{\pm})^{-1}\right)^{t_2} M_1 = \mathbb{I}^{\otimes 2}\,, \tag{4.16}$$

which can be rearranged to give the desired result (4.11). □

We are now ready to prove the main result (4.3), which is equivalent to the following

Proposition.

$$\left[T_1^{\pm}\,, t(u)\right] = 0\,. \tag{4.17}$$

Proof. Recalling the expression (2.17) for the transfer matrix, we obtain

$$\begin{aligned} T_1^{\pm}\, t(u) &= \mathrm{tr}_2 \left\{T_1^{\pm}\, M_2\, T_2(u)\, \hat{T}_2(u)\right\} \\ &= \mathrm{tr}_2 \left\{M_1^{-1}\, M_1\, M_2\, (R_{12}^{\pm})^{-1}\, R_{12}^{\pm}\, T_1^{\pm}\, T_2(u)\, \hat{T}_2(u)\right\} \\ &= \mathrm{tr}_2 \left\{M_1^{-1}\, (R_{12}^{\pm})^{-1}\, M_1\, M_2\, T_2(u)\, \hat{T}_2(u)\, R_{12}^{\pm}\, T_1^{\pm}\right\} = \ldots \end{aligned} \tag{4.18}$$

In passing to the third line, we have used the fact $\left[M_1\, M_2\,, R_{12}^{\pm}\right] = 0$ as well as the first lemma (4.4). Then

$$\begin{aligned} \ldots &= \mathrm{tr}_2 \left\{M_1^{-1}\, (R_{12}^{\pm})^{-1}\, M_1\, M_2\, T_2(u)\, \hat{T}_2(u)\, R_{12}^{\pm}\right\} T_1^{\pm} \\ &= \mathrm{tr}_2 \left\{A_{12}\, Z_2\, R_{12}^{\pm}\,\right\} T_1^{\pm} \\ &= \mathrm{tr}_2 \left\{A_{12}^{t_2}\, R_{12}^{\pm\, t_2}\, Z_2^{t_2}\right\} T_1^{\pm} = \ldots \end{aligned} \tag{4.19}$$

In passing to the second line we have made the identifications $A_{12} = M_1^{-1}\, (R_{12}^{\pm})^{-1}\, M_1$ and $Z_2 = M_2\, T_2(u)\, \hat{T}_2(u)$. Finally, we obtain

$$\begin{aligned} \ldots &= \mathrm{tr}_2 \left\{M_1^{-1}\, \left((R_{12}^{\pm})^{-1}\right)^{t_2}\, M_1\, R_{12}^{\pm\, t_2}\, Z_2^{t_2}\right\} T_1^{\pm} \\ &= \mathrm{tr}_2 \left\{Z_2^{t_2}\right\} T_1^{\pm} \\ &= t(u)\, T_1^{\pm}\,. \end{aligned} \tag{4.20}$$

In passing to the second line we have used the second lemma (4.11). □

4.2. Degeneracies and multiplicities

The $U_q(2s+1)$ symmetry of the transfer matrix implies that its eigenstates form representations of this algebra. The space of states has the decomposition (see e.g. [40,12,44,41,15] and references therein)

$$\left(\mathbb{C}^{2s+1}\right)^{\otimes N} = \bigoplus_{k=0(1)}^{N} \nu_k\, V_k\,, \tag{4.21}$$

where the summation is over even (odd) integers for even (odd) N, respectively; V_k are representations of $U_q(2s+1)$; and ν_k are the multiplicities. The dimensions of the representations are given by

$$\dim V_k = p_k(2s+1)\,, \tag{4.22}$$

where $p_k(x)$ are Chebyshev polynomials of the second kind, which are defined by the recurrence relations

$$p_{k+1}(x) + p_{k-1}(x) = x\, p_k(x)\,, \qquad p_0(x) = 1\,, \qquad p_{-1}(x) = 0\,. \tag{4.23}$$

Table 4
Dimensions (4.22) and multiplicities ν_k (4.24) of representations V_k for $N = 2$ and $s = \frac{1}{2}, 1, \frac{3}{2}$.

k	ν_k	$\dim V_k$		
		$s = \frac{1}{2}$	$s = 1$	$s = \frac{3}{2}$
0	1	1	1	1
2	1	3	8	15

The multiplicities are given by[3]

$$\nu_k = \begin{cases} \binom{N}{\frac{N-k}{2}} - \binom{N}{\frac{N-k}{2}-1} & k \neq 0, N \\ \frac{1}{\frac{N}{2}+1}\binom{N}{\frac{N}{2}} & k = 0 \\ 1 & k = N \end{cases}, \tag{4.24}$$

which are the dimensions of representations W_k of the TL algebra TL_N. As a check on (4.21)–(4.24), one can verify that the sum rule

$$\sum_{k=0(1)}^{N} \nu_k \dim V_k = (2s+1)^N \tag{4.25}$$

is satisfied.

For given values of N and s, let $\mathcal{N}(N, M)$ denote the number of solutions of the Bethe equations (3.19) with M roots, and let $\mathcal{D}(N, s, M)$ denote the corresponding degeneracy, i.e., the number of transfer-matrix eigenvalues (3.18) corresponding to each solution of the Bethe equations with M roots. We propose that $\mathcal{N}(N, M)$ and $\mathcal{D}(N, s, M)$ are related to ν_k and $\dim V_k$ in the following simple way[4]:

$$\mathcal{N}(N, M) = \nu_k\,, \tag{4.26}$$

$$\mathcal{D}(N, s, M) = \dim V_k\,, \tag{4.27}$$

with

$$M = \frac{1}{2}(N-k)\,. \tag{4.28}$$

We have verified these relations for small values of N and s. See e.g. Tables 4–6, and compare with Tables 1–3, respectively.

5. Algebraic Bethe ansatz

We present here several conjectures related to the algebraic Bethe ansatz solution of the TL chain. The conjecture for the off-shell equation has been proved for $s = 1$ [45], while the other conjectures have been checked numerically (up to $M = 3$, $N = 6$ and $s = \frac{3}{2}$).

[3] Note that the multiplicities ν_k are independent of s.

[4] We note that [15] does not discuss either the open-chain Bethe equations (3.19) or the open-chain transfer matrix (2.17), and therefore does not contain the results (4.26)–(4.28).

Table 5
Dimensions (4.22) and multiplicities ν_k (4.24) of representations V_k for $N=3$ and $s=\frac{1}{2}, 1, \frac{3}{2}$.

k	ν_k	$\dim V_k$		
		$s=\frac{1}{2}$	$s=1$	$s=\frac{3}{2}$
1	2	2	3	4
3	1	4	21	56

Table 6
Dimensions (4.22) and multiplicities ν_k (4.24) of representations V_k for $N=4$ and $s=\frac{1}{2}, 1, \frac{3}{2}$.

k	ν_k	$\dim V_k$		
		$s=\frac{1}{2}$	$s=1$	$s=\frac{3}{2}$
0	2	1	1	1
2	3	3	8	15
4	1	5	55	209

5.1. *Off-shell equation*

In order to implement the algebraic Bethe ansatz, we need to choose a convenient representation in the auxiliary space for the double-row monodromy matrix $T_0(u)\,\hat{T}_0(u)$. We choose

$$T_0(u)\,\hat{T}_0(u) = \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}_{1,2}(u) & \cdots & \mathcal{B}_{1,2s}(u) & \mathcal{B}(u) \\ \mathcal{C}_{2,1}(u) & \mathcal{A}_2(u) & \cdots & \mathcal{B}_{2,2s}(u) & \mathcal{B}_{2,2s+1}(u) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{C}_{2s,1}(u) & \mathcal{C}_{2s,2}(u) & \cdots & \mathcal{A}_{2s}(u) & \mathcal{B}_{2s,2s+1}(u) \\ \mathcal{C}(u) & \mathcal{C}_{2s+1,2}(u) & \cdots & \mathcal{C}_{2s+1,2s}(u) & \tilde{\mathcal{D}}(u) \end{pmatrix}_{(2s+1)\times(2s+1)}, \tag{5.1}$$

where each entry acts on the quantum space $\left(\mathbb{C}^{2s+1}\right)^{\otimes N}$. Let us also introduce the $(2s+1)$-dimensional reference state

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^{\otimes N} \tag{5.2}$$

and its dual

$$\langle 0| = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^{\otimes N} \tag{5.3}$$

such that $\langle 0|0\rangle = 1$. We have found that the (dual) Bethe vectors are generated by the action of a single double-row operator, namely $(\mathcal{C}(u))\,\mathcal{B}(u)$. Indeed, let us define the Bethe vector as

$$|u_1, \ldots, u_M\rangle = \prod_{k=1}^{M} \mathcal{B}(u_k)|0\rangle \tag{5.4}$$

as well as its dual

$$\langle u_1, \ldots, u_M| = \langle 0| \prod_{k=1}^{M} \mathcal{C}(u_k)\,. \tag{5.5}$$

We conjecture that the action of the transfer matrix (2.17) on (5.4) is given by

$$\begin{aligned} t(u)|u_1, \ldots, u_M\rangle = \Lambda(u; u_1, \ldots, u_M)|u_1 \ldots u_M\rangle \\ + \sum_{k=1}^{M} \lambda_k |u_1, \ldots, u_{k-1}, u, u_{k+1}, \ldots, u_M\rangle\,, \end{aligned} \tag{5.6}$$

while the action on (5.5) is given by,

$$\begin{aligned} \langle u_1, \ldots, u_M| t(u) = \langle u_1, \ldots, u_M|\Lambda(u; u_1, \ldots, u_M) \\ + \sum_{k=1}^{M} \langle u_1, \ldots, u_{k-1}, u, u_{k+1}, \ldots, u_M|\lambda_k\,, \end{aligned} \tag{5.7}$$

where

$$\begin{aligned} \lambda_k = -\frac{\omega(q)\omega(u^2q^2)\omega(u_k^2)}{\omega(uu_k^{-1})\omega(uu_kq)\omega(u_k^2q)} \\ \times \left[\omega(u_kq)^{2N} \prod_{\substack{j\neq k\\ j=1}}^{M} \frac{\omega(u_ku_j^{-1}q^{-1})\omega(u_ku_j)}{\omega(u_ku_j^{-1})\omega(u_ku_jq)} - \omega(u_k)^{2N} \prod_{\substack{j\neq k\\ j=1}}^{M} \frac{\omega(u_ku_j^{-1}q)\omega(u_ku_jq^2)}{\omega(u_ku_j^{-1})\omega(u_ku_jq)} \right], \end{aligned} \tag{5.8}$$

and $\Lambda(u; u_1, \ldots, u_M)$ is given by (3.18). Since the equations (5.6) and (5.7) are valid for arbitrary $\{u_k\}$, we write explicitly the dependence of Λ on $\{u_k\}$. Evidently, $\lambda_k = 0$ when the Bethe equations (3.19) are satisfied, in which case the Bethe states (5.4) and (5.5) are right and left eigenstates of the transfer matrix $t(u)$, respectively, with corresponding eigenvalue $\Lambda(u; u_1, \ldots, u_M)$. For the $s = \frac{1}{2}$ case the results (5.6) and (5.7) are known [29]; for the $s = 1$ case a proof will be reported in a separate paper [45].

5.2. *Highest-weight property*

When the Bethe states (5.4) are on shell (i.e., when $\{u_1, \ldots, u_M\}$ satisfy the Bethe equations (3.19)), we conjecture that

$$T^+_{ii}|u_1, \ldots, u_M\rangle = h_i|u_1, \ldots, u_M\rangle\,, \qquad i = 1, 2, \ldots, 2s+1\,, \tag{5.9}$$

$$T^+_{ij}|u_1, \ldots, u_M\rangle = 0\,, \qquad i > j\,, \tag{5.10}$$

where T^+_{ij} are quantum group generators defined in section 4.1. That is, on-shell Bethe states are highest-weight states of the quantum group, in the sense that they are eigenstates of the diagonal elements of T^+, and are annihilated by the lower triangular elements of T^+. This would help account for the observations in section 4.2 that the degeneracies and multiplicities are given by group theory.

5.3. Scalar products

Let us suppose that $\{u_1, \dots, u_M\}$ are Bethe roots. We propose that the scalar product between the on-shell state $\langle u_1, \dots, u_M|$ and an arbitrary off-shell state $|v_1, \dots, v_M\rangle$ is given by

$$\langle u_1, \dots, u_M|v_1, \dots, v_M\rangle = \left(\frac{1}{2Q^{2s}}\right)^M \prod_{i=1}^{M} \frac{\omega(u_i)^{2N} u_i\, \omega(u_i^2)}{\omega(u_i^2 q)\omega(v_i^2 q^2)} \prod_{j<i}^{M} \frac{\omega(u_i u_j q^2)}{\omega(u_i u_j)} \frac{\mathrm{Det}_M\left(\frac{\partial}{\partial u_i}\Lambda(v_j; u_1, \dots, u_M)\right)}{\mathrm{Det}_M\left(\frac{1}{\omega(v_i u_j^{-1})\omega(v_i u_j q)}\right)}. \tag{5.11}$$

The formula (5.11) was proved in [46] for the $s = \frac{1}{2}$ case with (diagonal) boundary fields (see also [47] for the XXX chain). Performing the limit $v_k \to u_k$, we obtain the square of the norm, namely,

$$\langle u_1, \dots, u_M|u_1, \dots, u_M\rangle = \left(\frac{\omega(q)\omega(-q^2)}{Q^{2s}}\right)^M \prod_{i=1}^{M} \omega(u_i)^{4N} \omega(u_i^2)^2 \prod_{j<i}^{M} \frac{\omega(u_i u_j q^2)}{\omega(u_j u_i^{-1})\omega(u_i u_j^{-1})\omega(u_i u_j)\omega(u_i u_j q)^2} \times \mathrm{Det}_M(G)\,, \tag{5.12}$$

where G is a $M \times M$ matrix with elements

$$G_{ij} = \frac{\prod_{k\neq i,j}^{M} \omega(u_j u_k^{-1} q)\omega(u_j u_k q^2)}{\omega(u_j u_i^{-1} q^{-1})\omega(u_i u_j)} \Bigg[1 - \delta_{i,j} + \delta_{i,j} \frac{\omega(q)\omega(u_i^2)}{\omega(q^2)\omega(u_i^2 q)^2} \times \left(-\frac{2N\omega(q)}{\omega(u_i)\omega(u_i q)} + \omega(q^2) \sum_{k\neq i}^{M} \frac{1}{\omega(u_i q^{-1} u_k^{-1})\omega(u_i q u_k^{-1})} + \frac{1}{\omega(u_i u_k)\omega(u_i u_k q^2)}\right)\Bigg]. \tag{5.13}$$

6. Discussion

We have considered the TL open quantum spin chain associated with the spin-s representation of quantum-deformed $sl(2)$. We have constructed the transfer matrix (2.17), and we have seen that its eigenvalues (3.18) and the corresponding Bethe equations (3.19) do not depend on the value of the spin. Due to the quantum-group invariance of the transfer matrix (4.3), (4.17), the number of solutions of the Bethe equations (4.26) and the degeneracies of the transfer-matrix eigenvalues (4.27) can be inferred from group theory.

We have proposed an algebraic Bethe ansatz construction of the Bethe vectors (5.4) and (5.5), and the corresponding off-shell equations (5.6) and (5.7), respectively. Remarkably, despite the fact that the auxiliary space has dimension greater than 2 for $s > 1/2$, a single creation operator suffices to construct all the Bethe states – no nesting is needed. We have also proposed a determinant formula for the scalar products between off-shell and on-shell Bethe states (5.11). Remarkably, these results are also universal in the sense that they depend on the value of the spin only through a constant factor. It is important to find proofs for these conjectures. So far, we have been able to prove only the off-shell equations for $s = 1$ [45].

We have seen that appropriate boundary conditions are necessary for the TL model to have a universal solution. Indeed, for periodic boundary conditions, the solution (A.11) is no longer universal. This solution has the unusual feature that it has a twist that is "dynamically" generated (i.e., the twist is not a fixed parameter of the model, as is typically the case). An algebraic Bethe ansatz solution for this model remains to be found. It may be interesting to consider generalizations of the open TL chain (1.2) which are still integrable but have boundary terms that break the quantum group symmetry [48,49].

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Appendix A. Closed TL chain

Let $t(u)$ now denote the transfer matrix for the closed TL chain with periodic boundary conditions

$$t(u) = \mathrm{tr}_0\, T_0(u)\,, \tag{A.1}$$

where the monodromy matrix $T_0(u)$ is given by (2.12). To determine the eigenvalues of $t(u)$, we follow the same approach used in Section 3 to analyze the open chain. Hence, we consider the inhomogeneous transfer matrix

$$t(u; \{\theta_j\}) = \mathrm{tr}_0\, T_0(u; \{\theta_j\})\,, \tag{A.2}$$

where $T_0(u; \{\theta_j\})$ is given by (3.2). Using the fusion procedure, we arrive at the functional relations

$$t(q^{-1}\theta_i; \{\theta_j\})\, t(\theta_i; \{\theta_j\}) = F(q^{-1}\theta_i)\, \mathbb{I}^{\otimes N}\,, \qquad i = 1, \ldots, N\,, \tag{A.3}$$

where $F(u)$ is now given by (cf. (3.9))

$$F(u) = \prod_{i=1}^{N} \left[(-1)^{2s}\omega(u/\theta_i)\,\omega(uq^2/\theta_i)\right]. \tag{A.4}$$

The corresponding eigenvalues $\Lambda(u; \{\theta_j\})$ therefore obey the same functional relations

$$\Lambda(q^{-1}\theta_i; \{\theta_j\})\, \Lambda(\theta_i; \{\theta_j\}) = F(q^{-1}\theta_i)\,, \qquad i = 1, \ldots, N\,. \tag{A.5}$$

To solve these equations, we introduce the functions

$$a(u; \{\theta_j\}) = \kappa \prod_{i=1}^{N} (-1)^s \omega(uq/\theta_i)\,, \qquad d(u; \{\theta_j\}) = \frac{1}{\kappa} \prod_{i=1}^{N} (-1)^s \omega(u/\theta_i)\,, \tag{A.6}$$

where the twist parameter κ is still to be determined. We observe that

$$a(q^{-1}\theta_i; \{\theta_j\}) = 0 = d(\theta_i; \{\theta_j\})\,, \qquad a(\theta_i; \{\theta_j\})\, d(q^{-1}\theta_i; \{\theta_j\}) = F(q^{-1}\theta_i)\,. \tag{A.7}$$

Hence, the functional relations (A.5) are satisfied by

$$\Lambda(u;\{\theta_j\}) = a(u;\{\theta_j\})\,\frac{\mathcal{Q}(uq^{-1})}{\mathcal{Q}(u)} + d(u;\{\theta_j\})\,\frac{\mathcal{Q}(uq)}{\mathcal{Q}(u)}\,, \tag{A.8}$$

where $\mathcal{Q}(u)$ is now given by

$$\mathcal{Q}(u) = \prod_{k=1}^{M} \omega(u/u_k)\,. \tag{A.9}$$

Setting the inhomogeneities to unity $\theta_j = 1$, we conclude that the eigenvalues $\Lambda(u)$ of the closed transfer matrix (A.1) are given by

$$\Lambda(u) = \kappa\,(-1)^{sN}\,\omega(uq)^N \prod_{j=1}^{M} \frac{\omega(uq^{-1}/u_j)}{\omega(u/u_j)} + \frac{1}{\kappa}\,(-1)^{sN}\,\omega(u)^N \prod_{j=1}^{M} \frac{\omega(uq/u_j)}{\omega(u/u_j)}\,, \tag{A.10}$$

and the Bethe roots are given by the Bethe equations

$$\left[\frac{\omega(u_k q)}{\omega(u_k)}\right]^N = \kappa^{-2} \prod_{\substack{j\neq k\\ j=1}}^{M} \frac{\omega(u_k u_j^{-1} q)}{\omega(u_k u_j^{-1} q^{-1})}\,. \tag{A.11}$$

A similar solution was proposed in [12], except with a trivial twist (i.e., with $\kappa = 1$). Such a twist is not expected, since the transfer matrix (A.1) corresponds to periodic boundary conditions. Nevertheless, from numerical studies (see below), we find that a nontrivial twist ($\kappa \neq 1$) is necessary in order to obtain the complete set of eigenvalues from the Bethe ansatz solution. The presence of an effective twist was already noted in earlier work, see e.g. [10,15,50,51].

We remark that the twist is characterized by an integer in $\mathbb{Z}_N$. Indeed, we observe from (2.5) that $R(1) = \omega(q)\,\mathcal{P}$. Hence, from (A.1) we obtain

$$t(1) = \omega(q)^N\, U\,, \tag{A.12}$$

where $U = \mathcal{P}_{12}\,\mathcal{P}_{23} \ldots \mathcal{P}_{N-1,N}$ is the one-site shift operator, which satisfies $U^N = \mathbb{I}^{\otimes N}$. From (A.10) we have

$$\Lambda(1) = \kappa\,(-1)^{sN}\,\omega(q)^N \prod_{j=1}^{M} \frac{\omega(qu_j)}{\omega(u_j)}\,. \tag{A.13}$$

It follows from (A.12) and (A.13) that the eigenvalue of U (which we also denote by U) is given by

$$U = \kappa\,(-1)^{sN} \prod_{j=1}^{M} \frac{\omega(qu_j)}{\omega(u_j)}\,. \tag{A.14}$$

Since $U^N = 1$, we conclude that the twist κ and the Bethe roots $\{u_j\}$ must satisfy the following constraint

$$\kappa = \frac{e^{i2\pi l/N}}{(-1)^{sN}} \prod_{j=1}^{M} \frac{\omega(u_j)}{\omega(qu_j)}\,, \qquad l = 0, 1, \ldots, N-1\,. \tag{A.15}$$

In particular, the twist is characterized by an integer $l \in \mathbb{Z}_N$

Table 7
Solutions $\{u_k\}$ of the Bethe equations (A.11), twist κ, and degeneracies $\mathcal{D}$ of the corresponding eigenvalues (A.10) for $N = 2$ and $s = \frac{1}{2}, 1, \frac{3}{2}$ with $q = 0.5$.

$s = \frac{1}{2}$				$s = 1$				$s = \frac{3}{2}$			
M	$\{u_k\}$	κ	$\mathcal{D}$	M	$\{u_k\}$	κ	$\mathcal{D}$	M	$\{u_k\}$	κ	$\mathcal{D}$
0	–	-1	2	0	–	1	5	0	–	-1	9
1	$1.41421i$	1	1	0	–	-1	2	0	–	1	5
1	1.41421	1	1	1	0.540182	0.381966	1	1	0.732051	0.267949	1
				1	1.21699	0.381966	1	1	1.1638	0.267949	1
		total:	4				9				16

Table 8
Solutions $\{u_k\}$ of the Bethe equations (A.11), twist κ, and degeneracies $\mathcal{D}$ of the corresponding eigenvalues (A.10) for $N = 3$ and $s = \frac{1}{2}, 1, \frac{3}{2}$ with $q = 0.5$.

$s = \frac{1}{2}$				$s = 1$				$s = \frac{3}{2}$			
M	$\{u_k\}$	κ	$\mathcal{D}$	M	$\{u_k\}$	κ	$\mathcal{D}$	M	$\{u_k\}$	κ	$\mathcal{D}$
0	–	i	2	0	–	-1	8	0	–	$-i$	20
1	$i\sqrt{2}e^{-i\pi/3}$	$-i$	2	0	–	$e^{i\pi/3}$	5	0	–	$ie^{i\pi/3}$	16
1	$-i\sqrt{2}e^{i\pi/3}$	$-i$	2	0	–	$e^{-i\pi/3}$	5	0	–	$ie^{-i\pi/3}$	16
1	$\sqrt{2}$	$-i$	2	1	$i\sqrt{2}$	-1	3	1	$-i\sqrt{2}e^{i\pi/3}$	i	4
				1	$(3\sqrt{3}+i)/\sqrt{14}$	-1	3	1	$i\sqrt{2}e^{-i\pi/3}$	i	4
				1	$(3\sqrt{3}-i)/\sqrt{14}$	-1	3	1	$\sqrt{2}$	i	4
		total:	8				27				64

We have verified numerically for small values of N and s that every distinct eigenvalue of the transfer matrix (A.1) can be expressed in the form (A.10). See Tables 7 and 8, and note that all $(2s+1)^N$ eigenvalues are accounted for. Note also that, in contrast with the case of the open chain, the solutions of the closed-chain Bethe equations (A.11) are not universal: the Bethe roots depend on the value of the spin s (cf. Tables 1–3).

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