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Distance in cone metric spaces and common fixed point theorems

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ABSTRACT

In this paper, we define a distance called c-distance on a cone metric space and prove a new common fixed point theorem by using the distance.

1. Introduction and preliminaries

Since the concept of cone metric space was introduced by Huang and Zhang [1], many fixed point theorems have been proved in normal or non-normal cone metric spaces by some authors; see [2–17] and references contained therein.

In this paper, we consider a new concept of c-distance on cone metric spaces, which is a cone version of the ω-distance of Kada et al. [18], and prove a new common fixed point theorem in a cone metric space by using the c-distance. Note that Saadati et al. in [19] introduced a distance called r-distance in a Menger probabilistic metric space which may be regarded as a probabilistic version of the ω-distance of Kada et al. [18].

Let E be a real Banach space and θ denote the zero element in E. A *cone* P is a subset of E such that

(i) P is nonempty closed and $P \neq \{\theta\}$;
(ii) if a, b are nonnegative real numbers and $x, y \in P$, then $ax + by \in P$,
(iii) $P \cap (-P) = \{\theta\}$.

For any cone $P \subset E$, the *partial ordering* $\preceq$ with respect to P is defined by $x \preceq y$ if and only if $y - x \in P$. The notation $x \prec y$ stands for $x \preceq y$, but $x \neq y$. Also, we use $x \ll y$ to indicate that $y - x \in \text{int}P$, where int P denotes the interior of P. A cone P is called *normal* if there exists a number $K > 0$ such that

$$\theta \preceq x \preceq y \Longrightarrow \|x\| \leq K\|y\|$$

for all $x, y \in E$. The least positive number K satisfying the above condition is called the *normal constant* of P.

Using the notations, we have the following definition of a cone metric space.

Definition 1.1 (*[1]*). Let X be a nonempty set and E be a real Banach space equipped with the partial ordering $\preceq$ with respect to the cone $P \subset E$. Suppose that the mapping $d : X \times X \to E$ satisfies the following conditions:

(d_1) $\theta \prec d(x, y)$ for all $x, y \in X$ with $x \neq y$ and $d(x, y) = \theta$ if and only if $x = y$;

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(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d_3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a *cone metric* on X and (X, d) is called a *cone metric space*.

Definition 1.2 ([1]). Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$.

(1) For all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x) \ll c$ for all $n > N$, then $\{x_n\}$ is said to be *convergent* to x and the point x is the limit of $\{x_n\}$. We denote this by $x_n \to x$.
(2) For all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x_m) \ll c$ for all $m, n > N$, then $\{x_n\}$ is called a *Cauchy sequence* in X.
(3) A cone metric space (X, d) is said to be *complete* if every Cauchy sequence in X is convergent.

Lemma 1.3 ([1]). *Let (X, d) be a cone metric space and P be a normal cone with normal constant K. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X with $x_n \to x$ and $y_n \to y$. Then $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$.*

The following remark is useful for the main results in this paper; see [1,11].

Remark 1.4. (1) If E is a real Banach space with a cone P and $a \preceq \lambda a$, where $a \in P$ and $0 < \lambda < 1$, then $a = \theta$.
(2) If $c \in \operatorname{int} P$, $\theta \preceq a_n$ and $a_n \to \theta$, then there exists a positive integer N such that $a_n \ll c$ for all $n \geq N$.
(3) If $a \preceq b$ and $b \preceq c$, then $a \preceq c$; if $a \ll b$ and $b \ll c$, then $a \ll c$.

For other basic properties on cone metric spaces, the authors refer to the paper [1].

Now, we introduce the concept of c-distance on a cone metric space (X, d), which is a generalization of the ω-distance of Kada et al. [18].

Definition 1.5. Let (X, d) be a cone metric space. Then the mapping $q : X \times X \to E$ is called a *c-distance* on X if the following are satisfied:

(q1) $\theta \preceq q(x, y)$ for all $x, y \in X$;
(q2) $q(x, z) \preceq q(x, y) + q(y, z)$ for all $x, y, z \in X$;
(q3) for all $x \in X$, if $q(x, y_n) \preceq u$ for some $u = u_x \in P$ and all $n \geq 1$, then $q(x, y) \preceq u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$;
(q4) for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll c$.

Remark 1.6. If $E = \mathbb{R}$ and $P = \mathbb{R}^+$ ($\mathbb{R}$ denotes the set of all real numbers and $\mathbb{R}^+$ denotes the set of all nonnegative real numbers), then (X, d) is an ordinary metric space. (1) If (q3) is replaced with the following condition:

$(q3')$ For any $x \in X$, $q(x, \cdot) \to \mathbb{R}^+$ is lower semi-continuous,

then the c-distance q is a w-distance on X due to Kada et al. [18]. (2) It is easy to see that, if $q(x, \cdot)$ is lower semi-continuous, then (q3) holds. Hence it is obvious that every ω-distance is a c-distance, but the converse does not hold. Therefore, the c-distance is a generalization of the ω-distance.

Now, we give some examples of the c-distance as follows:

Example 1.7. Let (X, d) be a cone metric space and P be a normal cone. Put $q(x, y) = d(x, y)$ for all $x, y \in X$. Then q is a c-distance. In fact, (q1) and (q2) are immediate. Lemma 1.3 shows that (q3) holds. Let $c \in E$ with $\theta \ll c$ be given and put $e = c/2$. Suppose that $q(z, x) \ll e$ and $q(z, y) \ll e$. Then $d(x, y) = q(x, y) \preceq q(x, z) + q(z, y) \ll e + e = c$. This shows that q satisfies (q4) and hence q is a c-distance.

Example 1.8. Let (X, d) be a cone metric space and P be a normal cone. Put $q(x, y) = d(u, y)$ for all $x, y \in X$, where $u \in X$ is a fixed point. Then q is a c-distance. In fact, (q1) and (q3) are immediate. Since $d(u, z) \preceq d(u, y) + d(u, z)$, i.e., $q(x, z) \preceq q(x, y) + q(y, z)$, (q2) holds. Let $c \in E$ with $\theta \ll c$ and put $e = c/2$. If $q(z, x) \ll e$ and $q(z, y) \ll e$, then we have

$$\begin{aligned} d(x, y) &\preceq d(x, u) + d(u, y) \\ &= d(u, x) + d(u, y) \\ &= q(z, x) + q(z, y) \\ &\ll e + e = c. \end{aligned}$$

This shows that (q4) holds. Hence q is a c-distance.

Example 1.9. Let $E = C^1_{\mathbb{R}}[0, 1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and $P = \{x \in E : x(t) \geq 0 \text{ on } [0, 1]\}$ (this cone is not normal). Define an order $x \preceq y$ by $x(t) \leq y(t)$ for all $t \in [0, 1]$. Let $X = [0, \infty)$ and define a mapping $d : X \times X \to E$ by $d(x, y) = |x - y|\varphi$ for all $x, y \in X$, where $\varphi : [0, 1] \to \mathbb{R}$ such that $\varphi(t) = e^t$. Then (X, d) is a cone metric space (see [10]). Define a mapping $q : X \times X \to E$ by $q(x, y) = (x+y)\varphi$ for all $x, y \in X$. Then q is a c-distance. In fact, (q1)–(q3) are immediate. Note that

$$d(x, y) = |x - y|\varphi \leq (x + z)\varphi + (y + z)\varphi = q(z, x) + q(z, y)$$

for all $x, y, z \in X$. This implies that (q4) holds. Hence q is a c-distance.

Example 1.10. Let $E = \mathbb{R}$ and $P = \{x \in E : x \geq 0\}$. Let $X = [0, \infty)$ and define a mapping $d : X \times X \to E$ by $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a cone metric space. Define a mapping $q : X \times X \to E$ by $q(x, y) = y$ for all $x, y \in X$. Then q is a c-distance. In fact, (q1)–(q3) are immediate. Let $\varepsilon > 0$ be given. Set $\delta = \varepsilon/2$. If $q(z, x) = x < \delta$ and $q(z, y) = y < \delta$, then $d(x, y) = |x + y| \leq x + y < 2\delta = \varepsilon$. It follows that (q4) holds. Hence q is a c-distance.

On c-distance, we have the following important remark:

Remark 1.11. (1) $q(x, y) = q(y, x)$ does not necessarily hold for all $x, y \in X$.
(2) $q(x, y) = \theta$ is not necessarily equivalent to $x = y$ for all $x, y \in X$.

Lemma 1.12. *Let (X, d) be a cone metric space and q be a c-distance on X. Let $\{x_n\}$ be a sequence in X. Suppose that $\{u_n\}$ is a sequence in P converging to θ. If $q(x_n, x_m) \preceq u_n$ for all $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.*

Proof. Let $c \in E$ with $\theta \ll c$. Then there exists $\delta > 0$ such that $c - x \in \text{int}P$ for any $x \in P$ with $\|x\| < \delta$. Since $\{u_n\}$ converges to θ, there exists a positive integer N such that $\|u_n\| < \delta$ for all $n \geq N$ and so $c - u_n \in \text{int}P$, i.e., $u_n \ll c$ for all $n \geq N$. By the hypothesis, $q(x_n, x_m) \preceq u_n \ll c$ for all $m > n$ with $n \geq N$. This implies that $q(x_n, x_{n+1}) \preceq u_n \ll c$ and $q(x_n, x_{m+1}) \preceq u_n \ll c$ for all $m > n$ with $n > N$. From (q4) with $e = c$ it follows that $d(x_{n+1}, x_{m+1}) \ll c$ for all $m > n$ with $n > N$. By the definition of Cauchy sequence, we conclude that $\{x_n\}$ is a Cauchy sequence. This completes the proof. □

2. Main result

In [2], Abbas and Jungck proved some common fixed point theorems in a normal cone metric space. We state Theorems 2.1 and 2.3 in [2] as follows.

Theorem 2.1. *Let (X, d) be a cone metric space, and P a normal cone with normal constant K. Suppose mappings $f, g : X \to X$ satisfy*

$$d(fx, fy) \preceq kd(gx, gy), \quad k \in [0, 1)$$

or

$$d(fx, fy) \preceq k(d(fx, gx) + d(fy, gy)), \quad k \in \left[0, \frac{1}{2}\right)$$

for all $x, y \in X$. If the range of g contains the range of f and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, f and g have a unique common fixed point.

The following is the main result of this paper. We prove a common fixed point theorem by using c-distance and we do not require that f and g are weakly compatible.

Theorem 2.2. *Let (X, d) be a cone metric space, and P a normal cone with normal constant K. Let $a_i \in (0, 1) (i = 1, 2, 3, 4)$ be constants with $a_1 + 2a_2 + a_3 + a_4 < 1$, and $f, g : X \to X$ be two mappings satisfying the condition*

$$q(fx, fy) \preceq a_1 q(gx, gy) + a_2 q(gx, fy) + a_3 q(gx, fx) + a_4 q(gy, fy), \tag{2.1}$$

for all $x, y \in X$. Suppose that the range of g contains the range of f and $g(X)$ is a complete subspace of X. If f and g satisfy

$$\inf\{\|q(fx, y)\| + \|q(gx, y)\| + \|q(gx, fx)\| : x \in X\} > 0$$

for all $y \in X$ with $y \neq fy$ or $y \neq gy$, then f and g have a common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point. Since $f(X) \subset g(X)$, there exists an $x_1 \in X$ such that $fx_0 = gx_1$. By induction, a sequence $\{x_n\}$ can chosen such that $fx_n = gx_{n+1}$, $n = 0, 1, 2, \ldots$. By (2.1) and (q2), for any natural number n, we have

$$\begin{aligned} q(gx_n, gx_{n+1}) &= q(fx_{n-1}, fx_n) \\ &\preceq a_1 q(gx_{n-1}, gx_n) + a_2 q(gx_{n-1}, fx_n) + a_3 q(gx_{n-1}, fx_{n-1}) + a_4 q(gx_n, fx_n) \\ &= a_1 q(gx_{n-1}, gx_n) + a_2 q(gx_{n-1}, gx_{n+1}) + a_3 q(gx_{n-1}, gx_n) + a_4 q(gx_n, gx_{n+1}) \\ &\preceq a_1 q(gx_{n-1}, gx_n) + a_2 [q(gx_{n-1}, gx_n) + q(gx_n, gx_{n+1})] + a_3 q(gx_{n-1}, gx_n) + a_4 q(gx_n, gx_{n+1}) \\ &= (a_1 + a_2 + a_3) q(gx_{n-1}, gx_n) + (a_2 + a_4) q(gx_n, gx_{n+1}). \end{aligned}$$

So,

$$q(gx_n, gx_{n+1}) \preceq bq(gx_{n-1}, gx_n), \quad n = 1, 2, \ldots,$$

where $b = \frac{a_1+a_2+a_3}{1-a_2-a_4} \in (0, 1)$. By induction, we get

$$q(gx_n, gx_{n+1}) \preceq b^n q(gx_0, gx_1), \quad n = 0, 1, 2, \ldots. \tag{2.2}$$

Let m, n with $m > n$ be arbitrary integers. From (2.2) and (q2) it follows that

$$\begin{aligned} q(gx_n, gx_m) &\preceq q(gx_n, gx_{n+1}) + q(gx_{n+1}, gx_{n+2}) + \cdots + q(gx_{m-1}, gx_m) \\ &\preceq b^n q(gx_0, gx_1) + b^{n+1} q(gx_0, gx_1) + \cdots + b^{m-1} q(gx_0, x_1) \\ &\preceq \frac{b^n}{1-b} q(gx_0, gx_1). \end{aligned} \tag{2.3}$$

By using Lemma 1.12, we conclude that the sequence $\{gx_n\}$ is a Cauchy sequence in X. Since $g(X)$ is complete, there exists some point $y \in g(X)$ such that $gx_n \to y$ as $n \to \infty$. By (2.3) and (q3) we have

$$q(gx_n, y) \preceq \frac{b^n}{1-b} q(gx_0, gx_1), \quad n = 0, 1, 2, \ldots. \tag{2.4}$$

Since P is a normal cone with normal constant K, from (2.4) it follows that

$$\|q(gx_n, y)\| \le \frac{Kb^n}{1-b} \|q(gx_0, gx_1)\|, \quad n = 0, 1, 2, \ldots. \tag{2.5}$$

From (2.3) we have

$$\|q(gx_n, gx_m)\| \le \frac{Kb^n}{1-b} \|q(gx_0, gx_1)\|$$

for all $m > n$. In particular, we have

$$\|q(gx_n, gx_{n+1})\| \le \frac{Kb^n}{1-b} \|q(gx_0, gx_1)\| \tag{2.6}$$

for all $n = 0, 1, \ldots$.

Suppose that $y \neq gy$ or $y \neq fy$. Then by hypothesis, (2.5) and (2.6), we have

$$\begin{aligned} 0 &< \inf\{\|q(fx, y)\| + \|q(gx, y)\| + \|q(gx, fx)\| : x \in X\} \\ &\le \inf\{\|q(fx_n, y)\| + \|q(gx_n, y)\| + \|q(gx_n, fx_n)\| : n \ge 1\} \\ &= \inf\{\|q(gx_{n+1}, y)\| + \|q(gx_n, y)\| + \|q(gx_n, gx_{n+1})\| : n \ge 1\} \\ &\le \inf\left\{\frac{Kb^{n+1}}{1-b}\|q(gx_1, gx_0)\| + \frac{Kb^n}{1-b}\|q(gx_1, gx_0)\| + \frac{Kb^n}{1-b}\|q(gx_0, gx_1)\| : n \ge 1\right\} \\ &= 0. \end{aligned}$$

This is a contradiction. Hence, $y = gy = fy$. This completes the proof. □

Example 2.3. Consider Example 1.10. Define the mapping $f : X \to X$ by $f(2) = \frac{3}{2}$ and $f(x) = \frac{x}{2}$ for all $x \in X$ with $x \neq 2$ and the mapping $g : X \to X$ by $gx = x$ for all $x \in X$. Since $d(f(1), f(2)) = d(g(1), g(2))$, there is not $k \in [0, 1)$ such that $d(fx, fy) \le kd(gx, gy)$ for all $x, y \in X$. Hence, Theorem 2.1 of Abbas and Jungck [2] cannot be applied to this example. In fact, Theorem 2.3 of Abbas and Jungck [2] also cannot be applied to this example. Let $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{32}$, $a_3 = \frac{3}{32}$ and $a_4 = \frac{5}{32}$. By simple checking, we see that f and g satisfy (2.1). For $y \neq fy$ or $y \neq gy$, i.e., $y \neq 0$, one has $\inf\{\|q(fx, y)\| + \|q(gx, y)\| + \|q(gx, fx)\| : x \in X\} = 2y > 0$. So, the hypothesis is satisfied. By Theorem 2.2 we conclude that f and g have a common fixed point in X. This common fixed point is $x = 0$.

Remark 2.4. It is of some interest to define such a c-distance as an auxiliary tool of the cone metric d to conclude the existence of (common) fixed points of some mappings. On the other hand, c-distance is a generalization of the ω-distance of Kada et al. [18].

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