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Weak Solutions for Linear Abstract Differential Equations in Banach Spaces

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Introduction

Let X be a Banach space, and let A and B be two infinitesimal generators of strongly continuous semigroups in X. In this paper we use the results contained in Refs. [9–12], which give conditions on A and B in order that $A + B$ be preclosed and its closure $\overline{A + B}$ have a spectrum different from C, to study the equation

$$\lambda u - Au - Bu = f \tag{I}$$

considering several realizations of A and B.

If we know that $A + B$ is preclosed and the resolvent set of $\overline{A + B}$ contains λ, then it is easy to see that Eq. (I) has one and only one weak solution; that is, there exists $x \in X$ and a sequence $\{x_n\}$ in $D_A \cap D_B$ such that

$$x_n \to x, \lambda x_n - Ax_n - Bx_n \to y. \tag{II}$$

For example, let Y be a Banach space, $[0, T]$ be an interval, and $\{B(t)\}_{t\in[0,T]}$ be a set of infinitesimal generators of strongly continuous semigroups in Y. If we then put

$$\begin{aligned} &A = -\frac{d}{dt} \quad \text{with domain} \quad D_A = \{u \in H^{1,p}(0, T; Y), u(0) = 0 \\ &\qquad \text{(or } u(0) = Eu(t), E \text{ linear bounded operator in } Y)\}, \\ &(Bu)(t) = B(t)\,u(t) \quad \text{with domain} \\ &D_B = \{u \in L^p(0, T; Y), u(t) \in D_{B(t)} t \text{ a.e. in } [0, T]\ t \to B(t)\,u(t) \in L^p(0, T, Y)\}, \end{aligned} \tag{III}$$

then Eq. (I) is equivalent to the problem

$$\begin{aligned} &\lambda u + \frac{du}{dt} - B(t)\,u(t) = f, \\ &u(0) = 0 \qquad \text{(or } u(0) = Eu(T)). \end{aligned} \tag{IV}$$

This problem, with the condition $u(0) = 0$ in a general Banach space, has been studied by many authors (Kato [23–27], Tanabe [40–42], and Lions [19–22] if Y is a Hilbert space and $p = 2$). Our results are similar. For the boundary condition $u(0) = Eu(T)$ (for example the periodic case), and in the general case of a Banach space Y, our results seem to be new.

We can also add in (IV) an integral term and study the problem

$$\begin{gathered}\lambda u + \frac{du}{dt} - B(t)\,u(t) - \int_0^t K(t, s)\,u(s)\,ds = f,\\ u(0) = 0,\end{gathered} \tag{V}$$

thus generalizing some results of Marti [28].

We have also studied some equations of higher order in t such as

$$\frac{d^2u}{dt^2} + B(t)\,u(t) = f \tag{VI}$$

with Dirichlet or periodic conditions, the wave equation, and the Schrödinger equation in $L^p(0, T; H)$, $p > 1$, H being a Hilbert space. (If $p = 2$ this problem has been previously studied by Lions [20]. Also see Baiocchi [6, 7].)

Finally, we have considered systems having the form

$$\begin{gathered}\frac{du_1}{dt} - B_1(t)\,u_1(t) - \sum_{k=1}^{n} Q_{1k}u_k(t) = f_1\,,\\ -\frac{du_h}{dt} - B_h(t)\,u_h(t) - \sum_{k=1}^{n} Q_{hk}u_k(t) = f_h\,, \qquad h = 2,..., n,\\ u_1(0) = 0, \qquad u_h(T) = 0, \qquad h = 2,..., n,\end{gathered} \tag{VII}$$

where Q is a linear operator in $\mathbf{C}^n$ such that $\operatorname{Re}(Qx, x) \leqslant 0$. This problem has been previously studied by Lions [12] in the Hilbert space case.

For the sake of simplicity we have considered only the case where A and B are the infinitesimal generators of contraction semigroups in X. But many results are true also when A is the infinitesimal generator of a strongly continuous semigroup and B is the infinitesimal generator of an analytic semigroup. For the same reason we have not looked for regularity results (for some results in this direction see Refs. [9, 10, 11]).

Chapter I. General Results

Let X be a real or complex Banach space, whose norm is denoted by $\| \ \|$.

Let X' be the dual of X, the pairing between X and X' being denoted by $\langle \ , \ \rangle$.

Let $K(X)$ be the set of all infinitesimal generators of contraction semigroups on X [18, 45].

Finally let A and B be linear operators belonging to $K(X)$; we will now give the necessary and sufficient conditions in order that $A + B$ be preclosed and $\overline{A + B}$ belong to $K(X)$.

1. Principal Theorems

The two following theorems are proved in [12]:

Theorem 1.1. *Let A and B be linear operators belonging to $K(X)$; then the following inequality holds*:

$$\| x \| \leqslant \frac{1}{\operatorname{Re} \lambda} \| \lambda x - Ax - Bx \| \ \forall x \in D_A \cap D_B, [1] \qquad \lambda \in \mathbf{C}_+ [2] \tag{1.1}$$

Theorem 1.2. *Let A and B be linear operators belonging to $K(X)$; then the following properties are equivalent*:

(i) *$A + B$ (with domain $D_A \cap D_B$) is preclosed and $\overline{A + B}$ belongs to $K(X)$*;

(ii) *$D_A \cap D_B$ is dense in X and there exists $\omega \in \mathbf{C}_+$ such that $(\lambda - A - B)(D_A \cap D_B)$ is dense in X for all $\lambda \in \omega + \mathbf{C}_+$.*

We now prove

Theorem 1.3. *Let X and Y be Banach spaces such that $X \subset Y$ algebrically and topologically and let A and B be linear operators in Y belonging to $K(Y)$. Assume that*

[1] Let L be a linear operator in X. Then D_L denotes the domain of L, $\sigma(L)$ (resp. $\rho(L)$) denotes the spectrum (resp. the resolvent set) of L, if L is preclosed L denotes its closure, and, finally, if $L \in K(X)$, then e^{tL} represents the semigroup generated by L.

[2] $\mathbf{C}$ denotes the set of all complex numbers and $\mathbf{C}_+$ (resp. $\bar{\mathbf{C}}_+$) the set of all complex numbers whose real part is positive (resp. nonnegative).

(i) *X is an invariant subspace for* e^{tA} *and* e^{tB} $\forall t \in \mathbf{R}_+$ *and the restrictions of* e^{tA} *and* e^{tB} *to* Y *define two contraction semigroups,* e^{tA} *and* e^{tB}, *respectively;*

(ii) $A_1 + B_1$ *is preclosed and* $\overline{A_1 + B_1} \in K(X)$;

(iii) $D_A \cap D_B$ *is dense in* Y;

then $A + B$ *is preclosed and* $\overline{A + B} \in K(Y)$.

Proof. A is an extension of A_1 and B is an extension of B_1 ; in fact, if, e.g., $x \in D_{A_1}$, we have

$$A_1 x = \lim_{h \to 0} \frac{1}{h} (e^{hA}x - x) \quad \text{in} \quad X, \tag{1.2}$$

and, since $X \subset Y$ topologically, the limit (1.2) exists also in Y; thus $x \in D_A$ and $A_1 x = Ax$.

Owing to (iii) $D_A \cap D_B$ is dense in X, moreover $(\lambda - A - B)(D_A \cap D_B) \supset (\lambda - A - B)(D_{A_1} \cap D_{B_1}) = (\lambda - A_1 - B_1)(D_{A_1} \cap D_{B_1})$, i.e., it is dense in X, thus Theorem 1.2 implies that $A + B$ is preclosed and $\overline{A + B} \in K(Y)$.

2. Weak Solutions

Definition 1.1. Let L be a linear operator in X and y be an element of X; x will be called a weak solution of the equation

$$Lx = y \tag{1.3}$$

if there exists a sequence $\{x_n\}$ in D_L such that

$$x_n \to x, \qquad Lx_n \to y.^3 \tag{1.4}$$

The following theorem of uniqueness shows that if an a priori estimate for L holds, then Definition 1.1 is independent of the choice of the sequence $\{x_n\}$.

Theorem 1.4. *Let L be a linear operator in X such that*

$$\text{there exists} \quad K \in \mathbf{R}_+{}^4 \quad \text{such that} \quad \| x \| \leqslant K \| Lx \|, \tag{1.5}$$

[3] The arrow denotes convergence in X.

[4] $\mathbf{R}$ denotes the set of all real numbers and $\mathbf{R}_+$ (resp. $\bar{\mathbf{R}}_+$) the set of all positive (resp. nonnegative) numbers.

and let x be a weak solution of the equation

$$Lx = 0. \tag{1.6}$$

Then $x = 0$.

Proof. Let x be a weak solution of (1.6) and $\{x_n\}$ be a sequence in $D_A \cap D_B$ such that

$$x_n \to x \qquad \text{and} \qquad Lx_n \to 0. \tag{1.7}$$

According to (1.4) we have

$$\| x_n \| \leqslant K \| Lx_n \| \; \forall n \in \mathbf{N};^5 \tag{1.8}$$

therefore, passing to the limit for $n \to \infty$ we obtain $x = 0$. Using Theorems 1.1 and 1.2 we now obtain the following results:

THEOREM 1.5. *Let A and B be linear operators belonging to $K(X)$ and let x be a weak solution of the equation*

$$\lambda x - Ax - Bx = 0, \qquad \lambda \in \mathbf{C}_+ . \tag{1.9}$$

Then $x = 0$.

THEOREM 1.6. *Let A and B be linear operators satisfying hypotheses* (i) *or* (ii) *of Theorem* 1.2. *Then the equation*

$$\lambda x - Ax - Bx = y, \qquad \lambda \in \mathbf{C}_+ , \qquad y \in X \tag{1.10}$$

has one and only one solution x. Moreover $x \in D_{\overline{A+B}}$.

Proof. Let $y \in X$ and $x = R(\lambda, \overline{A + B})\, y$; since $(\lambda - A - B)(D_A \cap D_B)$ is dense in X, there exists a sequence $\{x_n\}$ in X such that

$$\lambda x_n - Ax_n - Bx_n \to y. \tag{1.11}$$

We then have $x_n \to R(\lambda, \overline{A + B})\, y = x$ and thus x is a weak solution of (1.10).

Since $A + B$ is preclosed, $x \in D_{\overline{A+B}}$, uniqueness follows from Theorem 1.5.

The following theorem is useful for comparing different definitions of weak solutions.

[5] N denotes the set of all natural numbers.

THEOREM 1.7. *Let X be reflexive and let A and B be linear operators satisfying hypotheses* (i) *or* (ii) *of Theorem* 1.2. *Then if x is a weak solution of* (1.10) *we have*

$$\langle x, \lambda x' - A'x' - B'x' \rangle = \langle y, x' \rangle \qquad \forall x' \in D_{A'} \cap D_{B'} \tag{1.12}$$

A' and B' being respectively the adjoints of A and B.

Proof. Let x be a weak solution of (1.10); then $x = R(\lambda, \overline{A + B})\, y$. It follows that

$$\langle x, \lambda x' - Ax' - B'x' \rangle = \langle y, R'(\lambda, \overline{A + B})(\lambda x' - A'x' - B'x') \rangle. \tag{1.13}$$

Due to a theorem of Phillips [33] we have

$$R'(\lambda, \overline{A + B}) = R(\lambda, \overline{A + B'}), \tag{1.14}$$

$\overline{A + B'}$ and $R'(\lambda, \overline{A + B})$ being, respectively, the adjoints of $\overline{A + B}$ and $R(\lambda, \overline{A + B})$. Moreover, $D_{\overline{A+B}'} \supset D_{A'} \cap D_{B'}$; in fact, if $x' \in D_{A'} \cap D_{B'}$ the map

$$D_A \cap D_B \to X,\ x \to \langle Ax + Bx, x' \rangle = \langle x, A'x' \rangle + \langle x, B'x' \rangle$$

is continuous; therefore $x' \in D_{\overline{A+B}}$ and, recalling (1.14),

$$R'(\lambda, \overline{A + B})(\lambda x' - A'x' - B'x') = x'. \tag{1.15}$$

This implies

$$\langle x, \lambda x' - A'x' - B'x' \rangle = \langle y, x' \rangle. \tag{1.16}$$

3. Sufficient Conditions

We are going to give some sufficient conditions on A and B in order that hypothesis (ii) of Theorem 1.2 be satisfied.

We first consider the case where A and B "commute," i.e.,

$$R(\lambda, A)\, R(\mu, B) - R(\mu, B)\, R(\lambda, A) = 0, \qquad \forall \lambda, \mu \in \mathbf{C}_+ . \tag{1.17}$$

We now have

THEOREM 1.7. *Let A and B be linear operators belonging to $K(X)$. Assume that* (1.17) *holds; then $A + B$ is preclosed and $\overline{A + B} \in K(X)$.*

Proof. We have [18, 45]

$$x = \lim_{n\to\infty} n^2 R(n, A)\, R(n, B)x \qquad \forall x \in X; \tag{1.18}$$

since $R(n, A)\, R(n, B)\, x = R(n, B)\, R(n, A)\, x \in D_A \cap D_B$, we first deduce that $D_A \cap D_B$ is dense in X. We have also

$$(\lambda - A - B)\, R(\lambda, A + B_n)x = x + R(\lambda, A + B_n)(B_n x - Bx)\,^6 \qquad \forall x \in D_A. \tag{1.19}$$

Since

$$\| R(\lambda, A + B_n)\| \leqslant \frac{1}{\operatorname{Re} \lambda} \qquad \forall \lambda \in C_+ \quad [9], \tag{1.20}$$

we obtain

$$\lim_{n\to\infty} (\lambda - A - B)\, R(\lambda, A + B_n)x = x \qquad \forall x \in D_B. \tag{1.21}$$

Therefore the closure of $(\lambda - A - B)(D_A \cap D_B)$ contains D_B and thus $(\lambda - A - B)(D_A \cap D_B)$ is dense in X.

Let us now consider the case where B is a "perturbation" of A, i.e.,

$$D_A \subset D_B, \qquad \lim_{n\to\infty} \| BR(n, A)\| = 0. \tag{1.22}$$

We now prove

THEOREM 1.8. *Let A and B be linear operators belonging to $K(X)$. Assume that* (1.22) *holds; then $A + B$ is preclosed and $\overline{A + B} \in K(X)$.*

Proof. Let $\omega \in \mathbf{R}_+$ such that

$$\| BR(\lambda, A)\| < 1 \qquad \forall \lambda \in \omega + \mathbf{C}_+. \tag{1.23}$$

In view of the identity

$$\lambda x - Ax - Bx = (1 - BR(\lambda, A))(\lambda x - Ax) \quad \forall x \in D_A, \qquad \lambda \in C_+, \tag{1.24}$$

we deduce that $\rho(A + B) \supset \omega + \mathbf{C}_+$ and

$$R(\lambda, A + B) = R(\lambda, A)(1 - BR(\lambda, A))^{-1}. \tag{1.25}$$

Therefore we have

$$(\lambda - A - B)(D_A \cap D_B) = (\lambda - A - B)(D_A) = X \qquad \forall \lambda \in \omega + C_+, \tag{1.26}$$

and A and B satisfy hypothesis (ii) of Theorem 1.2.

[6] $B_n = nBR(n, B)$.

Remark 1.1. If B is bounded, then, on the base of a result of Phillips [32], $A + B$ is the infinitesimal generator of a (C_0) class semigroup. Theorem 1.8 also assures that $A + B \in K(X)$. The next two theorems give sufficient conditions involving respectively the commutator $[A, e^{tB}]$ and the semigroup $B^2 e^{tA} B^{-2}$ which is deduced by a similarity transformation from e^{tA}. The former is proved in Ref. [12] and the latter in Ref. [19].

THEOREM 1.9. *Let A and B be linear operators belonging to $K(X)$. Assume that*

$$\left.\begin{array}{l} \textit{If } x \in D_A \textit{ then } e^{tB}x \in D_A \ \forall t \in \mathbf{R}_+ \textit{, the map } \bar{R}_+ \to X,\ t \to Ae^{tB}x \\ \textit{is continuous, and there exists } k \in \mathbf{R}_+ \textit{ and } \alpha \in [0, 1] \textit{ such that} \\ \qquad \| Ae^{tB}x - e^{tB}Ax \| \leqslant Kt^{-\alpha} \| x \| \qquad \forall x \in D_A . \end{array}\right\} \tag{1.27}$$

Then $A + B$ is preclosed and $\overline{A + B} \in K(X)$.

THEOREM 1.10. *Let A and B be linear operators belonging to $K(X)$. Assume that*

$$\left.\begin{array}{l} \textit{If } x \in D_{B^2}\,{}^{7} \textit{ then } e^{tA}x \in D_{B^2} \ \forall t \in \mathbf{R}_+ \textit{ and there exists } \omega \in \mathbf{R} \textit{ such} \\ \textit{that the semigroup } t \to e^{-\omega t} B^2 e^{tA} B^{-2} \textit{ is a contraction semigroup.} \end{array}\right\} \tag{1.28}$$

Then $A + B$ is preclosed and $\overline{A + B} \in K(X)$.

If B is the infinitesimal generator of an analytic semigroup we have the following theorem [11]:

THEOREM 1.11. *Let A and B be linear operators belonging to $K(X)$. Assume that one of the following conditions is satisfied:*

$$\left.\begin{array}{l} \textit{If } x \in D_B \textit{ then } e^{tA}x \in D_B \textit{ and there exists } K \in \mathbf{R}_+ \textit{ and } \alpha \in]0, 1] \\ \textit{such that} \\ \qquad \| Be^{tA}x - e^{tA}Bx \| \leqslant Kt^{\alpha} \| Bx \| \qquad \forall x \in D_B . \end{array}\right\} \tag{1.29}$$

$$\left.\begin{array}{l} \textit{If } x \in D_A \textit{ then } e^{tB}x \in D_A \textit{ and there exists } K \in \mathbf{R}_+ \textit{ and } \alpha \in]0, 1] \\ \textit{such that} \\ \qquad \| Ae^{tB}x - e^{tB}Ax \| \leqslant Kt^{\alpha} \| Ax \| \qquad \forall x \in D_A . \end{array}\right\} \tag{1.30}$$

Then $A + B$ is preclosed and $\overline{A + B} \in K(X)$.

[7] If L is a linear operator and $n \in \mathbf{N}$, L^n is the n-th power of L, its domain is

$$D_{L^n} = \{x \in D_L \,;\, L^k x \in D_L \,,\, k = 1, 2, \ldots, n - 1\}.$$

Finally, if X is reflexive we have the result [10]:

THEOREM 1.12. *Let X be reflexive and let A and B be linear operators belonging to $K(X)$. Assume that*

$$\left.\begin{array}{l}\textit{If } x \in D_B \textit{ then } e^{tA}x \in D_B \ \forall t \in \overline{\mathbf{R}}_+ \textit{ and there exists } \omega \in R \textit{ such} \\ \textit{that the semigroup } t \to e^{-\omega t} B e^{tA} B^{-1} \textit{ is a contraction semigroup.}\end{array}\right\} \tag{1.31}$$

Then $A + B$ is preclosed and $\overline{A + B} \in K(X)$.

4. AN EXAMPLE

The aim of this section is to show that condition (ii) of Theorem 1.2 is not necessarily fulfilled.

Let us put $X = L^2(\mathbf{R}_+)$ [8] and define two linear operators A and B by writing

$$\begin{aligned} &D_A = H^2(\mathbf{R}_+) \cap H_0^{\,1}(\mathbf{R}_+), \\ &D_B = \{u \in H^2(\mathbf{R}_+);\ u'(0) = 0\},^9 \\ &Au = u'' \quad \forall u \in D_A \qquad \text{and} \qquad Bu = u'' \quad \forall u \in D_B\,. \end{aligned} \tag{1.32}$$

It can be easily checked that [10]

$$\begin{aligned} (R(\lambda, A)u)(t) &= \frac{1}{\sqrt{\lambda}}\, e^{-\sqrt{\lambda}t} \int_0^t \sinh(\sqrt{\lambda}\, s)\, u(s)\, ds \\ &\quad + \frac{1}{\sqrt{\lambda}} \cosh(\sqrt{\lambda}\, t)\, e^{-\sqrt{\lambda}s} u(s)\, ds \\ \forall \lambda \in C_+\,, &\qquad u \in L^2(\mathbf{R}_+), \qquad t \text{ a.e. in } \mathbf{R}_+\,, \end{aligned} \tag{1.33}$$

$$\begin{aligned} (R(\lambda, B)u)(t) &= \frac{1}{\sqrt{\lambda}}\, e^{-\sqrt{\lambda}t} \int_0^t \sinh(\sqrt{\lambda}\, s)\, u(s)\, ds \\ &\quad + \frac{1}{\sqrt{\lambda}} \cosh(\sqrt{\lambda}\, t) \int_t^\infty e^{-\sqrt{\lambda}s} u(s)\, ds \\ \forall \lambda \in C_+\,, &\qquad u \in L^2(\mathbf{R}_+), \qquad t \text{ a.e. in } \mathbf{R}_+\,. \end{aligned} \tag{1.34}$$

[8] For the definition of $L^2(\mathbf{R}_+)$, $H^k(\mathbf{R}_+)$, and $H_0^{\,k}(\mathbf{R}_+)$, $k \in \mathbf{N}$, see Ref. [22].

[9] u' denotes du/dt and u'' denotes d^2u/dt^2. Note that if $u \in H^2(\mathbf{R}_+)$, the trace of u' is defined.

[10] We put sihn $\xi = \frac{1}{2}(e^\xi - e^{-\xi})$, cosh $\xi = \frac{1}{2}(e^\xi + e^{-\xi}) \quad \forall \xi \in \mathbf{R}$.

Moreover, since

$$(Ax, x) \leqslant 0 \quad \forall x \in D_A \qquad \text{and} \qquad (Bx, x) \leqslant 0 \quad \forall x \in D_B{}^{11} \tag{1.35}$$

we know [1] that A and B belong to $K(X)$.

Notice that $(\lambda - A - B)(D_A \cap D_B)$ is not dense in X, in fact, the function $t \to e^{-\lambda t}$ is orthogonal to $(\lambda - A - B)(D_A \cap D_B)$.

Finally, let P be the linear operator in X defined by

$$D_P = H_0{}^1(R_+), \qquad Pu = -u' \quad \forall u \in D_P\,. \tag{1.36}$$

We then obtain $A + B = P^2/2$, and therefore $A + B$ is closed because $\rho(P) \supset \mathbf{C}_+$ (Ref. [14], p. 704), but $\sigma(A + B) = C$, and therefore $A + B \notin K(X)$.

Chapter II. Translation and Multiplication Operators

Let $\mathbf{R}^n$ be the Euclidean space of dimension n, Ω be a bounded open subset of R^n, and $\bar{\Omega}$ be its closure.

Let Y be a complex or a real Banach space whose norm is denoted by $|\ |$. If $k \in \mathbf{N}$ then $C^k(\bar{\Omega}; Y)$ is the Banach space of all functions on $\bar{\Omega}$ to Y continuous with its partial derivatives of order $\leqslant k$. $\epsilon(\bar{\Omega}; Y)$ is the locally convex space of all functions on $\bar{\Omega}$ to Y continuous with its partial derivatives of arbitrary order. $C^k(\bar{\Omega}; Y)$ and $\epsilon(\bar{\Omega}; Y)$ are endowed with the usual topology. We put $C(\bar{\Omega}; Y) = C^0(\bar{\Omega}; Y)$ and denote the norm in $C^0(\bar{\Omega}; Y)$ (resp. $C^k(\bar{\Omega}; Y)$, $k \in \mathbf{N}$) by $\|\ \|_\infty$ (resp. by $\|\ \|_{k,\infty}$), viz.,

$$\begin{aligned} \| u \|_\infty &= \sup\{| u(t)|;\ t \in \Omega\}, \\ \| u \|_{k,\infty} &= \sup\{\| D^\alpha u \|_\infty\,,\ \alpha \in \Delta_k\}, \end{aligned} \tag{2.1}$$

Δ_k being the set of partial derivatives of order $\leqslant k$.

If $p \geqslant 1$ and $L^p(\Omega; Y)$ is the Banach space of the functions a.e. on $\bar{\Omega}$ to Y measurable and such that the p-th power of the norm is summable, $L^p(\Omega; Y)$ is endowed with the usual norm,

$$\| u \|_p = \left(\int_\Omega | u(t)|^p\, dt\right)^{1/p}. \tag{2.2}$$

[11] (,) denotes the inner product in $L^2(\mathbf{R}_+)$.

The completion of $C^k(\bar{\Omega})$ with respect to the norm

$$\| u \|_{k,p} = \sum_{\alpha \in \Delta_k} \| D^\alpha u \|_p \tag{2.3}$$

will be called $H^{k,p}(\Omega)$.

Let us assume that Ω is smooth enough so that the space $H^{k,p}(\Omega)$ coincides with the space of vectorial distributions u such that $D^\alpha u \in L^p(\Omega)$, $0 \leqslant | \alpha | \leqslant k$.

1. Translations in an Interval

Let $[0, T]$ be a bounded interval in $\mathbf{R}_+$, and for sake of simplicity we put

$$\begin{aligned} &\epsilon([0, T]; Y) = \epsilon(0, T; Y), \\ &C^k([0, T]; Y) = C^k(0, T; Y) \qquad \text{for} \quad k + 1 \in \mathbf{N}, \\ &L^p(]0, T[; Y) = L^p(0, T; Y), \\ &H^{k,p}(]0, T[; Y) = H^{k,p}(0, T; Y) \qquad \text{for} \quad p \geqslant 1, \quad k \in \mathbf{N}. \end{aligned} \tag{2.4}$$

The closed subspace of $C^k(0, T; Y)$ (resp. $\epsilon(0, T; Y)$) of all the functions vanishing at zero with all derivatives of order $\leqslant k$ (resp. of arbitrary order) will be called $C_0{}^k(0, T; Y)$ (resp. $\epsilon_0(0, T; Y)$), and the closed subspace of $H^{k,p}(0, T; Y)$ of all the functions vanishing at zero with all derivatives of order $< k$ [12] will be called $H_0^{k,p}(0, T; Y)$.

Let us now introduce in $\epsilon_0(0, T; Y)$ the semigroup $R : t \to R_t$ of right translations,[13]

$$(R_t u)(s) \begin{cases} = u(s - t) & \text{if} \quad s - t \in]0, T], \\ = 0 & \text{if} \quad s - t \notin]0, T]. \end{cases} \tag{2.5}$$

It is easy to prove that R is an equicontinuous semigroup of class (C_0) in $\epsilon_0(0, T; Y)$ [45] and that its infinitesimal generator A is given by

$$Au = -u' \,{}^{9} \qquad \forall u \in \epsilon_0(0, T; Y). \tag{2.6}$$

[12] Recall that if $u \in H^{k,p}(0, T; Y)$, then u coincides a.e. with a function belonging to $C^{k-1}(0, T; Y)$.

[13] We could analogously define the semigroup $L : t \to L_t$ of left translations.

We now prove the following result:

THEOREM 2.1. *Let R be the semigroup defined by* (2.5). *Then* $\forall t \in \mathbf{R}_+$ *there exists one and only one bounded extension of R_t to $C_0(0, T; Y)$ (resp. $L^p(0, T; Y)$, resp. $H_0^{1,p}(0, T; Y)$) which will be called $\bar{R}_t$. Moreover, $\bar{R} : t \to \bar{R}_t$ is a contraction semigroup on $C_0(0, T; Y)$ (resp. $H_0^{1,p}(0, T; Y)$). Finally, the infinitesimal generator $\bar{A}$ of $\bar{R}$ is an extension of A, and we have*

$$D_{\bar{A}} = C_0^1(0, T; Y) \qquad (\textit{resp. } H_0^{1,p}(0, T; Y), \textit{resp. } H_0^{2,p}(0, T; Y)). \quad (2.7)$$

Proof. For sake of simplicity we call ν one of the following norms on $\epsilon_0(0, T; Y) : \| \|_\infty , \| \|_p , \| \|_{k,p}$, and ν_1 the norm

$$\nu_1(u) = \nu(u) + \nu(u'). \qquad (2.8)$$

Finally we call $L_0^\nu(0, T; Y)$ (resp. $H_0^{1,\nu}(0, T; Y)$ the completion of $\epsilon_0(0, T; Y)$ with respect to the norm ν (resp. ν_1).

We remark that ν satisfies the following properties:

$$\left.\begin{array}{ll} \text{(i)} & \text{The topology induced by } \nu \text{ on } \epsilon_0(0, T; Y) \text{ is coarser than} \\ & \text{that of } \epsilon_0(0, T; Y); \\ \text{(ii)} & \text{The following inequality holds:} \\ & \qquad \nu(R_t u) \leqslant \nu(u). \end{array}\right\} \qquad (2.9)$$

Let $t \in \mathbf{R}_+$; then R_t is clearly extensible to a bounded operator $\bar{R}_t$ on $L_0^\nu(0, T; Y)$, moreover

$$\nu(\bar{R}_t u) \leqslant \nu(u) \qquad \forall u \in L_0^\nu(0, T; Y). \qquad (2.10)$$

If $u \in \epsilon_0(0, T; Y)$ the map $\bar{\mathbf{R}}_+ \to L_0^\nu(0, T; Y)$, $t \to R_t u$ is continuous in view of (2.9, i); therefore, owing to (3.10), R is a contraction semigroup on $L_0^\nu(0, T; Y)$.

It is evident that $\bar{A}$ is an extension of A, and thus we have only to prove that

$$D_A = H_0^{1,\nu}(0, T; Y). \qquad (2.11)$$

We first note that the graph norm of $\bar{A}$ is equivalent to ν_1 , and therefore $H_0^{1,\nu}(0, T; Y)$ is a closed subspace of $D_{\bar{A}}$ (endowed with the $\bar{A}$ graph norm); on the other hand, if $\lambda \in C$, $R(\lambda, \bar{A})$ is a topological isomorphism between $L_0^\nu(0, T; Y)$ and $D_{\bar{A}}$ and $\epsilon_0(0, T; Y)$ is clearly an invariant subspace of $R(\lambda, \bar{A})$; consequently $\epsilon_0(0, T; Y)$ is dense in $D_{\bar{A}}$ and $D_{\bar{A}} = H_0^{1,\nu}(0, T; Y)$.

2. Periodic Translations

$\Pi_T(Y)$ is the closed subspace of $\epsilon(\mathbf{R}; Y)$ of the functions having all continuous periodic derivatives of period T.

Let E be a bounded operator in Y such that

$$\begin{aligned} &\text{(i)} \quad \sigma(E) \subset C_+ , \\ &\text{(ii)} \quad | E^{-1} | \leqslant 1. \end{aligned} \tag{2.12}$$

In view of (2.12, i) there exist $E^\alpha \; \forall \alpha \in \mathbf{R}$ such that

$$E^\alpha = \frac{1}{2\pi i} \int_\Gamma \lambda^\alpha R(\lambda, E)\, d\lambda, \tag{2.13}$$

Γ being a closed regular contour in C_+ around $\sigma(E)$.

We call Γ_E the map: $\Pi_T(\mathrm{Y}) \to \Pi_T(\mathrm{Y})$,

$$(\Gamma_E u)(t) = E^{t/T} u(t) \qquad \forall t \in \mathbf{R}, \qquad u \in \Pi_T(Y). \tag{2.14}$$

Γ_E is clearly a topological isomorphism of $\Pi_T(Y)$.

We now put

$$\Pi_{T,E}(Y) = \Gamma_E(\Pi_T(Y)). \tag{2.15}$$

$\Pi_{T,E}(Y)$ is a closed subspace of $\epsilon(\mathbf{R}; Y)$.

We remark that

$$\Pi_{T,E}(Y) = \left\{ u \in \epsilon(R; Y); \frac{d^h u}{dt^h}(t + T) = E \frac{d^h u}{dt^h}(t) \; \forall t \in \mathbf{R}, \; h + 1 \in \mathbf{N} \right\}. \tag{2.16}$$

Therefore we can identify $\Pi_{T,E}(Y)$ with the following subspace of $\epsilon(0, T; Y)$:

$$\left\{ u \in \epsilon(0, T; Y), \frac{d^h u}{dt^h}(T) = E \frac{d^h u}{dt^h}(0), \; \forall h + 1 \in \mathbf{N} \right\}. \tag{2.17}$$

In what follows we shall make this identification.

We now introduce the right translations semigroup on $\Pi_{T,E}(Y)$, $R : t \to R_t$:

$$(R_t u)(s) = u(s - t) \qquad \forall u \in \Pi_{T,E}(Y), \quad t, s \in \mathbf{R}. \tag{2.18}$$

It is easy to show that R is an equicontinuous semigroup of class (C_0) in $\Pi_{T,E}(Y)$ and that its infinitesimal generator A is given by

$$Au = -u' \qquad \forall u \in \Pi_{T,E}(Y). \tag{2.19}$$

We note the following identities:

$$u(t) = R_{-nT}E^{-n}u(t) = E^{-n}R_{nT}u(t) \qquad \forall t \in \mathbf{R}, \quad u \in \Pi_{T,E}(Y). \tag{2.20}$$

Finally, let us put

$$C_{\Pi,E}(0, T; Y) = \{u \in C(0, T; Y); u(T) = Eu(0)\}, \tag{2.21}$$

and if $k \in \mathbf{N}$ and $p > 1$,

$$\begin{gathered} C^k_{\Pi,E}(0, T; Y) = \left\{u \in C^k(0, T; Y); \frac{d^h u}{dt^h}(T) = E\frac{d^h u}{dt^h}(0), h = 0, 1, \ldots, k\right\}, \\ H^{k,p}_{\Pi,E}(0, T; Y) = \left\{u \in H^{k,p}(0, T; Y); \frac{d^h u}{dt^h}(T) = E\frac{d^h u}{dt^h}(0), h = 0, 1, \ldots, k-1\right\}. \end{gathered} \tag{2.22}$$

We now prove the following result:

THEOREM 2.2. *Let R be the semigroup defined by* (2.18). *Then* $\forall t \in \mathbf{R}$ *there exists one and only one bounded extension of R_t to $C_{\Pi,E}(0, T; Y)$ (resp. $L^p(0, T; Y)$, resp. $H^{1,p}_{\Pi,E}(0, T; Y)$), which will be called $\bar{R}_t$. Moreover, $\bar{R} : t \to \bar{R}_t\ \forall t \in \bar{\mathbf{R}}$ is a contraction semigroup on $C_{\Pi,E}(0, T; Y)$ (resp. $L^p(0, T; Y)$, resp. $H^{1,p}_{\Pi,E}(0, T; Y)$).*[14] *Finally, the infinitesimal generator $\bar{A}$ of $\bar{R}$ is an extension of A and we have*

$$D_{\bar{A}} = C^1_{\Pi,E}(0, T; Y) \quad (\textit{resp. } H^{1,p}_{\Pi,E}(0, T; Y), \textit{resp. } H^{2,p}_{\Pi,E}(0, T; Y)). \tag{2.23}$$

Proof. If $n \in \mathbf{N}$, using (2.20) we have

$$\| R_{nT}u \|_\infty = \| E^{-n}u \|_\infty \leqslant \| u \|_\infty \qquad \forall u \in \Pi_{T,E}(Y), \tag{2.24}$$

and if $0 < r < T$,

$$\begin{gathered} \| R_r u \|_\infty = \sup\{| u(t - r), t \in [0, T]|\} = \sup\{| E^{-1}u(t)|\}, \\ t \in [T - r, T] \cup \sup\{| u(t)|, t \in [0, T - r]\}. \end{gathered} \tag{2.25}$$

[14] $\bar{\mathbf{R}} : t \to \bar{R}_t$ is a class (C_0) group but it is not, in general, a contraction group.

Using (2.12, ii) we have

$$\| R_r u \|_\infty \leqslant \| u \|_\infty \qquad \forall u \in \Pi_{T,E}(Y), \quad 0 < r < T. \tag{2.26}$$

Therefore (2.24) and (2.26) give

$$\| R_t u \|_\infty \leqslant \| u \|_\infty \qquad \forall u \in \Pi_{T,E}(Y), \quad t \in \bar{\mathbf{R}}_+ . \tag{2.27}$$

Similarly we prove that

$$\| R_t u \|_{k,p} \leqslant \| u \|_{k,p} \qquad \forall u \in \Pi_{T,E}(Y), \quad p \geqslant 1, \quad k+1 \in \mathbf{N}. \tag{2.28}$$

The last part of this proof is very similar to that of Theorem 2.1 and therefore will be omitted.

3. Translations in Product Spaces

Let $[0, T]$ be a bounded interval of $\bar{R}_+$ and $\forall t \in [0, T]$ let Y_t be a Banach space (norm $|\ |_t$), and assume that

$$\left.\begin{array}{rl} \text{(i)} & \text{If } t \leqslant s,\ Y_t \subset Y_s\,; \\ \text{(ii)} & \text{There exists } \omega \in \mathbf{R} \text{ such that} \\ & |u|_{t+s} \leqslant e^{\omega s} |u|_t \quad \forall t, t+s \in [0, T], \quad u \in Y_t\,; \\ \text{(iii)} & \text{The map: } [0, T] \to \bar{\mathbf{R}}_+,\ t \to |u(t)|_t \text{ is continuous} \\ & \forall u \in \epsilon_0(0, T; Y_0). \end{array}\right\} \tag{2.29}$$

The completion of $\epsilon_0(0, T; Y_0)$ with respect to the norm

$$\| u \|_\infty = \sup\{| u(t)|_t\ t \in [0, T]\}$$
$$\left(\text{resp. } \| u \|_{k,\infty} = \sup \left\{\left| \frac{d^h u}{dt^h}(t) \right|_t, \ t \in [0, T], h = 0, 1,..., k\right\}\right) \tag{2.30}$$

will be called $C_0(0, T; Y)$ (resp. $C_0{}^k(0, T; Y)$). The completion of $\epsilon_0(0, T; Y)$ with respect to the norm

$$\| u \|_p = \left(\int_0^T | u(t)|_t^p \, dt\right)^{1/p}$$
$$\left(\text{resp. } \| u \|_{k,p} = \left(\sum_{h=0}^k \int_0^T \left| \frac{d^h u}{dt^h}(t) \right|_t^p dt\right)^{1/p}\right) \tag{2.31}$$

will be called $L^p(0, T; Y_t)$ (resp. $H_0^{k,p}(0, T; Y_t)$).

We now prove

THEOREM 2.3. *Let R be the semigroup on $\epsilon_0(0, T; Y_0)$ defined by* (2.5). *Then $\forall t \in \bar{\mathbf{R}}_+$ there exists one and only one bounded extension of R_t to $C_0(0, T; Y_t)$ (resp. $L^p(0, T; Y_t)$, resp. $H_0^{1,p}(0, T; Y_t)$) which will be called $\bar{R}_t$. Moreover $\bar{R} : t \to e^{-\omega t} \bar{R}_t$ is a contraction semigroup on $C_0(0, T; Y_t)$ (resp. $L^p(0, T; Y_t)$, resp. $H^{1,p}(0, T; Y_t)$). Finally, the infinitesimal generator $\bar{A}$ of $\bar{R}$ is an extension of A and we have*

$$D_A = C_0^1(0, T; Y_t) \quad (resp.\ H_0^{1,p}(0, T; Y_t),\ resp.\ H_0^{2,p}(0, T; Y_t)). \tag{2.32}$$

Proof. If $u \in \epsilon_0(0, T; Y_0)$ and $t \in \bar{\mathbf{R}}_+$ we have

$$\begin{aligned} \| D_t u \|_\infty &= \sup\{| u(s-t)|_s ,\ s \in [t, T]\} \\ &= \sup\{| u(s)|_{s+t} ,\ s \in [0, T-t]\}, \end{aligned} \tag{2.33}$$

and using (2.29, ii) we obtain

$$\| D_t u \|_\infty \leqslant e^{\omega t} \| u \|_\infty . \tag{2.34}$$

Similarly we find

$$\| D_t u \|_{k,p} \leqslant e^{\omega t} \| u \|_{k,p} . \tag{2.35}$$

The last part of the proof is very similar to that of Theorem 2.1 and therefore will be omitted.

4. MULTIPLICATION OPERATORS

Let Y be a Banach space (norm $|\ |$) and let $[0, T]$ be a subinterval of $\bar{\mathbf{R}}_+$. Let us introduce the "polynomial space" $P(0, T; Y)$, by putting

$$P(0, T; Y) = \Big\{ u \in C(0, T; Y);\ u(t) = \sum_{i=1}^{n} \varphi_i(t)\, x_i ,\quad \varphi_1, \varphi_2, \ldots, \varphi_n \text{ polynomials in } t \text{ and } x_1, x_2, \ldots, x_n \in Y,\ n \in \mathbf{N} \Big\}. \tag{2.36}$$

The subspace of $P(0, T; Y)$ of polynomials vanishing at zero, with all their derivatives of order $\leqslant k$, will be called $P_0^k(0, T; Y)$.

We now prove

LEMMA 2.1. *$P(0, T; Y)$ is dense in $C(0, T; Y)$.*

Proof. Let $u \in P(0, T; Y)$, and put

$$\begin{aligned} u_n(t) &= \sum_{p=0}^{n} r_{np}(tT)\, u(p/nT), \\ r_{np}(t) &= \binom{n}{p} t^p (1-t)^{n-p} \qquad p = 0,\ldots, n. \end{aligned} \tag{2.37}$$

Then, with a proof very similar to that one of Bernstein's theorem (Ref. [45], p. 8) we obtain the thesis.

The following corollary is trivial:

COROLLARY 2.1. *$P_0^k(0, T; Y)$ is dense in $C_0^k(0, T; Y)$ and in $H_0^{k,p}(0, T; Y)$, $k = 0, 1;\ldots$.*

Finally, let $\{B(t)\}_{t\in[0,T]}$ be a set of linear operators in Y such that

$$\left.\begin{array}{ll} \text{(i)} & B(t) \in K(Y) \quad \forall t \in [0, T]; \\ \text{(ii)} & \text{The map } [0, T] \to Y,\ t \to R(\lambda, B(t))y \text{ is continuous} \\ & \text{(resp. measurable and bounded, resp. having the first} \\ & \text{derivative measurable and bounded) } \forall y \in Y,\ \lambda \in C_+ . \end{array}\right\} \tag{2.38}$$

In what follows X (norm $\| \|$) is indifferently one of the spaces $C_0(0, T; Y)$, $L^p(0, T; Y)$, and $H_0^{1,p}(0, T; Y)$.

Let us introduce the linear operator B in X:

$$\begin{aligned} D_B &= \{u \in X;\ u(t) \in D_{B(t)}\ ,\ \forall t \in [0, T],\ t \to B(t)\, u(t) \in X\}, \\ (Bu)(t) &= B(t)\, u(t) \qquad \forall u \in D_B\ , \quad t \in [0, T].^{15} \end{aligned} \tag{2.39}$$

We now prove

THEOREM 2.4. *Assume that the set $\{B(t)\}_{t\in[0,T]}$ of linear operators in Y satisfies* (2.38). *Then the linear operator B, defined by* (2.39), *belongs to $K(X)$.*

Proof. We consider only the case $X = C_0(0, T; Y)$, the other cases being similar.

[15] If $X = L^p(0, T; Y)$ the notation $\forall \in [0, T]$ must be substituted by a.e. in $[0, T]$.

We first prove that D_B is dense in X; let $u \in X$ and $\epsilon \in \mathbf{R}$. By virtue of Lemma 2.1 there exists $n \in \mathbf{N}$ and $u_n \in P(0, T; Y)$ such that

$$\| u - u_n \| < \epsilon/2, \tag{2.40}$$

u_n being given by (2.37).

Since $D_{B(t)}$ is dense in $Y \, \forall t \in [0, T]$, then $\forall n, p \in \mathbf{N}$ there exists $w_{n,p} \in D_{B(p/nT)}$ such that

$$\| u(p/nT) - w_{np} \| < \epsilon/2. \tag{2.41}$$

Put

$$w_n(t) = \sum_{p=0}^{n} r_{np}(tT)\, w_{np} \qquad \forall t \in [0, T]. \tag{2.42}$$

Clearly $w_n \in D_B$ and

$$\| u_n - w_n \| < \epsilon/2, \tag{2.43}$$

since

$$\sum_{p=0}^{n} r_{np}(tT) = 1. \tag{2.44}$$

It follows that

$$\| u - w_n \| < \epsilon, \tag{2.45}$$

and therefore D_B is dense in X.

Now let $\lambda \in \mathbf{C}_+$, $v \in X$. The equation

$$\lambda u - Bu = v \tag{2.46}$$

is equivalent to

$$\lambda u(t) - B(t)\, u(t) = v(t) \qquad \forall t \in [0, T], \tag{2.47}$$

so that

$$\begin{aligned} u(t) &= R(\lambda, B(t))\, v(t) - v(t) \qquad \forall t \in [0, T], \\ B(t)\, u(t) &= \lambda R(\lambda, B(t))\, v(t) - v(t). \end{aligned} \tag{2.48}$$

Due to (2.38, ii) we deduce that $\lambda \in \rho(B)$ and

$$(R(\lambda, B)u)(t) = R(\lambda, B(t))\, u(t) \qquad \forall t \in [0, T]. \tag{2.49}$$

Since $B(t) \in K(Y)$ we find

$$\| R(\lambda, B)\| \leqslant \frac{1}{\operatorname{Re}\lambda} \qquad \forall\lambda \in \mathbf{C}_+, \tag{2.50}$$

and the thesis follows from the Hille–Yosida theorem.

5. The Semigroup Generated by d^2/dt^2 in $[0, T]$ (Dirichlet Conditions)

We first consider the case $Y = \mathbf{C}$. We put $L^p(0, T; \mathbf{C}) = L^p(0, T)$, $H^{1,p}(0, T; \mathbf{C}) = H^{1,p}(0, T)$, and introduce the semigroup G on $L^p(0, T)$,

$$\begin{aligned} &(G(t)\varphi)(s) = \int_0^T K(\xi, s, t)\, \varphi(\xi)\, d\xi, \qquad s \in [0, T], \\ &K(\xi, s, t) = 2T/\pi \sum_{n=1}^{\infty} e^{-n^2 t} \sin(n\pi\xi/T) \sin(n\pi s/T). \end{aligned} \tag{2.51}$$

We recall the following properties of $K(\xi, s, t)$ [43]:

$$\begin{aligned} &K(\xi, s, t) \geqslant 0 \qquad \forall t \in \mathbf{R}_+, \quad s \in]0, T[, \\ &\int_0^T K(\xi, s, t)\, d\xi \leqslant 1. \end{aligned} \tag{2.52}$$

It is well known [3] that $G : t \to G(t)$ is an analytic semigroup and that its infinitesimal generator A is given by

$$\begin{aligned} &D_A = \{u \in H^{2,p}(0, T); u(0) = u(T) = 0\}, \\ &Au = d^2u/dt^2 \qquad \forall u \in D_A. \end{aligned} \tag{2.53}$$

We also have

$$\begin{aligned} (R(\lambda, A)\varphi)(s) = {} & \frac{(\sinh(\rho - \rho s/T))}{(T\rho \sinh \rho)} \\ & \times \int_0^s \sinh(\rho y/T)\, \rho(y)\, dy + \frac{(\sinh \rho s)}{(T\rho \sinh \rho)} \\ & \times \int_s^T \sinh(\rho - \rho y/T)\, \varphi(y)\, dy, \\ & \text{for} \quad \lambda \in \overline{\mathbf{R}}_+, \quad \rho = \sqrt{\lambda}, \quad \varphi \in L^p(0, T). \end{aligned} \tag{2.54}$$

As is shown in Chapter V, Section 2, for every $p > 1$ there exists $\epsilon(p) \in \mathbf{R}_+$ such that $t \to e^{\epsilon(p)t}G(t)$ is an analytic contraction semigroup in $L^p(0, T)$.

We now prove

THEOREM 2.5. *Let us put* $\forall t \in \overline{\mathbf{R}}_+$

$$(G(t)u)(s) = \int_0^T K(\xi, s, t)\, u(\xi)\, d\xi \qquad \forall u \in \epsilon(0, T; Y), \quad s \in [0, T]. \tag{2.55}$$

Then there exists a bounded extension on $L^p(0, T; Y)$ *of* $G(t)$ *which we denote by* $\overline{G}(t)$ *and* $\overline{G} : t \to e^{-\epsilon(p)t}\overline{G}(t)$ *is an analytic contraction semigroup. Finally, the infinitesimal generator* A *of* G *is given by*

$$\begin{aligned} D_{\bar{A}} &= \{u \in H^{2,p}(0, T; Y); u(0) = u(T) = 0\}, \\ \bar{A}u &= d^2u/dt^2 \qquad \forall u \in D_{\bar{A}}\,. \end{aligned} \tag{2.56}$$

Proof. Let $u \in \epsilon(0, T; Y)$ and $t \in \overline{\mathbf{R}}$. Then we have, using (2.52),

$$|(G(t)u)(s)| \leqslant \int_0^T K(\xi, s, t)\, |\, u(\xi)|\, d\xi = (G(t)(|\, u(\cdot)|)(s), \tag{2.57}$$

from which we have

$$\|\, G(t)u\, \|_p \leqslant \|\, G(t)(|\, u(\cdot)|)\|_p \leqslant e^{-\epsilon(p)t}\, \|\, u\, \|_p\,, \tag{2.58}$$

and therefore there exists an extension of $G(t)$, $\overline{G}(t)$ on $L^p(0, T; Y)$. If $u \in P(0, T; Y)$ it is very easy to check that

$$\overline{G}(t + s)u = \overline{G}(t)\, \overline{G}(s)u \qquad \forall t, s \in \overline{\mathbf{R}}_+\,, \tag{2.59}$$

and that the map $\overline{R}_+ \to X$, $t \to \overline{G}(t)\, u$ is continuous; then, on the basis of (2.58) and of the density of $P(0, T; Y)$ in $L^p(0, T; Y)$ we see that $e^{\epsilon(p)}\overline{G}$ is an analytic contraction semigroup. Also, if $u \in P(0, T; Y)$, in view of (2.54) we have

$$\begin{aligned} (R(\lambda, \bar{A})u)(s) = {} & \frac{(\sinh(\rho - \rho s/T))}{(T\rho \sinh \rho)} \\ & \times \int_0^s \sinh(\rho y/T)\, u(y)\, dy + \frac{(\sinh \rho s)}{(T\rho \sinh \rho)} \\ & \times \int_s^T \sinh(\rho - \rho y/T)\, u(y)\, dy \qquad \forall \lambda\, \mathbf{R}\in{}_+\,, \quad \rho = \sqrt{\lambda}. \end{aligned} \tag{2.60}$$

Since $P(0, T; Y)$ is dense in $L^p(0, T; Y)$ we see that (2.60) is true for every $u \in L^p(0, T; Y)$; we then have

$$D_{\bar{A}} \subset \{u \in H^{2,p}(0, T; Y); u(0) = u(T) = 0\}. \tag{2.61}$$

Conversely, if $u \in \{u \in H^{2,p}(0, T; Y); u(0) = u(T) = 0\}$ there exists (Corollary 2.1) a sequence $\{u_n\}$ in $P_0^2(0, T; Y)$ such that

$$u_n \to u, \qquad \bar{A}u_n \to u''; \tag{2.62}$$

therefore $u \in D_{\bar{A}}$ and the theorem is proved.

6. The Semigroup Generated by d^n/dt^n in $[0, T]$ (with Periodic Conditions)

Let H be a Hilbert space whose inner product and norm, respectively, are denoted by $(\, , \,)$ and $| \; |$.

Let A be the linear operator defined by Theorem 2.2 (with $E = 1$) in $L^2(0, T; Y)$, viz.,

$$\begin{aligned} D_A &= \{u \in H^{1,2}(0, T; H), u(0) = u(T)\}, \\ Au &= -u' \qquad \forall u \in D_A . \end{aligned} \tag{2.63}$$

Lemma 2.1. *iA is self-adjoint.*

Proof. We only need to observe that the linear operator e^{tA} is unitary for every $t \in \mathbf{R}$.

We now prove

Theorem 2.6. *Put*

$$Q_n \begin{cases} = (-1)^{n/2+1} A^n & \text{if } n \text{ is even}, \\ = A^n & \text{if } n \text{ is odd}; \end{cases} \tag{2.64}$$

$$D_{Q_n} = \{u \in H^{n,2}(0, T; Y); u^{(k)}(0) = u^{(k)}(T), k = 0, 1, \dots, n-1\}.$$

Then Q_n belongs to $K(L^2(0, T; Y))$, and if n is even it is the infinitesimal generator of an analytic semigroup. Moreover, we have

$$R(\lambda, Q_n) = \prod_{i=1}^{n} R(| \lambda |^{1/n} x_1 , A) \qquad \text{for} \qquad \lambda \in C_+ , \tag{2.65}$$

where $x_1, x_2, \dots, x_n$ are the n-th roots of $(-1)^{n/2+1}$ if n is even and the n-th roots of 1 if n is odd.

Proof. Let n be odd, $n = 2p + 1$, and consider the equation

$$\lambda x - A^n x = y \qquad \text{for} \qquad \lambda \in C - i\mathbf{R}. \tag{2.66}$$

We can easily check that (2.65) is true, and furthermore if $x \in D_A$, we have

$$\lambda \| x \|^2 - (A^n x, x) = (x, y). \tag{2.67}$$

Considering Lemma 2.1 we have

$$\operatorname{Re}(A^n x, x) = \operatorname{Re}(AA^p x, A^p x) = 0, \tag{2.68}$$

and therefore from (2.68) we deduce that

$$\| x \| = \| R(\lambda, A^n)\, y \| \leqslant \frac{1}{\operatorname{Re}\lambda} \| y \|, \tag{2.69}$$

and that $A^n \in K(L^2(0, T; Y))$.

Now let n be even. We can easily check that (2.65) is true, and that the equation

$$\lambda x - Q_n x = y, \qquad \text{where} \qquad x = \rho e^{i\theta}, \quad \theta \neq \pm\pi, \tag{2.70}$$

is equivalent to

$$\rho e^{i\theta/2} x - (-1)^{n/2+1}\, e^{-i\theta/2} A^n x = y, \tag{2.71}$$

whence

$$\rho e^{i\theta/2} \| x \|^2 + e^{-i\theta/2}((iA)^{n/2}\, x, (iA)^{n/2}\, x) = (y, x). \tag{2.72}$$

It follows that

$$(\rho \cos \theta/2) \| x \|^2 \leqslant \| x \| \, \| y \|, \tag{2.73}$$

from which we have

$$\| R(\lambda, Q_n) \| \leqslant 1/(\rho \cos \theta/2), \tag{2.74}$$

and the thesis follows.

Remark 2.1. If n is odd and $p \neq 2$ it is well-known [14] that A^n is not the infinitesimal generator of a semigroup of class (C_0) in $L^p(0, T; Y)$. This is also true if $H = C$.

7. Powers of the Laplace Operator in R^n

Let R^n be the Euclidean space. We introduce the semigroup $t \to G(t)$ on $L^p(R^n)$, $p > 1$:

$$\begin{aligned} &G(t)\varphi = g_t * \varphi, \\ &g_t(x) = a_{mn} t^{-n/(2m)} e^{-|x|^{2m}/t}, \\ &a_{mn}^{-1} = \int_{R^n} e^{-|x|^{2m}}\, dx \qquad \text{for} \qquad m \in \mathbf{N}, \end{aligned} \tag{2.75}$$

where $*$ denotes convolution.

It is well-known [3] that $G : t \to G(t)$ is an analytic semigroup of contraction in $L^p(R^n)$ and that its infinitesimal generator is given by

$$\begin{aligned} &D_A = H^{2m,p}(R^n), \\ &Au = (-1)^{m+1} \Delta^m u \qquad \forall u \in H^{2m,p}(R^n) \end{aligned} \tag{2.76}$$

Δ^m being the m-th power of the Laplace operator in R^n. Reasoning similarly as in the proof of Theorem 2.5 we obtain

THEOREM 2.7. *Let us put*

$$\bar{G}(t)u = g_t * u \qquad \forall u \in D(R^n; Y), \quad t \in \mathbf{R}_+ .\text{[16]} \tag{2.77}$$

Then $\bar{G}(t)$ has a bounded extension $\tilde{G}(t)$ on $L^p(R^n; Y)$ and $\tilde{G} : t \to \tilde{G}(t)$ is an analytic semigroup of contraction. Finally, the infinitesimal generator $\tilde{A}$ of $\tilde{G}$ is given by

$$\begin{aligned} &Au = \Delta^m u \qquad \forall u \in H^{2m,p}(\mathbf{R}^n, Y), \\ &D^{\tilde{A}} \supset H^{2m,p}(\mathbf{R}^n; Y). \end{aligned} \tag{2.78}$$

[16] $D(R^n; Y)$ is the subspace of $\epsilon(R^n; Y)$ of functions having the support compact.

Chapter III. Abstract Differential Equations of the First Order

1. Cauchy's Problem (Uniqueness)

Let Y be a Banach space (norm $|\ |$) and $[0, T]$ be a subinterval of $\mathbf{R}_+$. Let $\{B(t)\}_{t\in[0,T]}$ be a set of linear operators in Y such that

$$\begin{aligned}
&\text{(i)}\quad B(t) \in K(Y) \qquad \forall t \in [0, T];\\
&\text{(ii)}\quad \rho(B(t)) \supset 0 \text{ and there exists } M \in \mathbf{R}_+ \text{ such that } |B^{-1}(t)| \leqslant M.
\end{aligned} \tag{3.1}$$

In what follows X (norm $\|\ \|$) denotes the space $C_0(0, T; Y)$ (resp. $L^p(0, T; Y)$), $p > 1$) and A and B, respectively, denote the operators on X:

$$\begin{aligned}
&D_A = C_0^1(0, T; Y) \qquad (\text{resp. } H_0^{1,p}(0, T; Y)),\\
&Au = -\frac{du}{dt} \qquad \forall u \in D_A,
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
&D_B = \{u \in X;\ u(t) \in D_{B(t)},\ \forall t \in [0, T],\ t \to B(t)\, u(t) \in X\},\\
&(Bu)(t) = B(t)\, u(t) \qquad \forall u \in D_B, \quad t \in [0, T].^{17}
\end{aligned} \tag{3.3}$$

We know that $A \in K(Y)$ (Theorem 2.1) and that if $\{B(t)\}_{t\in[0,T]}$ satisfies (2.38) then $B \in K(X)$ (Theorem 2.4). Finally the completion of $C_0^1(0, T; Y) \cap D_B$ (resp. $H_0^{1,p}(0, T; Y) \cap D_B$) with respect to the norm,

$$\| u \| + \left\| -\frac{du}{dt} + Bu \right\|, \tag{3.4}$$

will be called $\Lambda_0^\infty(0, T; Y; B)$ (resp. $\Lambda_0^p(0, T; Y; B)$).

We now consider Cauchy's problem:

$$\begin{aligned}
&\lambda u + \frac{du}{dt} - B(t)\, u(t) = f(t),\\
&u(0) = 0, \qquad f \in X, \quad \lambda \in C, \quad t \in [0, T].
\end{aligned} \tag{3.5}$$

[17] If $\{B(t)\}_{t\in[0,T]}$ satisfies (3.17) we must assume that $X = L^p(0, T; Y)$ and Y is reflexive. Then $L^p(0, T; Y)$ is reflexive as shown in Ref. [14].

We shall say that $u \in X$ is a weak solution of the problem (3.5) if there exists a sequence $\{u_n\}$ in X such that

$$\left.\begin{array}{ll} \text{(i)} & u_n \in C_0^1(0, T; Y) \ (\text{resp.}\ H_0^{1,p}(0, T; Y)) \qquad \forall n \in N; \\ \text{(ii)} & u_n(t) \in D_{B(t)} \quad \forall t \in [0, T] \text{ and } t \to B(t)\, u_n(t) \in C_0(0, T; Y) \\ & (\text{resp.}\ L^p(0, T; Y)); \\ \text{(iii)} & u_n \to u \text{ and } \lambda u_n + u_n' - Bu_n \to f \text{ in } C_0(0, T; Y) \\ & (\text{resp.}\ L^p(0, T; Y)). \end{array}\right\} \tag{3.6}$$

It is clear that u is a weak solution of (3.6) if and only if it is a weak solution of the equation

$$\lambda u - Au - Bu = f. \tag{3.7}$$

We have the following result of uniqueness:

THEOREM 3.1. *Assume that the set* $\{B(t)\}_{t \in [0,T]}$ *satisfies* (2.38). *Then if* $u \in \mathbf{C}_0^1(0, T; Y) \cap D_B$ (*resp.* $H_0^{1,p}(0, T; Y) \cap D_B$) *we have*

$$\| u \| \leqslant x(\lambda) \left\| \lambda u + \frac{du}{dt} - Bu \right\| \qquad \forall \lambda \in \mathbf{C}, \tag{3.8}$$

where

$$x(\lambda) \begin{cases} = 1/\mathrm{Re}\,\lambda & \text{if } \lambda \in C_+, \\ = T e^{(1 - \mathrm{Re}\lambda/T)} & \text{if } \lambda \notin C_+, \end{cases} \tag{3.9}$$

and the problem (3.5) *has at most one weak solution.*

Proof. If $\lambda \in \mathbf{C}_+$ the thesis is a consequence of Theorems 1.1 and 1.5. Now let $\lambda \notin \mathbf{C}_+$. Then the equation

$$\lambda u - Au - Bu = f \tag{3.10}$$

is equivalent to

$$\epsilon \bar{u} - A\bar{u} - B\bar{u} = \bar{f} \qquad \text{for} \qquad \epsilon \in \mathbf{R}_+, \tag{3.11}$$

where

$$\bar{u}(t) = u(t)\, e^{(\lambda - \epsilon)t}, \qquad \bar{f}(t) = f(t)\, e^{(\lambda-\epsilon)t}. \tag{3.12}$$

It follows that

$$\| u \| \leqslant e^{-(\mathrm{Re}\lambda - \epsilon)T} \| \bar{u} \| \leqslant \frac{1}{\epsilon} e^{-(\mathrm{Re}\lambda-\epsilon)T} \| f \|, \tag{3.13}$$

and, taking the maximum of the function

$$g(\epsilon) = \frac{1}{\epsilon}\, e^{-(\mathrm{Re}\lambda-\epsilon)T} \tag{3.14}$$

we obtain the thesis.

Remark 3.1. If u is a weak solution of the problem (3.5) then it is not true, in general, that $u \in \Lambda_0^\infty(0, T; Y; B)$(resp. $\Lambda_0^p(0, T; Y; B)$).

2. Cauchy's Problem (Existence)

We give now some conditions on the set $\{B(t)\}_{t\in[0,T]}$ which assure that A and B satisfy the hypotheses of one of the Theorems 1.9–1.12, and therefore that the problem (3.5) has one and only one weak solution. We shall assume that $\{B(t)\}_{t\in[0,T]}$ satisfies, besides (3.1), one of the following conditions:

$$\left.\begin{array}{ll}\text{(i)} & B(t) \text{ is the infinitesimal generator of an analytic semigroup} \\ & \text{in } X \quad \forall t \in [0, T]; \\ \text{(ii)} & \text{If } t \leqslant s,\ D_{B(t)} \subset D_{B(s)}, \text{ and there exists } K \in \mathbf{R}_+ \text{ and } \alpha \in\,]0, 1] \\ & \text{such that} \\ & |\, B(t)\, B^{-1}(s) - 1 \,| \leqslant K \,|\, t - s \,|^\alpha \qquad \forall t, s \in [0, T], \quad t \geqslant s;\end{array}\right\} \tag{3.15}$$

$$\left.\begin{array}{ll}\text{(i)} & B(t) \text{ is the infinitesimal generator of an analytic semigroup} \\ & \text{in } X \quad \forall t \in [0, T]; \\ \text{(ii)} & \text{For every } \lambda \in \mathbf{C}_+ \text{ and } x \in Y \text{ the map: } [0, T] \to Y, \\ & t \to R(\lambda, B(t)x) \text{ is derivable, its derivative is continuous} \\ & \text{(resp. measurable and bounded), and there exists } K \in \mathbf{R}_+ \\ & \text{and } \alpha \in\,]0, 1] \text{ such that} \\ & \qquad \left\| \dfrac{dR(\lambda, B(t))}{dt} \right\| \leqslant K t^{-\alpha};\end{array}\right\} \tag{3.16}$$

$$\left.\begin{array}{ll}\text{(i)} & \{B(t)\}_{t\in[0,T]} \text{ satisfies (2.38);} \\ \text{(ii)} & \text{If } t \leqslant s,\ D_{B^2(t)} \subset D_{B^2(s)}, \text{ and there exists } K \in \mathbf{R}_+ \text{ such that} \\ & \quad |\, B^2(t)\, B^{-2}(s) - 1 \,| \leqslant K \,|\, t - s \,| \qquad \forall t, s \in [0, T].\end{array}\right\} \tag{3.17}$$

If Y is reflexive and $X = L^p(0, T; Y)$ we shall also consider the following conditions:

$$\begin{array}{l}\text{If } t \leqslant s,\ D_{B(t)} \subset D_{B(s)} \text{ and there exists } K \in \mathbf{R}_+ \text{ such that} \\ \qquad |\, B(t)\, B^{-1}(s) - 1 \,| \leqslant K \,|\, t - s \,| \qquad \forall t, s \in [0, T].\end{array} \tag{3.18}$$

We now show the result:

THEOREM 3.2. *Let A and B be the operators defined by* (3.2) *and* (3.3); *assume that the set* $\{B(t)\}_{t\in[0,T]}$ *satisfies* (3.1) *and one of the conditions* (3.15)–(3.18).[17] *Then $A + B$ is preclosed and $\overline{A + B} \in K(X)$.*

Proof. We consider only the case $X = C_0(0, T; Y)$, the case $X = L^p(0, T; Y)$ being completely analogous.

First assume that (3.14) be satisfied; we remark that the set $\{B(t)\}_{t\in[0,T]}$ satisfies (2.38) in view of (3.1), (3.15), and the identity

$$R(\lambda, B(t)) - R(\lambda, B(s)) = R(\lambda, B(t))(B(t)\, B^{-1}(s) - 1)\, B(s)\, R(\lambda, B(s)) \qquad \forall s, t \in [0, T], \quad s \geqslant t, \tag{3.19}$$

and therefore $B \in K(X)$ (Theorem 2.4).

Now let $u \in D_B$ and $v = Bu$. We then have

$$(e^{tA}u)(s) \begin{cases} = B^{-1}(s-t)\, v(s-t) & \text{if } s - t \in]0, T], \\ = 0 & \text{if } s - t \notin]0, T], \end{cases} \tag{3.20}$$

and therefore, in view of (3.15), $e^{tB}u \in D_B$ and

$$(Be^{tA}u)(s) \begin{cases} = B(s)\, B^{-1}(s-t)\, v(s-t) & \text{if } s - t \in]0, T], \\ = 0 & \text{if } s - t \notin]0, T]. \end{cases} \tag{3.21}$$

It follows that

$$(Be^{tA}u - e^{tA}Bu)(s) \begin{cases} = (B(s)\, B^{-1}(s-t) - 1)\, v(s-t) & \text{if } s - t \in]0, T], \\ = 0 & \text{if } s - t \notin]0, T], \end{cases} \tag{3.22}$$

and, recalling (3.15),

$$\| Be^{tA}u - e^{tA}Bu \| \leqslant Kt^{\alpha} \| Bu \|, \tag{3.23}$$

the thesis follows from Theorem 1.11.

Assume now that (3.16) is satisfied; then (2.38) is clearly satisfied and $B \in K(X)$ (Theorem 2.4).

Let $u \in D_A$. We then have [18, 45],

$$(e^{tB}u)(s) = e^{tB(s)}u(s) = \frac{1}{2\pi i} \int_{\Gamma} e^{t\lambda} R(\lambda, B(s))\, u(s)\, d\lambda, \tag{3.24}$$

Γ being a suitable contour in C around $\sigma(B(s))$. Due to (3.15) $e^{tB}u \in D_A$, and we have

$$(Ae^{tB}u)(s) = -\frac{1}{2\pi i}\int_\Gamma e^{t\lambda}\frac{dR(\lambda, B(s))}{ds}u(s)\,d\lambda + (e^{tB}Au)(s) \qquad \forall s \in [0, T], \tag{3.25}$$

from which, using (3.16),

$$\| Ae^{tB}u - e^{tB}Au \| \leqslant K't^{\alpha-1} \qquad \forall t \in [0, T], \tag{3.26}$$

K' being a suitable constant; the thesis follows now from Theorem 1.9. The other cases are proved analogously, and the last affirmation is a consequence of the identitites

$$\begin{aligned} R(-\lambda, \overline{A+B})u &= w_{2\lambda}R(\lambda, \overline{A+B})(w_{-2\lambda}u), \\ R(0, \overline{A+B})u &= w_\lambda R(\lambda, \overline{A+B})\, w_{-\lambda}u) \qquad \text{for} \qquad u \in X, \end{aligned} \tag{3.27}$$

where $w_\lambda(t) = e^{\lambda t}\ \forall t \in [0, T]$.

The following corollary is immediate:

COROLLARY 3.1. *Let A and B satisfy the hypotheses of Theorem 3.2. Then $\overline{A+B}$ is an isomorphism between $\Lambda_0^\infty(0, T; Y; B)$ (resp. $\Lambda_0^p(0, T; Y; B)$) and $C_0(0, T; Y)$ (resp. $L^p(0, T; Y)$). Moreover, the problem (3.5) has one and only one weak solution belonging to $\Lambda_0^\infty(0, T; Y; B)$ (resp. $\Lambda_0^p(0, T; Y; B)$).*

3. INTEGRODIFFERENTIAL EQUATIONS

We use here the same notations of Section 2. Let k be a strongly continuous map $[0, T] \times [0, T] \to \mathscr{L}(Y, Y)$.[18] We now consider the problem

$$\begin{aligned} \lambda u(t) + \frac{du(t)}{dt} - B(t)\,u(t) - \int_0^t K(t, s)\,u(s)\,ds &= f(t), \\ u(0) = 0 \qquad \text{for} \qquad \lambda \in C, \qquad & f \in X. \end{aligned} \tag{3.28}$$

[18] $\mathscr{L}(Y, Y)$ is the Banach algebra of the linear bounded operators on Y.

We say that u is a weak solution of (3.27) if it is a weak solution of the problem

$$\lambda u(t) + \frac{du(t)}{dt} - B(t)\,u(t) = f(t) + \int_0^t K(t, s)\,u(s)\,ds, \qquad u(0) = 0. \tag{3.29}$$

Let us introduce the linear operator K on X,

$$(Ku)(t) = \int_0^t k(t, s)\,u(s)\,ds \qquad \text{for} \qquad t \in [0, T], \quad \forall u \in X. \tag{3.30}$$

We prove the following result:

LEMMA 3.3. *K is bounded and*

$$\| K \| \leqslant MT, \tag{3.31}$$

where

$$M = \sup\{| k(t, s)|, t, s \in [0, T]\}. \tag{3.32}$$

Proof. We first remark that for the Banach–Steinhaus theorem we have $M < +\infty$.

Now let $X = C_0(0, T; Y)$ and $u \in X$; for every $t \in [0, T]$ the map $s \to | k(t, s)\,u(s)|$ is measurable (Ref. [14] p. 616), and therefore we have

$$|(Ku)(t)| \leqslant M \int_0^t | u(s)|\,ds \leqslant MT \| u \|_\infty \qquad \forall t \in [0, T], \tag{3.33}$$

and the thesis follows.

Finally, let $X = L^p(0, T; Y)$ and $u \in L^p(0, T; Y)$. We have

$$\| Ku \|_p^p \leqslant \int_0^T \left(\int_0^t | K(t, s)|\,| u(s)|\,ds\right)^p dt \leqslant M^p T^p \| u \|_p^p, \tag{3.34}$$

and (3.31) follows.

We now prove

THEOREM 3.3. *Let A, B, and K be the operators defined respectively by* (3.2), (3.3), *and* (3.29); *assume that the set $\{B(t)\}_{t\in[0,T]}$ satisfies* (3.1) *and one of the conditions* (3.15)–(3.18).[19] *Then $A + B + K$ is preclosed and $\overline{A + B + K} = \overline{A + B} + K \in K(X)$.*

[19] $L^p(0, T; K_t)$ is the space defined in Chapter II, Section 3.

Proof. Let $\lambda \in \mathbf{C}$, $f \in X$. Put

$$(K_\mu u)(t) = \int_0^t e^{-\mu(t-s)} k(t, s)\, u(s)\, ds \qquad \forall u \in X, \quad \mu \in \mathbf{C}_+ . \tag{3.35}$$

Due to Lemma 3.1, we have

$$\| K_\mu \| \leqslant MT. \tag{3.36}$$

Let $\mu > MT - \operatorname{Re} \lambda$ and put

$$\lambda_1 = \lambda + \mu - MT, \qquad H = K_\mu - MT. \tag{3.37}$$

Then $\lambda_1 \in \mathbf{C}_+$ and $\| e^{tH} \| \leqslant 1\ \forall t \in \mathbf{R}_+$. Therefore H belongs to $K(X)$ and in view of Theorem 1.8 $A + B + H$ is preclosed and $\overline{A + B + H} \in K(X)$.

It is now easy to verify that $A + B + K$ is preclosed:

$$R(\lambda, \overline{A + B} + K) = w_\mu R(\lambda_1, \overline{A + B} + H)\, w_{-\mu}, \qquad w_\mu(t) = e^{\mu t}. \tag{3.38}$$

COROLLARY 3.2. *Let A, B, and K satisfy the hypotheses of Theorem* 3.3. *Then $\overline{A + B} + K$ is an isomorphism between $\Lambda_0^\infty(0, T; Y; B)$ (resp. $\Lambda_0^p(0, T; Y; B)$) and $C_0(0, T; Y)$ (resp. $L^p(0, T; Y)$). Moreover the problem* (3.28) *has one and only one weak solution belonging to $\Lambda_0^\infty(0, T; Y; B)$ (resp. $\Lambda_0^p(0, T; Y; B)$).*

4. PERIODIC PROBLEMS

Let Y be a Banach space (norm $|\ |$) and $[0, T]$ be a subinterval of $\overline{\mathbf{R}}_+$. Let E be a linear operator in Y satisfying (2.12). We shall use the notations of Chapter II, Section 2 and denote by X (norm $\|\ \|$) the space $\mathbf{C}_{\Pi,E}(0, T; Y)$ (resp. $L^p(0, T; Y)$).

Let $\{B(t)\}_{t \in \mathbf{R}}$ be a set of linear operators in Y such that

$$\left.\begin{array}{ll} \text{(i)} & B(t) \in K(Y) \qquad \forall t \in [0, T]; \\ \text{(ii)} & \rho(B(t)) \supset 0 \text{ and there exists } M \in \mathbf{R}_+ \text{ such that } | B^{-1}(t) | \leqslant M; \\ \text{(iii)} & \text{If } X = C_{\Pi,E}(0, T; Y) \text{ then } B(t + T) = B(t) \qquad \forall t \in \mathbf{R}. \end{array}\right\} \tag{3.39}$$

In what follows A and B are respectively the operators on X,

$$\begin{aligned} &D_A = C^1_{\Pi,E}(0, T; Y) \qquad (\text{resp. } H^{1,p}_{\Pi,E}(0, T; Y)), \\ &Au = -\frac{du}{dt} \qquad \forall u \in D_A , \end{aligned} \tag{3.40}$$

$$\begin{aligned} D_B &= \{u \in X;\, u(t) \in D_{B(t)}\} \qquad \forall t \in \mathbf{R}, \quad t \to B(t)\, u(t) \in X, \\ (Bu)(t) &= B(t)\, u(t) \qquad \forall u \in D_B , \quad t \in \mathbf{R}. \end{aligned} \tag{3.41}$$

We know that $A \in K(X)$ (Theorem 2.1) and that if $\{B(t)\}_{t\in\mathbf{R}}$ satisfies (2.38) then $B \in K(X)$ (Theorem 2.4).

Finally, the completion of

$$C^1_{\Pi,E}(0, T; Y) \cap D_B \qquad (\text{resp. } H^1_{\Pi,E}(0, T; Y) \cap D_B)$$

with respect to the norm,

$$\| u \| + \left\| -\frac{du}{dt} + Bu \right\|, \tag{3.42}$$

will be called $\Lambda^\infty_{\Pi,E}(0, T; Y; B)$ (resp. $\Lambda^p_{\tau,E}(0, T; Y; B)$).

We now consider the problem

$$\begin{aligned} &\lambda u + \frac{du}{dt} - B(t)\, u(t) = f(t), \\ &u(0) = u(T) \qquad \text{for} \qquad f \in X, \quad \lambda \in \mathbf{C}_+ . \end{aligned} \tag{3.43}$$

We shall say that $u \in X$ is a weak solution of the problem (3.43) if there exists a sequence $\{u_n\}$ in X such that

$$\left.\begin{aligned} &\text{(i)} \quad u_n \in C^1_{\Pi,E}(0, T; Y) \quad (\text{resp. } L^p(0, T; Y)) \qquad \forall n \in \mathbf{N}; \\ &\text{(ii)} \quad u_n(t) \in D_{B(t)} \qquad \forall t \in \mathbf{R} \text{ and } t \to B(t)\, u_n(t) \in C^1_{\Pi,E}(0, T; Y) \\ &\qquad (\text{resp. } H^{1,p}_{\Pi,E}(0, T; Y)); \\ &\text{(iii)} \quad u_n \to u \text{ and } \lambda u_n + \frac{du_n}{dt} - Bu_n \to f \text{ in } C^1_{\Pi,E}(0, T; Y) \\ &\qquad (\text{resp. } H^1_{\Pi,E}(0, T; Y)). \end{aligned}\right\} \tag{3.44}$$

It is clear that u is a weak solution of (3.43) if and only if it is a weak solution of the equation

$$\lambda u - Au - Bu = f. \tag{3.45}$$

The following result of uniqueness is a consequence of Theorems 1.1 and 1.5.

THEOREM 3.4. *Assume that the set* $\{B(t)\}_{t\in\mathbf{R}}$ *satisfies* (2.38) *and* (3.39, iii). *Then if* $u \in C^1_{\Pi,E}(0, T; Y) \cap D_B$ (resp. $H^{1,p}_{\pi,E}(0, T; Y) \cap D_B$), *we have*

$$\| u \| \leqslant \frac{1}{\operatorname{Re}\lambda} \left\| \lambda u + \frac{du}{dt} - Bu \right\| \qquad \forall \lambda \in \mathbf{C}_+ , \tag{3.46}$$

and the problem (3.43) *has at most one weak solution.*

We now give some conditions on the set $\{B(t)\}_{t\in\mathbf{R}}$ which assure that A and B satisfy the hypothesis of one of the theorems 1.9–1.12, and therefore the problem (3.43) has one and only one weak solution.

We shall assume that $\{B(t)\}_{t\in\mathbf{R}}$ satisfies, besides (3.38), one of the following conditions:

$$\left.\begin{array}{ll}
\text{(i)} & B(t) \text{ is the infinitesimal generator of an analytic semigroup} \\
& \text{in } X \quad \forall t \in \mathbf{R}; \\
\text{(ii)} & \text{The domain of } D_{B(t)} \text{ is independent of } t \text{ and there exists} \\
& K \in \mathbf{R}^+ \text{ and } \alpha \in]0, 1] \text{ such that} \\
& \qquad | B(t) B^{-1}(s) - 1 | \leqslant K | t - s |^{\alpha} \quad \forall t, s \in \mathbf{R};
\end{array}\right\} \tag{3.47}$$

$$\left.\begin{array}{ll}
\text{(i)} & B(t) \text{ is the infinitesimal generator of an analytic semigroup} \\
& \text{in } X \quad \forall t \in [0, T]; \\
\text{(ii)} & \text{For every } \lambda \in \mathbf{C}_+ \text{ and } x \in Y \text{ the map: } \mathbf{R} \to Y,\ t \to R(\lambda, B(t))x \\
& \text{is differentiable, its derivative is continuous (resp. measurable} \\
& \text{and bounded) and there exists } K \in \mathbf{R}_+ \text{ and } \alpha \in]0, 1] \text{ such} \\
& \text{that} \\
& \qquad \left\| \dfrac{dR(\lambda, B(t))}{dt} \right\| \leqslant K t^{-\alpha}; \\
\text{(iii)} & \text{If } X = C_{\Pi,E}(0, T; Y) \text{ then } \dfrac{dR(\lambda, B(t))}{dt} \text{ is periodic with} \\
& \text{period } T.
\end{array}\right\} \tag{3.48}$$

$$\left.\begin{array}{ll}
\text{(i)} & \{B(t)\}_{t\in[0,T]} \text{ satisfies (2.38);} \\
\text{(ii)} & D_{B^2(t)} \text{ is independent of } t \text{ and there exists } K \in \mathbf{R}_+ \text{ such that} \\
& \qquad | B^2(t) B^{-2}(s) - 1 | \leqslant K | t - s | \quad \forall t, s \in \mathbf{R}.
\end{array}\right\} \tag{3.49}$$

If Y is reflexive and $X = L^p(0, T; Y)$ we shall also consider the following condition:

$$\begin{aligned} &D_B(t) \text{ is independent of } t \text{ and there exists } K \in \mathbf{R}_+ \text{ such that} \\ &\qquad | B(t)\, B^{-1}(s) - 1 | \leqslant K \,| t - s | \qquad \forall t, s \in \mathbf{R}. \end{aligned} \tag{3.50}$$

The following theorem is proved in the same way as Theorem 3.2.

THEOREM 3.5. *Let A and B be the operators defined by* (3.40) *and* (3.41). *Assume that the set $\{B(t)\}_{t\in\mathbf{R}_+}$ satisfies* (3.39) *and one of the conditions* (3.47)–(3.50).[17] *Then $\overline{A + B}$ is preclosed and $A + B \in K(X)$.*

The following corollary is immediate.

COROLLARY 3.3. *Let A and B satisfy the hypotheses of Theorem* 3.5. *Then $\overline{A + B}$ is an isomorphism between $\Lambda^{\infty}_{\Pi,E}(0, T; Y; B)$* (*resp.* *$\Lambda^{p}_{\Sigma,E}(0, T; Y; B)$*) *and $C_{\Pi,E}(0, T; Y)$* (*resp.* *$L^p(0, T; Y)$*). *Moreover the problem* (3.43) *has one and only one weak solution belonging to $\Lambda^{\infty}_{\Pi,E}(0, T; Y; B)$* (*resp.* *$\Lambda^{p}_{\Pi,E}(0, T; Y; B)$*).

Chapter IV.
Abstract Differential Equations of Higher Order and Systems

1. WAVE EQUATIONS (PRELIMINARIES)

Let H be a Hilbert space, whose inner product andnor m are denoted by ( ) and $|\ |$.

Let $[0, T]$ be a bounded subinterval of $\overline{\mathbf{R}}_+$ and $\{C(t)\}_{t\in[0,T]}$ be a set of positive self-adjoint operators in H such that

$$\left.\begin{aligned} &\text{(i) If } t \leqslant s,\ D_{C(t)} \subset D_{C(s)} \text{ and there exists } K \in \mathbf{R}_+ \text{ such that} \\ &\qquad\qquad | \mathbf{C}(s)\, \mathbf{C}^{-1}(t) - 1 | \leqslant K \,| s - t |; \\ &\text{(ii) There exists } \omega \in \mathbf{R} \text{ such that} \\ &(C(t + s)x, x) \leqslant e^{2\omega s}(C(t)x, x) \quad \forall s, t, s + t \in [0, T], \quad x \in D_{C(t)}\,. \end{aligned}\right\} \tag{4.1}$$

Let us put

$$K_t = D_{C^{1/2}(t)} \qquad \forall t \in [0, T] \tag{4.2}$$

where K_t is a Hilbert space with respect to the inner product,

$$(x, y)_t = (C^{1/2}(t)x, C^{1/2}(t)y) \qquad \forall x, y \in K_t, \tag{4.3}$$

and we shall denote by $| \ |_t$ the norm in K_t.

We first prove the following two lemmas:

LEMMA 4.1. *Let* $\{C(t)\}_{t\in[0,T]}$ *be a set of positive self-adjoint linear operators in* H *satisfying* (4.1) *and* $\{K_t\}_{t\in[0,T]}$ *be the set of spaces defined by* (4.2). *Then the set* $\{C(t)\}_{t\in[0,T]}$ *satisfies* (2.29).

Proof. We proceed in the proof by successive steps:

(i) $| C^{1/2}(t+s)x | \leqslant e^{\omega s} | C^{1/2}(t)x | \qquad \forall x \in K_t$.

We first note that if $x \in D_{C(t)}$ then (i) is equivalent to (4.1, ii). Now let $x \in D_{C^{1/2}(t)}$. Since $D_{C(t)}$ is dense in K_t there exists a sequence $\{x_n\}$ in $D_{C(t)}$ converging to x in K_t. It follows that

$$C^{1/2}(t)\, x_n \to C^{1/2}(t)x \quad \text{in} \quad H. \tag{4.4}$$

If $n, m \in \mathbf{N}$ we have, due to (4.1, ii),

$$| C^{1/2}(t+s)\, x_n - C^{1/2}(t+s)\, x_m | \leqslant e^{\omega s} | C(t)\, x_n - C(t)\, x_m |. \tag{4.5}$$

Therefore $x \in D_{C(t+s)}$ and

$$C^{1/2}(t+s)\, x_n \to C^{1/2}(t+s)x \quad \text{in} \quad H. \tag{4.6}$$

Passing to the limit for $n \to \infty$ in the inequality

$$| C^{1/2}(t+s)\, x_n | \leqslant e^{\omega s} | C^{1/2}(t)\, x_n |, \tag{4.7}$$

we obtain (i).

(ii) The map: $[0, T] \to \mathscr{L}(H, H)$, $t \to C(t)\, C^{-1}(0)$ is continuous. If $t \leqslant s$ we have

$$C(s)\, C^{-1}(0) - C(t)\, C^{-1}(0) = (C(s)\, C^{-1}(t) - 1)\, C(t)\, C^{-1}(0), \tag{4.8}$$

whence, recalling (4.1, i),

$$| C(s)\, C^{-1}(0) - C(t)\, C^{-1}(0)| \leqslant K\, | t - s |\, (1 + Kt). \tag{4.9}$$

(iii) The map: $[0, T] \to \mathbf{C}$, $t \to (C(t)\, x, y)$ is continuous for every $x \in D_{C(0)}$, $y \in H$.

We only need to observe that if $x \in D_{C(0)}$ and $y \in H$ we have

$$(C(t)x, y) = (C(t)\, C^{-1}(0)\, C(0)x, y), \tag{4.10}$$

and then we use (ii).

(iv) If $x_n \to x$ in K_0 then $|\, C^{1/2}(t)\, x_n \,| \to |\, C^{1/2}(t)\, x \,|$ uniformly in t in $[0, T]$.

Let $\epsilon \in \mathbf{R}_+$. Then there exists $n_\epsilon \in \mathbf{N}$ such that

$$|\, C^{1/2}(0)x - C^{1/2}(0)\, x_n \,| < \epsilon e^{-\omega T} \qquad \forall n > n_\epsilon , \tag{4.11}$$

and, due to (i),

$$|\, C^{1/2}(t)x - C^{1/2}(t)\, x_n \,| < \epsilon \qquad \forall n > n_\epsilon , \quad t \in [0, T], \tag{4.12}$$

from which

$$\begin{aligned} &|\,|\, C^{1/2}(t)x \,|^2 - |\, C^{1/2}(t)\, x_n \,|^2 \,| \\ &\quad \leqslant |(C^{1/2}(t)(x - x_n), C^{1/2}(t)x)| + |\, C^{1/2}(t)\, x_n , C^{1/2}(t)(x - x_n)| \\ &\quad \leqslant \epsilon e^{\omega t}(|\, C^{1/2}(0)x \,| + |\, C^{1/2}(0)\, x_n \,|). \end{aligned} \tag{4.13}$$

(v) The map $[0, T] \to \mathbf{C}$, $t \to |\, C(t)\, x \,|$ is continuous for every $x \in K_0$.

Let $x_n \to x$ in K_0 and $t, s \in [0, T]$. Then

$$\begin{aligned} &|\,|\, C^{1/2}(t)x \,| - |\, C^{1/2}(s)x \,|\,| \\ &\quad \leqslant |\,|\, C^{1/2}(t)x \,| - |\, C^{1/2}(t)\, x_n \,|\,| + |\,|\, C^{1/2}(s)x \,| - |\, C^{1/2}(s)\, x_n \,|\,| \\ &\quad \leqslant |\,|\, C^{1/2}(t)\, x_n \,| - |\, C^{1/2}(s)\, x_n \,|\,|, \end{aligned}$$

and, using (iii) and (iv), the thesis follows.

(vi) The map: $[0, T] \to \mathbf{C}$, $t \to |\, C^{1/2}(t)\, u(t)|$ is continuous for every $u \in \epsilon_0(0, T; K_0)$.

If $u \in \epsilon_0(0, T; K_0)$ and $t, s \in [0, T]$ we have

$$\begin{aligned} &|\,|\, C^{1/2}(t)\, u(t)| - |\, C^{1/2}(s)\, u(s)|\,| \\ &\quad \leqslant |\,|\, C^{1/2}(t)\, u(t)| - |\, C^{1/2}(s)\, u(t)|\,| + \|\, C^{1/2}(s)\, u(t) - C^{1/2}(s)\, u(s)\| \\ &\quad \leqslant |\,|\, C^{1/2}(t)\, u(t)| - |\, C^{1/2}(s)\, u(t)|\,| + e^{ws}\,|\, C^{1/2}(0)(u(t) - u(s))|. \end{aligned} \tag{4.15}$$

Therefore, due to (v), the thesis follows. The lemma is completely proved.

LEMMA 4.2. *Let* $\{C(t)\}_{t\in[0,T]}$ *be a set of positive self-adjoint linear operators in* H *satisfying* (4.1). *Then* $L^p(0, T; K_t)$,[19] $p > 1$ *is reflexive.*

Proof. On the basis of a theorem of Milman [30] it suffices to prove that $L^p(0, T; K_t)$ is uniformly convex (Ref. [45], p. 126). Then let $x, y \in L^p(0, T; K_t)$, $p \geqslant 2$, we have

$$\| x + y \|_p^p + \| x - y \|_p^p = \int_0^T \{| x(t) + y(t)|_t^p + | x(t) - y(t)|_t^p \, dt. \tag{4.16}$$

Then, using the inequality

$$| x(t) + y(t)|_t^p + | x(t) - y(t)|_t^p \leqslant 2^{p/2}(| x(t)|_t^2 + | y(t)|_t^2)^{p/2}, \tag{4.17}$$

we find

$$\| x + y \|_p^p + \| x - y \|_p^p \leqslant 2^{p/2}(\| x \|_p^2 + \| y \|_p^2)^{p/2}. \tag{4.18}$$

Now let $\epsilon \in \mathbf{R}_+$ and x, y such that

$$\| x \|_p \leqslant 1, \qquad \| y \|_p \leqslant 1 \qquad \text{and} \qquad \| x - y \|_p \geqslant \epsilon. \tag{4.19}$$

From (4.18) it follows that

$$\| x + y \|_p \leqslant 2(1 - \delta_\epsilon), \tag{4.20}$$

where

$$\delta_\epsilon = 1 - \tfrac{1}{2}(2^p - \epsilon^p)^{1/p}. \tag{4.21}$$

Therefore $L^p(0, T; K_t)$ is reflexive if $p \geqslant 2$. Finally, it is easy to see that if $p \geqslant 2$ the dual of $L^p(0, T; K_t)$ is $L^q(0, T; K_t)$, where $1/p + 1/q = 1$.

The theorem is completely proved.

We define the linear operator C in $L^p(0, T; H)$,

$$D_C = \{x \in L^p(0, T; H);\ x(t) \in H \text{ a.e. in } [0, T],\ t \to C(t)\, x(t) \in L^p(0, T; H)\},$$

$$(Cx)(t) = C(t)\, x(t). \tag{4.22}$$

In view of (4.1) $C \in K(L^p(0, T; H))$.

We also define the space $Y_t\ \forall\ t \in [0, T]$ as the direct sum of K_t and H,

$$Y_t = K_t \oplus H, \tag{4.23}$$

and denote by $\binom{u}{v}$ the generic element of Y_t and by $|\binom{u}{v}|_t$ the $\binom{u}{v}$ norm,

$$\left|\begin{pmatrix} u \\ v \end{pmatrix}\right|_t = | C^{1/2}(t)u |^2 + | v |^2. \tag{4.24}$$

In view of lemmas 4.1 and 4.2 we get the following result:

LEMMA 4.3. *Let* $\{C(t)\}_{t\in[0,T]}$ *be a set of positive self-adjoint linear operators in* H *satisfying* (4.1). *Then the set* $\{Y_t\}_{t\in[0,T]}$ *defined by* (4.2) *and* (4.23) *satisfies* (2.29). *Moreover, if* $p > 1$, $L^p(0, T; Y_t)$ *is reflexive.*

In what follows X (norm $\| \ \|$) denotes the space $L^p(0, T; Y_t)$. Finally, let A and B be the linear operators in $L^p(0, T; Y_t)$ defined respectively by

$$D_A = H_0^{1,p}(0, T; Y_t), \quad A\begin{pmatrix} u \\ v \end{pmatrix} = -\begin{pmatrix} u' \\ v' \end{pmatrix} \qquad \forall \begin{pmatrix} u \\ v \end{pmatrix} \in D_A, \tag{4.25}$$

and

$$D_B = D_C \oplus L^p(0, T; Y_t), \quad B\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -Cu \end{pmatrix} \qquad \forall \begin{pmatrix} u \\ v \end{pmatrix} \in D_B. \tag{4.26}$$

We know (Theorem 2.3) that $A \in K(X)$. The following lemma shows that $B \in K(X)$ also.

LEMMA 4.4. $B \in K(X)$.

Proof. $\forall t \in [0, T]$ let $B(t)$ be the linear operator in Y_t defined by

$$D_{B(t)} = K_t \oplus H, \quad B(t)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -C(t)x \end{pmatrix} \qquad \forall \begin{pmatrix} x \\ y \end{pmatrix} \in D_{B(t)}. \tag{4.27}$$

It is well known [19, 45] that $B(t) \in K(Y_t)\ \forall t \in [0, T]$. Consider now the equation

$$\lambda\begin{pmatrix} u \\ v \end{pmatrix} - B\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \lambda \in \mathbf{C}_+, \qquad \begin{pmatrix} f \\ g \end{pmatrix} \in X, \tag{4.28}$$

which is equivalent to

$$\begin{aligned} u &= -\lambda R(\lambda^2, C) f + R(\lambda^2, C) g, \\ v &= C R(\lambda^2, C) f + \lambda R(\lambda^2, C) g. \end{aligned} \tag{4.29}$$

We then find that $\lambda \in \rho(B)$ and

$$R(\lambda, B)\begin{pmatrix} f \\ g \end{pmatrix}(t) = R(\lambda, B(t))\begin{pmatrix} f(t) \\ g(t) \end{pmatrix}. \tag{4.30}$$

Since $B(t) \in K(Y_t)$ we therefore have $B \in K(X)$.

2. Wave Equation (Cauchy's Problem)

We are now going to consider Cauchy's problem,

$$\begin{gathered}\frac{d^2u}{dt^2} + C(t)\,u(t) = f(t)_{f\in L^p(0,T;H)}, \qquad t \in [0, T], \\ u(0) = u'(0) = 0,\end{gathered} \tag{4.31}$$

We shall say that $u \in L^p(0, T; K_t) \cap H^{1,p}(0, T; H)$ is a weak solution of the problem (4.31) if there exists a sequence $\{u_n\}$ in $H^{1,p}(0, T; K_t) \cap H^{2,p}(0, T; H) \cap D_C$ such that

$$\left.\begin{array}{ll}\text{(i)} & u_n(0) = u_n'(0) = 0 \qquad \forall n \in \mathbf{N}; \\ \text{(ii)} & u_n \to u \quad \text{in} \quad L^p(0, T; K_t), \qquad u_n \to u' \quad \text{in} \quad L^p(0, T; H); \\ \text{(iii)} & u_n'' + Cu_n \to f \quad \text{in} \quad L^p(0, T; H).\end{array}\right\} \tag{4.32}$$

It is clear that u is a weak solution of (4.31) if and only if it is a weak solution of the equation

$$A\begin{pmatrix} u \\ u' \end{pmatrix} + B\begin{pmatrix} u \\ u' \end{pmatrix} = -\begin{pmatrix} 0 \\ f \end{pmatrix}. \tag{4.33}$$

We now show the following result:

Theorem 4.1. *Let A and B be the operators defined by* (4.25) *and* (4.26). *Assume that the set $\{C(t)\}_{t\in[0,T]}$ satisfies* (4.1). *Then $A + B$ is preclosed and $\overline{A + B} - \omega \in K(L^p(0, T; Y_t))$, Y_t being defined by* (4.23). *Finally*, $\rho(\overline{A + B}) = \mathbf{C}$.

Proof. Let $\binom{u}{v}: \in D_B$ and $B\binom{u}{v} = \binom{\bar{u}}{\bar{v}}$. Then we have

$$\left(e^{tA}\begin{pmatrix} u \\ v \end{pmatrix}\right)(s) \begin{cases} = \begin{pmatrix} C^{-1}(s-t)\,\bar{v}(s-t) \\ u(s-t) \end{pmatrix} & \text{if} \quad s - t \in [0, T[, \\ = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{if} \quad s - t \notin [0, T[. \end{cases} \tag{4.34}$$

It follows that $e^{tA}\binom{u}{v} \in D_B$ and

$$\left(Be^{tA}\begin{pmatrix} u \\ v \end{pmatrix}\right)(s) \begin{cases} = \begin{pmatrix} \bar{u}(s-t) \\ C(s)\,C^{-1}(s-t)\,\bar{v}(s-t) \end{pmatrix} & \text{if} \quad s - t \in [0, T[, \\ = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{if} \quad s - t \notin [0, T[. \end{cases} \tag{4.35}$$

Therefore we deduce

$$\left\| Be^{tA} \binom{u}{v} \right\|^p = \int_0^T \{| \bar{u}(s-t)|_t^2 + | C(s)\, C^{-1}(s-t)\, \bar{v}(s-t)|^2\}^{p/2}\, dt. \tag{4.36}$$

Due to (4.1, i) we deduce

$$| C(s)\, C^{-1}(s-t)\, \bar{v}(s-t)| \leqslant e^{Kt} \,| \bar{v}(s-t)| \qquad \forall s, t \in [0, T], \tag{4.37}$$

from which, recalling (4.36),

$$\begin{aligned} \left\| Be^{tA} \binom{u}{v} \right\|^p &\leqslant \int_t^T \{| \bar{u}(s-t)|_t^2 + e^{2kt} \,| \bar{v}(s-t)|^2\}^{p/2}\, dt \\ &\leqslant e^{kpt} \left\| \binom{u}{v} \right\|^p. \end{aligned} \tag{4.38}$$

Therefore we have

$$\| Be^{tA} B^{-1} \| \leqslant e^{kt}. \tag{4.39}$$

Then, in view of Theorem 2.3 and Lemma 4.2 we see that $A - \omega$ and B satisfy the hypothesis of Theorem 1.12; therefore $A + B$ is preclosed and $\overline{A + B} - \omega \in K(X)$.

Finally, the last affirmation is a consequence of identities (3.26).

The following corollary is immediate:

Corollary 4.1. *Assume that the set* $\{C(t)\}_{t\in[0,T]}$ *satisfies* (4.1). *Then the problem* (4.31) *has one and only one weak solution.*

Remark 4.1. We can study the wave equation in $C_0(0, T; Y_t)$; the results are analogous to those of Theorem 4.1 and Corollary 4.1.

3. Abstract Differential Equations of the Second Order (Dirichlet Conditions)

Let Y be a Banach space (norm $|\ |$), $[0, T]$ be a bounded interval of $\overline{\mathbf{R}}_+$, and $\{B(t)\}_{t\in[0,T]}$ be a set of linear operators in Y such that

$$\left.\begin{array}{ll} \text{(i)} & B(t) \in K(Y) \qquad \forall t \in [0, T]; \\ \text{(ii)} & 0 \in \rho(B(t)) \text{ and there exists } M \in \mathbf{R}_+ \text{ such that} \\ & | B^{-1}(t)| \leqslant M \qquad \forall t \in [0, T]. \end{array}\right\} \tag{4.40}$$

In what follows X (norm $\|\ \|$) denotes the space $L^p(0, T; Y)$.

Let A and B be the linear operators in X defined, respectively, by

$$\begin{aligned} D_A &= \{u \in H^{2,p}(0, T; Y); u(0) = u(T) = 0\}, \\ Au &= d^2u/dt^2 \qquad \forall u \in D_A , \end{aligned} \tag{4.41}$$

$$\begin{aligned} D_B &= \{u \in X, u(t) \in D_{B(t)} \text{ a.e. in } [0, T], t \to B(t)\, u(t) \in X\}, \\ (Bu)(t) &= B(t)\, u(t)t \text{ a.e. in } [0, T] \qquad \forall u \in D_B . \end{aligned} \tag{4.42}$$

We know that $A \in K(X)$ (Theorem 2.5) and that if $\{B(t)\}_{t\in[0,T]}$ satisfies (2.38), then $B \in K(X)$.

We now consider the problem

$$\begin{aligned} \lambda u(t) - d^2u/dt^2 - B(t)\, u(t) = f(t) \qquad \text{for} \qquad f \in L^p(0, T; Y), \quad \lambda \in \mathbf{C}, \\ u(0) = u(T) = 0. \end{aligned} \tag{4.43}$$

We will say that u is a weak solution of the problem (4.43) if there exists a sequence $\{u_n\}$ in $L^p(0, T; Y)$ such that

$$\left.\begin{array}{ll} \text{(i)} & u_n \in \{H^{2,p}(0, T; Y), u(0) = u(T) = 0\} \qquad \forall n \in \mathbf{N}; \\ \text{(ii)} & u_n(t) \in D_{B(t)}t \text{ a.e. in } [0, T] \text{ and } t \to B(t)\, u_n(t) \in L^p(0, T; Y) \\ & \forall n \in \mathbf{N}; \\ \text{(iii)} & u_n \to u \quad \text{and} \quad \lambda u_n - \dfrac{d^2u_n}{dt^2} - Bu_n \to f \quad \text{in} \quad L^p(0, T; Y). \end{array}\right\} \tag{4.44}$$

Clearly u is a weak solution of the problem (4.43) if and only if it is a weak solution of the equation

$$\lambda u - Au - Bu = f. \tag{4.45}$$

The following result of uniqueness is a consequence of Theorems (1.1), (1.5), and (2.5).

THEOREM 4.2. *Assume that the set* $\{B(t)\}_{t\in[0,T]}$ *satisfies* (2.38) *and* (4.40). *Then there exists* $\epsilon(p) \in \mathbf{R}_+$ *such that*

$$\|u\| \leqslant \frac{1}{\operatorname{Re}\lambda + \epsilon(p)} \|\lambda u - d^2u/dt^2 - Bu\| \qquad \forall u \in D_A \cap D_B , \quad \lambda \in \mathbf{C}_+ - \epsilon(p). \tag{4.46}$$

Moreover the problem (4.43) *has at most one weak solution.*

We now consider existence for the problem (4.43). We shall assume that the set $\{B(t)\}_{t\in[0,T]}$ satisfies the following condition:

$$\begin{gathered}\text{If } t \leqslant s \text{ then } D_{B(t)} \subset D_{B(s)} \text{ and there exists } K \in \mathbf{R}_+ \text{ such that} \\ | B(t) B^{-1}(s) - 1 | \leqslant K | t - s | \qquad \forall t, s \in [0, T].\end{gathered} \tag{4.47}$$

We first prove

LEMMA 4.5. *Let Γ be an integral transformation in $L^\infty(0, T)$ defined by*

$$\begin{aligned}(\Gamma\varphi)(s) = {} & \frac{(\sinh(\rho - \rho s/T))}{(T\rho \sinh \rho)} \int_0^s \sinh(\rho y/T)(s - y)\, \varphi(y)\, dy \\ & + \frac{(\sinh \rho s)}{(T\rho \sinh \rho)} \int_s^T \sinh(\rho - \rho y/T)(y - s)\, \varphi(y)\, dy \\ & \text{for} \quad \rho \in R_+, \quad \varphi \in C(0, T), \quad s \text{ a.e. in } [0, T].\end{aligned} \tag{4.48}$$

Then Γ can be uniquely extended to a bounded linear operator (which we also denote by Γ), and we have

$$\| \Gamma \|_p \leqslant 1/\rho^3. \tag{4.49}$$

Proof. If $\varphi \in L^\infty(0, T)$ we have

$$\begin{aligned}|(\Gamma\varphi)(s)| & \leqslant \| \varphi \|_\infty \Big\{ \frac{\sinh(\rho - \rho s/T)}{(T\rho \sinh \rho)} \int_0^s \sinh(\rho y/T)(s - y)\, dy \\ & \quad + \frac{(\sinh(\rho s))}{(T\rho \sinh \rho)} \int_s^T \sinh(\rho - \rho y/T)(y - s)\, dy \\ & \leqslant \frac{\| \varphi \|_\infty}{(\rho^2 \sinh \rho)} \{\sinh(\rho - \rho s/T) \Big(\frac{1}{\rho \sinh(\rho s/T)} - s/T \Big) \\ & \quad \times (\sinh(\rho s/T)) \Big(\frac{1}{\rho \sinh(\rho - \rho s)} - (1 - s/T) \Big) \\ & \leqslant 2 \| \varphi \|_\infty \sinh(\rho s/T) \sinh(\rho - \rho s/T)(\rho^3 \sinh \rho).\end{aligned} \tag{4.50}$$

In view of the inequality

$$\frac{2 \sinh(\rho x) \sinh(\rho - \rho x)}{\sinh \rho} \leqslant 1 \qquad \forall x \in [0, 1], \quad \rho \in \mathbf{R}_+, \tag{4.51}$$

we obtain

$$\| \Gamma\varphi \|_\infty \leqslant 1/\rho^3. \tag{4.52}$$

Now let $\varphi \in L^1(0, T)$. Then we have

$$\| \Gamma\varphi \|_1 \leqslant \int_0^T \frac{\sinh(\rho - \rho s/T)}{(\rho T \sinh \rho)}\, ds \int_0^s \sinh(\rho y)(s - y)\, | \varphi(y)|\, dy$$
$$+ \int_0^T \frac{\sinh(\rho s)}{(\rho \sinh \rho)}\, ds \int_s^T \sinh(\rho - \rho y)(y - s)\, | \varphi(y)|\, dy. \tag{4.53}$$

By exchanging the order of integration in the last integral and using (4.51) we find

$$\| \Gamma\varphi \|_1 \leqslant 1/\varphi^3. \tag{4.54}$$

The thesis follows from the Riesz interpolation theorem.

We now prove

THEOREM 4.3. *Let A and B be the operators defined, respectively, by* (4.41) *and* (4.42); *assume that the set $\{B(t)\}_{t\in[0,T]}$ satisfies* (4.40) *and* (4.47). *Then $A + B$ is preclosed and there exists $\epsilon(p) > 0$ such that $\overline{A + B} + \epsilon(p) \in K(L^p(0, T; Y))$.*

Proof. Let $u \in D_B$, $v = Bu$, $\lambda \in \mathbf{R}_+$, and $\rho = \sqrt{\lambda}$. In view of (2.60) we have

$$(R(\lambda, A)u)(s) = \frac{(\sinh(\rho - \rho s/T))}{(T\rho \sinh \rho)}$$
$$\times \int_0^s \sinh(\rho y/T)\, B^{-1}(y)\, v(y)\, dy + \frac{(\sinh \rho s)}{(T\rho \sinh \rho)}$$
$$\times \int_s^T \sinh(\rho - \rho y/T)\, B^{-1}(y)\, v(y)\, dy. \tag{4.55}$$

Therefore $R(\lambda, A)\, u \in D_B$ and we have

$$((BR(\lambda, A) - R(\lambda, A)B)u)(s)$$
$$= \frac{(\sinh(\rho - \rho s/T))}{(T\rho \sinh \rho)} \times \int_0^s \sinh(\rho y/T)(B(s)\, B^{-1}(y) - 1)\, v(y)\, dy$$
$$+ \frac{(\sinh \rho s)}{(T\rho \sinh \rho)} \times \int_0^T \sinh(\rho - \rho y/T)(B(s)\, B^{-1}(y) - 1)\, v(y)\, dy, \tag{4.56}$$

from which, using Hypothesis (4.47) and Definition (4.48), we obtain

$$|(BR(\lambda, A) - R(\lambda, A)B)\, u(s)| \leqslant K(\Gamma(|\, v(\cdot)|))(s) \qquad \forall s \in [0, T]. \quad (4.57)$$

Then, from Lemma (4.5), it follows that

$$|((BR(\lambda, A) - R(\lambda, A)B)\, u(s)| \leqslant \frac{K}{\lambda^{3/2}}\, |(Bu)(\cdot)|(s) \qquad \forall s \in [0, T], \quad (4.58)$$

whence

$$\|\, BR(\lambda, A)u - R(\lambda, A)\, Bu\, \| \leqslant \frac{K}{\lambda^{3/2}} \|\, Bu\, \| \qquad \forall u \in D_B\, . \qquad (4.59)$$

Since A is an infinitesimal generator of an analytic semigroup, we have [45]

$$e^{tA} = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} R(\lambda, A)\, d\lambda, \qquad (4.60)$$

Γ being a suitable contour in $\mathbf{C}$. Then, from (4.59), we easily obtain

$$\|\, Be^{tA}u - e^{tA}Bu\, \| \leqslant K' t^{1/2}\, \|\, Bu\, \| \qquad \forall u \in D_B\, , \qquad (4.61)$$

K' being a suitable constant. Therefore the thesis is a consequence of Theorem 1.11.

The following corollary is immediate:

COROLLARY 4.2. *Assume that the set* $\{B(t)\}_{t\in[0,T]}$ *satisfies* (4.40) *and* (4.47). *Then the problem* (4.43) *has one and only one weak solution.*

4. ABSTRACT EQUATIONS OF ARBITRARY ORDER (WITH PERIODIC CONDITIONS)

Let $\{B(t)\}_{t\in[0,T]}$ be a set of linear operators in the Hilbert space H (norm $|\ |$) such that

$$\left.\begin{array}{ll} \text{(i)} & B(t) \in K(H); \\ \text{(ii)} & \rho(B(t)) \supset 0 \text{ and there exists } M \in \mathbf{R}_+ \text{ such that} \\ & \qquad |\, B^{-1}(t)| \leqslant M. \end{array}\right\} \qquad (4.62)$$

In what follows A and B are, respectively, the operators on $X = L^2(0, T; H)$ (norm $\| \ \|$),

$$\begin{aligned} D_A &= \{u \in H^{1,2}(0, T; H);\ u(0) = u(T)\}, \\ Au &= -\frac{du}{dt}, \qquad \forall u \in D_A, \end{aligned} \tag{4.63}$$

$$\begin{aligned} D_B = \{u \in X;\ u(t) \in D_{B(t)} \qquad \forall t \in [0, T],\ t \to B(t)\, u(t) \in X\}, \\ (Bu)(t) = B(t)\, u(t) \qquad \forall u \in D_B. \end{aligned} \tag{4.64}$$

We know (Theorem 2.6) that $(-1)^{n+1}A^{2n}$ and A^{2n+1} belong to $K(X)$ $\forall n \in \mathbf{N}$; moreover, if the set $\{B(t)\}_{t\in[0,T]}$ satisfies (2.38) then $B \in K(X)$. We now consider the problem

$$\lambda u - \epsilon_n \frac{d^n u}{dt^n} - B(t)\, u(t) = f(t),$$

and

$$\begin{aligned} u^{(k)}(0) = u^{(k)}(T), \qquad k = 0, 1,\dots, n-1, \qquad \lambda \in \mathbf{C}_+, \\ \epsilon_n \begin{cases} = -1 & \text{if } n \text{ is odd}, \\ = (-1)^{n/2+1} & \text{if } n \text{ is even}. \end{cases} \end{aligned} \tag{4.65}$$

We shall say that $u \in X$ is a weak solution of the problem (4.65) if there exists a sequence $\{u_m\}$ in $L^2(0, T; H)$ such that

$$\left.\begin{aligned} &\text{(i)} \quad u_m \in H^{n,2}(0, T; H) \cap D_B, \qquad u_m^{(k)}(0) = u_m^{(k)}(T) \\ &\qquad\qquad \forall m \in \mathbf{N}, \qquad k = 0,\dots, n-1; \\ &\text{(ii)} \quad u_m \to u \quad \text{and} \quad \lambda u_m - \epsilon_n u_m^{(n)} + Bu_m \to f \quad \text{in} \quad L^2(0, T; H). \end{aligned}\right\} \tag{4.66}$$

Clearly u is a weak solution of (4.65) if and only if it is a weak solution of the equation

$$\lambda u - \epsilon_n A^n u - Bu = f. \tag{4.67}$$

The following result of uniqueness is a consequence of Theorems 1.1 and 1.5.

THEOREM 4.4. *Assume that the set* $\{B(t)\}_{t\in[0,T]}$ *satisfies* (2.38). *Then if* $u \in \{v \in H^{n,2}(0, T; H);\ u^{(k)}(0) = u^{(k)}(T),\ k = 0,\dots, n-1\} \cap D_B$, *we have the following inequality*:

$$\| u \| \leqslant \frac{1}{\operatorname{Re} \lambda} \| \lambda u - \epsilon_n A^n u - Bu \| \qquad \forall \lambda \in \mathbf{C}_+, \tag{4.68}$$

and the problem (4.65) *has at most one weak solution.*

We now prove

THEOREM 4.5. *Let A and B be the operators defined by* (4.63) *and* (4.64). *Assume that the set* $\{B(t)\}_{t\in[0,T]}$ *satisfies* (4.62) *and* (3.46) (*resp.* (4.62) *and one of the conditions* (3.46)–(3.48)) *if n is odd* (*resp. even*).[20] *Then $\epsilon_n A^n + B$ is preclosed and $\epsilon_n A^n + B$ belongs to $K(L^2(0, T; H))$. Moreover the problem* (4.65) *has one and only one weak solution.*

Proof. We assume that $\{B(t)\}_{t\in[0,T]}$ satisfies (3.46) and that n is odd, the proof in the other cases being similar.

Let $\rho \in \mathbf{R}_+$, $p = (n-1)/2$. In view of (2.65) we have

$$R(\rho^n, A^n) = (-1)^p R(\rho, A) \prod_{i=1}^{n} R(\alpha_i\rho, A) R(\beta_i\rho, -A), \tag{4.69}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_p$ (resp. $-\beta_1, -\beta_2, \ldots, -\beta_p$) are the n-th roots of one having a real positive (resp. negative) part. From (4.69) it follows that

$$R(\rho^n, A^n) = (-1)^p \int_0^\infty du \int_0^\infty dt_1 \cdots \int_0^\infty dt_p \int_0^\infty ds_1 \cdots \int_0^\infty ds_p$$
$$\times\, e^{-(u+\alpha_1 t_1+\cdots+\alpha_p t_p+\beta_1 s_1+\cdots\beta_p s_p)} G(u + t_1 + \cdots + t_p - s_1 - \cdots - s_p)x \quad \forall x \in X, \tag{4.70}$$

G being the semigroup $t \to e^{tA}$. Using (3.46) we now obtain

$$\| BR(\rho^n, A^n) B^{-1} - R(\rho^n, A^n)\|$$
$$\leqslant K \int_0^\infty du \int_0^\infty dt_1 \cdots \int_0^\infty dt_p \int_0^\infty ds_1 \cdots \int_0^\infty ds_p$$
$$\times\, e^{-\rho(u+\Sigma_{i=1}^p(\mathrm{Re}\,\alpha_i t_i+\mathrm{Re}\beta_i s_i))} \,| u + t_1 + \cdots + t_p - s_1 - \cdots - s_p |^\alpha. \tag{4.71}$$

If we put $\bar{u} = \rho u$, $\bar{t}_i = \rho t_i$, $\bar{s}_i = \rho s_i$, $i = 1, \ldots, p$ we see that there exists a positive constant $\bar{K}$ such that

$$\| BR(\rho^n, A^n) B^{-1} - R(\rho^n, A^n)\| \leqslant \bar{K}/\rho^{n+\alpha}, \tag{4.72}$$

which is equivalent (due to the Hille–Yosida theorem) to

$$\| Be^{tA^n}B^{-1} - e^{tA^n} \| \leqslant K' t^{\alpha/n}, \tag{4.73}$$

K' being a suitable constant.

The thesis follows now from Theorem 1.11.

[20] The set $\{B(t)\}_{t\in\mathbf{R}}$ is here obtained from $\{B(t)\}_{t\in[0,T]}$ by a periodic extension.

5. Laplacian Iterated in R^n

We use the notations of Chapter II, Section 7. Let $\{B(x)\}_{x\in R^n}$ be a set of linear operators in Y such that

$$\left.\begin{array}{ll}\text{(i)} & B(x)\in K(Y) \qquad \forall x\in R^n;\\ \text{(ii)} & \text{The map } R^n\to Y,\ x\to R(\lambda, B(x))u \text{ is continuous } \forall u\in Y.\end{array}\right\} \tag{4.74}$$

In what follows X (norm $\|\ \|$) is the Banach space $L^p(R^n; Y)$, A is the infinitesimal generator of the semigroup defined by Theorem 2.7 (with $\tilde{A} = A$), and B is the operator defined by

$$\begin{gathered} D_B = \{u\in L^p(R^n; Y),\ u(x)\in D_{B(x)}\ x \text{ a.e. in } R^n,\ x\to B(x)\,u(x)\in L^p(R^n; Y)\},\\ (Bu)(x) = B(x)\,u(x) \qquad \forall u\in D_B\,,\quad x \text{ a.e. in } R^n. \end{gathered} \tag{4.75}$$

The proof of the following lemma is very similar to that of Theorem 2.4 and therefore it will be omitted.

Lemma 4.6. *Assume that the set* $\{B(x)\}_{x\in\mathbf{R}^n}$ *satisfies* (4.74). *Then the linear operator* B, *defined by* (4.75), *belongs to* $K(L^p(R^n; Y))$.

We are going to study the following problem:

$$\lambda u + (-1)^m \Delta^m u - Bu = f \qquad \text{for} \qquad \lambda\in\mathbf{C}_+\,, \qquad f\in L^p(R^n; Y), \tag{4.76}$$

where Δ is the Laplacian in R^n.

The following result of uniqueness is a consequence of Theorems 1.1 and 1.5 and of Lemma 4.6.

Theorem 4.6. *Assume that the set* $\{B(x)\}_{x\in\mathbf{R}^n}$ *satisfies* (4.74). *Then if* $u\in H^{2m,p}(R^n; Y)\cap D_B$ *we have*

$$\|u\| \leqslant \frac{1}{\operatorname{Re}\lambda}\|\lambda u + (-1)^m \Delta^m u - Bu\| \qquad \forall\lambda\in\mathbf{C}_+\,, \tag{4.77}$$

and the problem (4.76) *has at most one weak solution.*

Finally, we prove

Theorem 4.7. *Assume that the set* $\{B(x)\}_{x\in\mathbf{R}^n}$ *satisfies* (4.74) *and the following conditions*:

$$\begin{gathered} D_{B(x)} \textit{ is independent of } x \textit{ and there exists } K\in\mathbf{R}_+ \textit{ and } \alpha\in\,]0, 1] \textit{ such that}\\ |B(x)\,B^{-1}(y) - 1| \leqslant K\,|x-y|^\alpha \qquad \forall x, y\in\mathbf{R}^n. \end{gathered} \tag{4.78}$$

Then $A + B$ *is preclosed and* $\overline{A + B}\in K(L^p(R^n; Y))$.

Proof. Let $u \in D_B$, $v = Bu$. In view of (2.73) we have

$$(e^{tA}u)(x) = a_{mn}t^{-n/(2m)} \int_{\mathbf{R}^n} e^{-|y|^{2m}/t} B^{-1}(x-y)\, v(x-y)\, dy. \tag{4.79}$$

Due to (4.78) we can see that $e^{tA} \in D_B$ and

$$(Be^{tA}u - e^{tA}Bu)(x) = a_{mn}t^{-n/(2m)} \int_{\mathbf{R}^n} e^{-|y|^{2m}/t}(B(x)\, B^{-1}(x-y) - 1)\, v(x-y)\, dy, \tag{4.80}$$

from which, in view of (4.78),

$$|(Be^{tA}u - e^{tA}Bu)(x)| \leqslant a_{mn}t^{-n/(2m)}K \int_{\mathbf{R}^n} e^{-|y|^{2m}/t}\, |\, y\,|^{\alpha}\, |\, v(x-y)|\, dy, \tag{4.81}$$

and, using Young's inequality,

$$\|\, Be^{tA}u - e^{tA}Bu\,\| \leqslant Ka_{mn}t^{-n/(2m)} \int_{\mathbf{R}^n} e^{-|y|^{2m}/t}\, |\, y\,|^{\alpha}\, dy\, \|\, v\,\|. \tag{4.82}$$

If we put $y = t^{i/2m}y$ we find

$$\|\, Be^{tA}u - e^{tA}Bu\,\| \leqslant K't^{1/2m}\, \|\, Bu\,\| \qquad \forall u \in D_B\,, \tag{4.83}$$

where

$$K' = Ka_{mn} \int_{\mathbf{R}^n} e^{-|y|^{2m}}\, |\, y\,|^{\alpha}\, dy. \tag{4.84}$$

The thesis follows from Theorem 1.11.

COROLLARY 4.3. *Assume that the set* $\{B(x)\}_{x \in \mathbf{R}^n}$ *satisfies* (4.74) *and* (4.78). *Then the problem* (4.76) *has one and only one weak solution.*

6. SOME SYSTEMS

Let $[0, T]$ be a subinterval of $\overline{\mathbf{R}}_+$, and A_r and A_l be the linear operators defined by

$$\begin{aligned} &D_{A_r} = \{u \in H^{1,p}(0, T;\, Y);\, u(0) = 0\},\\ &A_r u = -\frac{du}{dt} \qquad \forall u \in D_{A_r}\,;\\ &D_{A_l} = \{u \in H^{1,p}(0, T;\, Y);\, u(T) = 0\},\\ &A_l u = \frac{du}{dt} \qquad \forall u \in D_{A_l}\,. \end{aligned} \tag{4.85}$$

We know that A_r and A_l belong to $K(L^p(0, T; Y))$ (Chapter II, Section 1), and we have

$$\begin{aligned}
(e^{tA_r}u)(s) &\begin{cases} = u(s-t) & \text{if } s-t \in [0, T], \\ = 0 & \text{if } s-t \notin [0, T], \end{cases} \\
(e^{tA_l}u)(s) &\begin{cases} = u(s+t) & \text{if } s+t \in [0, T], \\ = 0 & \text{if } s+t \notin [0, T], \end{cases} \\
&\forall u \in L^p(0, T; Y) \qquad s \text{ a.e. in } [0, T].
\end{aligned} \tag{4.86}$$

In what follows we put $L^p(0, T; Y) = X$ (norm $\| \;\|$).

Now let $\{B_1(t)\}_{t\in[0,T]}$, $\{B_2(t)\}_{t\in[0,T]}$, $\cdots \{B_n(t)\}_{t\in[0,T]}$ be n sets of linear operators such that

$$\left.\begin{array}{ll}
\text{(i)} & \{B_1(t)\}_{t\in[0,T]} \text{ satisfies (3.1) and one of the conditions} \\
& \text{(3.15)–(3.18);} \\
\text{(ii)} & \text{If we put} \\
& \qquad \bar{B}_i(t) = B_i(T-t) \qquad \forall t \in [0, T], \quad i = 2, \dots, n, \\
& \text{then } \{\bar{B}_i(t)\}_{t\in[0,T]} \text{ satisfies (3.1) and one of the conditions} \\
& \text{(3.15)–(3.18), } i = 2, 3, \dots, n.
\end{array}\right\} \tag{4.87}$$

Then put

$$\begin{aligned}
D_{B_i} &= \{u \in X,\ B_i(t)\,u(t) \in D_{B_i(t)},\ t \text{ a.e. in } [0, T],\ t \to B_i(t)\,u(t) \in X\}, \\
(B_i u)(t) &= B_i(t)\,u(t) \qquad t \text{ a.e. in } [0, T], \qquad i = 1, 2, \dots, n.
\end{aligned} \tag{4.88}$$

We remember that $B_1, B_2, \dots, B_n$ belong to $K(X)$ (Theorem 2.4).

In the same manner as Theorem 3.2 we prove the following result:

THEOREM 4.8. *Let $A_r, A_l, B_1, B_2, \dots, B_n$ be the operators defined by* (4.85) *and* (4.88), *and assume that condition* (4.87) *is satisfied. Then $A_r + B_i$ (resp. $A_l + B_i$) is preclosed and $\overline{A_r + B_i}$ (resp. $\overline{A_l + B_i}$) belongs to $K(X)$, $i = 1, 2, \dots, n$.*

Finally, let Q be a linear operator in the Hilbert space C^n (inner product $(\;,\;)$) such that

$$\operatorname{Re}(Qx, x) \leqslant 0 \qquad \forall x \in \mathbf{C}^n. \tag{4.89}$$

We denote by Y^n (resp. X^n) the direct sum of n copies of Y (resp. X). In what follows we shall identify X^n to $L^p(0, T; Y^n)$ and denote by $\mathbf{u}$ the general element of X^n.

We also put

$$
\begin{aligned}
D_B &= D_{B_1} \oplus D_{B_2} \oplus \cdots \oplus D_{B_n}, \\
B\mathbf{u} &= (B_1 u_1, \dots, B_n u_n) \qquad \forall \mathbf{u} \in D_B, \\
D_A &= D_{A_r} \oplus D_{A_1} \oplus \cdots \oplus D_{A_r}, \\
A\mathbf{u} &= (A_r u_l, A_l u_2, \dots, A_l u_n) \qquad \forall \mathbf{u} \in D_A, \\
(Q\mathbf{u})(t) &= Q\mathbf{u}(t), \qquad t \text{ a.e. in } [0, T], \qquad \mathbf{u} \in X^n.
\end{aligned} \tag{4.90}
$$

We remark that, in view of (4.89), Q belongs to $K(X^n)$. The following corollary of Theorem 4.8 is immediate:

COROLLARY 4.4. *Let A and B be the operators defined by* (4.90), *and assume that condition* (4.87) *is satisfied. Then $A + B$ is preclosed and $\overline{A + B}$ belongs to $K(X^n)$.*

We now consider the following problem:

$$
\begin{aligned}
&\lambda u_1(t) + \frac{du_1(t)}{dt} - B_1(t)\, u_1(t) - \sum_{k=1}^{n} Q_{1k} u_k(t) = f_1(t), \\
&\lambda u_h(t) - \frac{du_h(t)}{dt} - B_h(t)\, u_h(t) - \sum_{k=1}^{n} Q_{hk} u_k(t) = f_h(t), \\
&u(0) = 0, \qquad u_h(T) = 0, \qquad h = 2, 3, \dots, n \\
&\text{for} \qquad \lambda \in \mathbf{C}_+, \qquad \mathbf{f} = (f_1, \dots, f_n) \in X^n.
\end{aligned} \tag{4.91}
$$

We shall say that $\mathbf{u}$ is a weak solution of the problem (4.91) if there exists a sequence $\{\mathbf{u}^{(m)}\}$ in X^n such that

$$
\left.
\begin{aligned}
&\text{(i)} \quad u_h^{(m)} \in H^{1,p}(0, T; Y), \quad h = 1, \dots, n, \quad u_1^{(m)}(0) = u_n^{(m)}(T) = 0, \\
&\qquad h = 2, \dots, n; \\
&\text{(ii)} \quad u_h^{(m)}(t) \in D_{B_h(t)} \quad t \text{ a.e. in } [0, T] \qquad \text{and} \\
&\qquad t \to B_h(t)\, u_h^{(m)}(t) \in L^p(0, T; Y) \qquad \forall m \in \mathbf{N}, \quad h = 2, \dots, n; \\
&\text{(iii)} \quad u_1^{(m)} \to u_h, \qquad \text{and} \\
&\qquad \lambda u_1^{(m)} + \frac{du_1^{(m)}}{dt} - B_1 u_1^{(m)} - \sum_{k=1}^{n} Q_{1k} u_k^{(m)} = f_1, \\
&\qquad \lambda u_h^{(m)} + \frac{du_h^{(m)}}{dt} - B_h u_h^{(m)} - \sum_{k=1}^{n} Q_{hk} u_k^{(m)} \to f_h, \quad h = 2, \dots, n \\
&\qquad \text{in } X.
\end{aligned}
\right\} \tag{4.92}
$$

Clearly $\mathbf{u}$ is a weak solution of the problem (4.92) if and only if it is a weak solution of the equation

$$\lambda \mathbf{u} - A\mathbf{u} - B\mathbf{u} - Q\mathbf{u} = \mathbf{f}. \tag{4.93}$$

We now prove

THEOREM 4.9. *Let A, B, and Q be the operators in X^n defined by* (4.90), *and assume that the conditions* (4.87) *and* (4.89) *are satisfied. Then $A + B + Q$ is preclosed and $\overline{A + B + Q} = \overline{A + B} + Q$ belongs to $K(X^n)$.*

Proof. In view of Theorem 4.8, $A + B$ is preclosed and $\overline{A + B} \in K(X^n)$. Then $\overline{A + B}$ and Q satisfy the hypotheses of Theorem 1.8 and therefore the thesis follows.

COROLLARY 4.5. *Under the same hypotheses of Theorem* 4.9 *the problem* (4.92) *has one and only one weak solution* $\mathbf{u}$. *Moreover* $\mathbf{u}$ *belongs to* $D_{\overline{A+B}}$.

Chapter V. Some Applications to Differential Operators

1. AN ABSTRACT CASE WHERE THE CONDITION $| B(t)\, B^{-1}(s) - 1 | \leqslant K \,| t - s |^{\alpha}$ IS SATISFIED

Let Y and Z be Banach spaces whose norms are denoted by $|\ |_Y$ and $|\ |_Z$, respectively. We shall assume that $Z \subset Y$, the immersion being algebrical and topological and denote by $\mathscr{L}(Z, Y)$ the Banach space of the linear bounded maps of Z into Y.

Let $[0, T]$ be a subinterval of $\bar{\mathbf{R}}_+$ and let $B : t \to B(t)$ be a map of $[0, T]$ in $\mathscr{L}(Z, Y)$ such that

$$\left.\begin{array}{ll} \text{(i) There exists } m_1, m_2 \in \mathbf{R}_+ \text{ such that} & \\ \qquad m_1 \,| x |_Z \leqslant | B(t)x |_Y \leqslant m_2 \,| x |_Z & \forall t \in [0, T]; \\ \text{(ii) There exists } N, \alpha \in \mathbf{R}_+, \alpha \leqslant 1 \text{ such that} & \\ \qquad | B(t)x - B(s)x |_Y \leqslant N \,| t - s |^{\alpha} \,| x |_Z & \forall t, s \in [0, T]. \end{array}\right\} \tag{5.1}$$

We first prove

LEMMA 5.1. *Put*

$$F(t) = B(t)\, B^{-1}(0) \qquad \forall t \in [0, T]. \tag{5.2}$$

Then the map $[0, T] \to \mathscr{L}(Y, Y)$, $t \to F(t)$ *has the following properties*:

$$\left.\begin{array}{ll}\text{(i)} & F \textit{ is Hölder continuous;} \\ \text{(ii)} & |F^{-1}(t)|_Y \leqslant m_2/m_1 .\end{array}\right\} \tag{5.3}$$

Proof. Let $x \in Y$, $y = B^{-1}(0)\, x$. Then we have

$$\begin{aligned}|F(t)x|_Y &= |B(t)\, B^{-1}(0)x|_Y \leqslant m_2 \,|B^{-1}(0)x|_Z = m_2\,|y|_Z \\ &\leqslant (m_2/m_1)|\, B(0)\, y\,|_Y = (m_2/m_1)|\, x\,|_Y ,\end{aligned} \tag{5.4}$$

so that $F(t) \in \mathscr{L}(Y, Y)$. Clearly we have that

$$F^{-1}(t) = B(0)\, B^{-1}(t), \tag{5.5}$$

whence

$$|F^{-1}(t)x|_Y = |B(0)\, B^{-1}(t)x|_Y \leqslant m_2\,|B^{-1}(t)x|_Z \leqslant (m_2/m_1)|\,x\,|_Y \tag{5.6}$$

and (ii) is proved.

Finally, let $x \in Y$, $y = B^{-1}(0)\, x$, $t, s \in [0, T]$. Then we have

$$\begin{aligned}|F(t)x - F(s)x|_Y &= |B(t)\, B^{-1}(0)x - B(s)\, B^{-1}(0)x|_Y \leqslant N\,|t-s|^\alpha\,|y|_Z \\ &\leqslant (N/m_1)|\,t-s\,|^\alpha\,|\,x\,|_Y ,\end{aligned} \tag{5.7}$$

and the lemma is completely proved.

We now prove

THEOREM 5.1. *Let* $B : t \to B(t)$ *be a map of* $[0, T]$ *in* $\mathscr{L}(Z, Y)$, *and assume that* B *satisfies* (5.1). *Then there exists* $K \in \mathbf{R}_+$ *such that*

$$|B(t)\, B^{-1}(s) - 1|_Y \leqslant K\,|t-s|^\alpha \qquad \forall t, s \in [0, T]. \tag{5.8}$$

Proof. The following identity is clear:

$$\begin{aligned}B(t)\, B^{-1}(s) - 1 &= (B(t) - B(s))\, B^{-1}(0)\, B(0)\, B^{-1}(s) \\ &= (F(t) - F(s))\, F^{-1}(s),\end{aligned} \tag{5.9}$$

and the thesis is a consequence of Lemma 5.1.

EXAMPLE 5.1. Let Ω be an open bounded subset of the Euclidean space R^n and Γ be its boundary, which we assume sufficiently smooth. Set $D_j = (1/i)(\partial/\partial x_i)$, $j = 1,\dots, m$, and $D^\alpha = D_n^{\alpha_1},\dots, D_1^{\alpha_n}$, $\alpha = (\alpha_1 \cdots \alpha_n)$

is a multiindex with integral components $\alpha_i \geqslant 0$ whose length $\alpha_1 + \alpha_2 + \cdots + \alpha_n$ is denoted by $|\alpha|$.

Let $[0, T]$ be a subinterval of $\mathbf{R}_+$ and let $B(x, t, D)$, $D = (D_1, ..., D_n)$, be an elliptic operator of order $2m$ in Ω,

$$B(x, t, D) = \sum_{|\alpha| \leqslant 2m} b_\alpha(x, t) D^\alpha. \tag{5.10}$$

The definition of ellipticity is the one given in Ref. [2].

Let there also be given a system of m differential boundary operators $\{B_j\}_1^m$ of respective order m_j. We shall assume that the elliptic problem $(B, \{B_j\}_1^m, \Omega)$ is regular [2]. We denote by $H^{2m,q}(\Omega, \{B_j\}_1^m)$ the completion of the subspace

$$\{u \in C^{2m}(\bar{\Omega}); B_j u = 0 \text{ on } \Gamma, j = 1, ..., m\} \tag{5.11}$$

with respect to the norm

$$\| u \|_{2m,q} = \sum_{|\alpha| \leqslant 2m} \| D^\alpha u \|_q, \qquad q \geqslant 1. \tag{5.12}$$

For every $t \in [0, T]$ $B(t)$ is the linear operator in $L^q(\Omega)$ defined by

$$\begin{aligned} D_{B(t)} &= H^{2m,q}(\Omega; \{B_j\}_1^m), \\ B(t)u &= B(x, t, D)u \qquad \forall u \in D_{B(t)}. \end{aligned} \tag{5.13}$$

Assume that $\rho(B(t))$ contains the sector

$$\omega + S_\theta = \{\lambda \in \mathbf{C}; \lambda = \omega + \mu, |\arg \mu| \leqslant \theta\}, \qquad \theta > \pi/2, \qquad \omega \in \mathbf{R}_+ \tag{5.14}$$

for every $t \in [0, T]$; then from Theorem 5.1 (where $Y = L^q(\bar{\Omega})$, $Z = H^{2m,q}(\Omega; \{B_j\}_1^m)$) it follows that

$$\| B(t) B^{-1}(s) - 1 \|_q \leqslant K | t - s |^\alpha \qquad \text{for} \qquad K \in \mathbf{R}_+. \tag{5.15}$$

We also remember that $B(t)$ is the infinitesimal generator of an analytic semigroup [3].

2. Elliptic Operators of the Second Order (Variational Form)

Let Ω be an open bounded subset in R^n with boundary sufficiently smooth, and let B be a differential operator of second order,

$$Bu = \sum_{ij=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial u}{\partial x_j} \right), \tag{5.16}$$

where a_{ij}, $i, j = 1, 2, \ldots, n$, is a real measurable bounded function on Ω.

We assume that B is uniformly elliptic, i.e., there exists $\nu \geqslant 1$ such that

$$\nu^{-1} | \xi |^2 \leqslant \sum_{ij=1}^{n} a_{ij}(x) \xi_i \xi_j \leqslant \nu | \xi |^2 \qquad \forall x \in \Omega, \quad \xi \in \mathbf{R}^n. \tag{5.17}$$

Let b be the bilinear form on $H^1(\Omega)$[21] associated to B,

$$b(u, v) = \int_\Omega \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} dx. \tag{5.18}$$

Finally, let V be a closed subspace of $H^1(\Omega)$ such that

$$\left.\begin{array}{rl} \text{(i)} & H_0^1(\Omega) \subset V \subset H^1(\Omega) \text{ algebrically and topologically;} \\ \text{(ii)} & \text{If } u, v \in C(\bar{\Omega}) \cap V \text{ then } |u| \in V \text{ and } uv \in V; \\ \text{(iii)} & \text{There exists } \alpha \in \mathbf{R} \text{ and } K \in \mathbf{R}_+ \text{ such that} \\ & \qquad b(u, u) + \alpha \| u \|_2^2 \geqslant K \| u \|_v^2 . \end{array}\right\} \tag{5.19}$$

The following theorem is due to G. Stampacchia (private communication).

Theorem 5.2. *Assume that V satisfies* (5.19). *Let $\lambda \in \mathbf{R}_+$, $f \in L^p(\Omega)$, $p \geqslant 2$, and u be the weak solution of the problem*[22]

$$\lambda(u, v) - b(u, v) = (f, v), \qquad u \in L^2(\Omega) \qquad \forall v \in V. \tag{5.20}$$

Then $u \in L^p(\Omega)$ and

$$\| u \|_p \leqslant \frac{1}{\lambda} \| f \|_p . \tag{5.21}$$

Proof. This theorem is well-known if $p = 2$ [1, 22], and therefore, in view of the Riesz interpolation theorem, it is sufficient to prove the theorem for $p > n/2$.

Let $p > n/2$ and u be the weak solution of (5.20). Put $v = | u |^{p-2} u$. Then

$$\frac{\partial v}{\partial x_i} = (p - 1) | u |^{p-2} \frac{\partial u}{\partial x_i}, \qquad i = 1, \ldots, n. \tag{5.22}$$

[21] We set $H^{1,2}(\Omega) = H^1(\Omega)$ and $H_0^1(\Omega) = \{u \in H^1(\Omega); u(x) = 0 \ \forall x \in \Gamma\}$. We also put $(u, v) = \int_\Omega uv dx \quad \forall u, v \in L^2(\Omega)$.

[22] See G. Stampacchia in Ref. [39].

In view of a result of Stampacchia [38] u belongs to $C(\bar{\Omega})$. Therefore $v \in V$ and from (5.20) it follows that

$$\lambda \int_\Omega | u |^p \, dx + \nu^{-1}(p-1) \int_\Omega | u |^{p-2} \sum_{i=1}^n dx \leqslant \int_\Omega | f | \, | u |^{p-1} \, dx, \tag{5.23}$$

from which

$$\lambda \int_\Omega | u |^p \, dx \leqslant \int_\Omega | f | \, | u |^{p-1} \, dx, \tag{5.24}$$

and using Hölder's inequality we obtain (5.21).

We now prove

THEOREM 5.3. *Assume that* V *satisfies* (5.19) *and the following condition*:

$$\exists K_V \in \mathbf{R}_+ \text{ such that } \quad \| u \|_2 \leqslant K_V \left\| \frac{\partial u}{\partial x_i} \right\|_2 \qquad \forall u \in V, \quad i = 1,\ldots, n, \text{ [23]} \tag{5.25}$$

there exists $\lambda_V \in \mathbf{R}_+$ *such that*

$$\| u \|_p \leqslant \frac{1}{\lambda + \lambda_V} \| f \|_p \qquad \forall f \in L^p(\Omega), \tag{5.26}$$

u being the weak solution of (5.5).

Proof. Remark first that (5.23) is equivalent to

$$\lambda \int_\Omega | u |^p \, dx + \frac{4\nu^{-1}(p-1)}{p^2} \sum_{i=1}^n \int_\Omega \left| \frac{\partial u^{p/2}}{\partial x_i} \right|^2 dx \leqslant \int_\Omega | f | \, | u |^{p-1} \, dx. \tag{5.27}$$

From (5.25) it follows that

$$(\lambda + \lambda_V) \int_\Omega | u |^p \, dx \leqslant \int_\Omega | f | \, | u |^{p-1} \, dx, \tag{5.28}$$

where

$$\lambda_V = \frac{4\nu^{-1}(p-1)}{p^2 K_V}. \tag{5.29}$$

The thesis follows from Hölder's inequality.

[23] The inequality in (5.25) is a Poincaré inequality. For general cases it is verified (see Ref. [39]).

We now define the operator $B_{V,p}$ in $L^p(\Omega)$ for $p \geqslant 2$ in the following manner:

$$\begin{gathered} D_{B_{V,p}} = \{u \in L^p(\Omega);\ \exists f \in L^p(\Omega) \text{ such that } b(u, v) = (f, v)\ \forall v \in V\}, \\ B_{V,p}u = f \quad \text{if} \quad b(u, v) = (f, v) \quad \forall v \in V. \end{gathered} \tag{5.30}$$

From theorems 5.2 and 5.3 we have

COROLLARY 5.1. *If V satisfies* (5.19) *then $B_{V,p}$ belongs to $K(L^p(\Omega))$. If V also satisfies* (5.25), *then there exists $\lambda_V \in \mathbf{R}_+$ such that*

$$\lambda_V + B_V \in K(L^p(\Omega)).$$

Finally we shall define $B_{V,p}$ also for $1 < p \leqslant 2$ for duality. In view of a theorem of Phillips on dual semigroups [33] we know that $B_{V,p}$ belongs to $K(L^p(\Omega))$.

Remark 5.1. If B is a differential operator (in variational form) of order greater than two and $B_{V,p}$ is defined analogously to (5.30), then $B_{V,p}$ does not belong, in general, to $K(L^p(\Omega))$ if $p \neq 2$, as is shown by the following example:

EXAMPLE 5.2. Let $\Omega =]0, 1[$, $p = 4$, and

$$\begin{aligned} D_B &= \{u \in H^{4,4}(0, 1);\ u(0) = u(1) = u'(0) = u'(1) = 0\}, \\ Bu &= -\frac{d^4u}{dt^4}. \end{aligned} \tag{5.31}$$

It is well known [3] that B is the infinitesimal generator of an analytic semigroup.

Consider the semiscalar product [45]

$$[u, v] = \frac{\int_0^1 u(t)\, v^3(t)\, dt}{(\int_0^1 v^4(t)\, dt)^{1/2}} \qquad \forall u, v \in L^4(0, 1). \tag{5.32}$$

If $t \to e^{tB}$ should be a contraction semigroup we would have

$$[Bu, u] \leqslant 0 \qquad \forall u \in D_B, \tag{5.33}$$

but this is wrong if

$$u(t) = t^3(1 - t)^3. \tag{5.34}$$

3. Parabolic Equations and Systems

Let $[0, T]$ be a subinterval of $\bar{\mathbf{R}}_+$ and $Q = \Omega X]0, T[$. Let $\{B(t)\}_{t\in[0,T]}$ be a set of second-order uniformly elliptic operators in variational form in Ω,

$$B(t)u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(b_{ij}(x, t)\frac{\partial u}{\partial x_j}\right) \qquad \forall t \in [0, T]. \tag{5.35}$$

We assume that the ellipticity is uniform with respect to t, i.e., there exists $\nu \geqslant 1$ such that

$$\nu^{-1} \mid \xi \mid^2 \leqslant \sum_{i,j=1}^{n} b_{ij}(x, t)\, \xi_i \xi_j \leqslant \nu \mid \xi \mid^2 \qquad \forall x \in \Omega, \quad \xi \in \mathbf{R}^2, \quad t \in [0, T]. \tag{5.36}$$

Let $b(t, u, v)$ be the bilinear form "associated" to $B(t)$, viz.,

$$b(t, u, v) = -\int_\Omega \sum_{i,j=1}^{n} b_{ij}(x, t)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\, dx. \tag{5.37}$$

Let V be a closed subspace of $H^1(\Omega)$ such that

$$\left.\begin{array}{ll} \text{(i)} & V \text{ satisfies (5.19, i, ii);} \\ \text{(ii)} & \text{There exist } \alpha \in \mathbf{R} \text{ and } K \in \mathbf{R}_+ \text{ such that} \\ & \qquad b(t, u, u) + \alpha \| u \|_2^2 \geqslant K \| u \|_{2,1}^2 . \end{array}\right\} \tag{5.38}$$

Let $B_{V,q}(t)$ be the operator defined by (5.30) (with $b(t, u, v)$ instead of $b(u, v)$) in $L^q(\Omega)$, and for simplicity we put

$$B_{V,q}(t) = B(t) \qquad \text{for} \qquad q > 1. \tag{5.39}$$

Finally, let $q > 1$ and E be a linear bounded operator on $L^q(\Omega)$ such that (2.12) is verified.

We now consider the problem

$$\begin{gathered} \frac{du}{dt} - B(t)u = f, \\ u(0) = 0 \qquad (\text{resp. } u(0) = Eu(t)). \end{gathered} \tag{5.40}$$

We now prove

THEOREM 5.4. *Let* $\{B(t)\}_{t\in[0,T]}$ *be a set of uniformly elliptic operators in* Ω, *and assume that*

$$\left.\begin{array}{ll}\text{(i)} & (5.36) \textit{ is satisfied;} \\ \text{(ii)} & \textit{There exists } \alpha \in]0,1] \textit{ such that } b_{ij} \textit{ is Hölder continuous with} \\ & \textit{exponent } \alpha \textit{ on } \bar{Q} \textit{ for every } i,j = 1,\dots,n.\end{array}\right\} \qquad (5.41)$$

Let V *be a closed subspace of* $H^1(\Omega)$ *such that* (5.38) (*resp.* (5.38) *and* (5.25)) *is satisfied. Then if* $f \in L^p(0, T; L^q(\Omega))$, $p > 1$, $p > 2$, *the problem* (5.40) *has one and only one weak solution* $u \in \Lambda_0{}^p(0, T; L^q(\Omega); B)$ (*resp.* $\Lambda^p_{\Pi,E}(0, T; L^q(\Omega); B)$).

Proof. On the basis of Theorem 5.3 if V satisfies (5.38) $B(t) \in K(L^q(\Omega))$ $\quad \forall t \in [0, T]$, and if V satisfies (5.38) and (5.25) there exists $\omega \in \mathbf{R}_+$ such that $B(t) + \omega \in K(L^q(\Omega))$. Then, using Theorem 5.1, $\{B(t)\}_{t\in[0,T]}$ (resp. $\{B(t) + \omega\}_{t\in[0,T]}$) satisfies the hypotheses of Theorem 3.2 (resp. of Theorem 3.5), and therefore the thesis follows from Corollary 3.1 (resp. Corollary 3.3).

Remark 5.2. If u is a weak solution of (5.40) then it is a weak solution (in the usual meaning) of the mixed problem

$$\begin{gathered}\frac{\partial u}{\partial t} - \sum_{ij=1}^{n} \frac{\partial}{\partial x_i}\left(b_{ij}\frac{\partial u}{\partial x_j}\right) = f \\ u(.,t) \in V \qquad \forall t \in [0, T], \\ u(x, 0) = 0 \quad (\text{resp. } u(x,0) = Eu(x,t)) \qquad \forall x \in \Omega,\end{gathered} \qquad (5.42)$$

and conversely.

Remark 5.3. We have not studied the problem of characterizing the spaces $\Lambda_0{}^p(0, T; L^q(\Omega); B)$ and $\Lambda^p_{\Pi,E}(0, T; L^q(\Omega); B)$ (regularization problem). We have proved only that they are contained in $L^p(0, T; L^q(\Omega))$.

Now let k be a strongly continuous map: $[0, T] \times [0, T] \to \mathscr{L}(L^q(\Omega), L^q(\Omega))$. We consider the problem

$$\frac{du}{dt} - B(t)u + \int_0^t K(t,s)\, u(s)\, ds = f, \qquad u(0) = 0. \qquad (5.43)$$

The following theorem is proved in the same way as Theorem 5.4, using corollary 3.2 instead of Corollary 3.1.

THEOREM 5.5. *Under the same hypotheses of Theorem* 5.4, *if* $u \in L^p(0, T; L^q(\Omega))$, $p > 1$, $q \geqslant 2$, *then the problem* (5.43) *has one and only one weak solution* $u \in \Lambda_0{}^p(0, T; L^q(\Omega); B)$.[24]

Remark 5.4. If u is a weak solution of (5.43) then it is a weak solution (in the usual meaning) of the mixed integrodifferential problem

$$\begin{gathered}\frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(b_{ij}\frac{\partial u}{\partial x_j}\right) + \int_0^t K(t, s)\, u(x, s)\, ds = f,\\ u(.,t) \in V \qquad \forall t \in [0, T],\\ u(x, 0) = 0 \qquad \forall x \in \Omega,\end{gathered} \tag{5.44}$$

and conversely.

Remark 5.5. Let, for example, $p = q$, $V = H_0{}^1(\Omega)$. Then it is known [5] that

$$\Lambda_0{}^p(0, T; L^p(\Omega); B) = \{u \in H^{1,p}(0, T; H^{2,p}(\Omega) \cap H_0{}^1(\Omega)); u(0) = 0\}.$$

Therefore if $f \in L^p(Q)$ the solution u of (5.44) belongs to

$$H^{1,p}(0, T; H^{2,p}(\Omega) \cap H_0{}^1(\Omega)) \qquad \text{and} \qquad u(0) = 0.$$

Finally, let $\{B_k(t)\}_{t\in[0,T]}$, $k = 1,\ldots, m$, be m sets of second-order uniformly elliptic operators in variational form in Ω,

$$B_k(t)u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(b_{ij}^{(k)}(x, t)\frac{\partial u}{\partial x_j}\right). \tag{5.45}$$

We assume that for every k, $k = 1,\ldots, m$, $\{B_k(t)\}_{t\in[0,T]}$ satisfies the hypotheses of $\{B(t)\}_{t\in[0,T]}$.

We put

$$b^{(k)}(u, v, t) = \int_\Omega \sum_{ij=1}^{n} b_{ij}^{(k)}(x, t)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}\, dx \qquad \forall u,v \in H^1(\Omega), \quad t \in [0, T]. \tag{5.46}$$

Let V a closed subspace of $H^1(\Omega)$ such that

$$\begin{gathered}\text{For every } k,\ k = 1, 2,\ldots, m,\ V \text{ satisfies (5.38) and (5.25)}\\ \text{(with } b^{(k)}(u, v, t) \text{ instead of } b(u, v, t)).\end{gathered} \tag{5.47}$$

[24] See Chapter II, Section 3.

Let $B_{V,q}^{(k)}(t)$ be the operator defined by (5.30) (with $b^{(k)}(t, u, v)$ instead of $b(u, v)$). For simplicity we put

$$B_{V,q}^{(k)}(t) = B_k(t). \tag{5.48}$$

Finally, let P be a linear operator in R^m such that

$$(Px, x) \leqslant 0 \qquad \forall x \in \mathbf{R}^m, \tag{5.49}$$

$(\, , \,)$ being the inner product in $\mathbf{R}^m$.

We now consider the problem

$$\begin{gathered} \frac{du_1}{dt} - B_1(t)\, u_1(t) - \sum_{k=1}^{m} P_{1k} u_k(t) = f_1(t), \\ -\frac{du_h}{dt} + B_h(t)\, u_h(t) + \sum_{k=1}^{m} P_{hk} u_k(t) = f_h(t), \qquad h = 2, 3,\dots, m, \\ u_1(0) = 0, \qquad u_h(T) = 0, \qquad h = 2, 3,\dots, m. \end{gathered} \tag{5.50}$$

The following theorem is proved in the same manner as Theorem 5.4, using Corollary 4.5 instead of Corollary 3.1.

THEOREM 5.6. *Let* $\{B_k(t)\}_{t\in[0,T]}$, $k = 1,\dots, m$, *be* m *sets of uniformly elliptic operators in* Ω *satisfying* (5.36) (*with* $b_{i,j}^{(k)}(x, t)$ *instead of* $b_{ij}(x, t)$), *let* V *be a closed subspace of* $H^1(\Omega)$ *satisfying* (5.47), *and let* P *be a linear operator in* R^m *satisfying* (5.49). *Then if* $\mathbf{f} = (f_1 ,\dots, f_m)$, $(L^p(0, T; L^q(\Omega))^m$, $p > 1$, $q \geqslant 2$, *the problem* (5.50) *has one and only one weak solution* $\mathbf{u} = (u_1 ,\dots, u_m)$, *and* $u_i \in \Lambda_0{}^p(0, T; L^q(\Omega)), B_i)$, $i = 1,\dots, m$.

Remark 5.6. If $\mathbf{u}$ is a weak solution of (5.50) then it is a weak solution (in the usual meaning) of the system

$$\begin{gathered} \frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(b_{ij}^{(1)} \frac{\partial u_1}{\partial x_j}\right) - \sum_{k=1}^{m} P_{1k} u_k = f_1 , \\ -\frac{\partial u_h}{\partial t} + \sum_{ij=1}^{n} \frac{\partial}{\partial x_i}\left(b_{ij}^{(h)} \frac{\partial u_h}{\partial x_j}\right) + \sum_{k=1}^{m} P_{hk} u_k = f_h , \qquad h = 2, 3,\dots, m, \\ u(0) = 0, \qquad u_h(T) = 0, \qquad h = 2, 3,\dots, m. \end{gathered} \tag{5.51}$$

Remark 5.7. If $p = q$, $V = H_0^1(\Omega)$. Then if $\mathbf{f} \in (L^p(Q))^n$ the solution $\mathbf{u}$ of the problem (5.51) belongs to $(H^{1,p}(0, T); H^{2,p}(\Omega) \cap H_0^1(\Omega))^m$ and $u_1(0) = U_h(T) = 0$, $h = 2, 3,..., m$.

4. Schrödinger and Wave Equations

Let Ω be an open bounded subset of the Euclidean space $\mathbf{R}^n$ with Γ its boundary, $[0, T]$ be a subinterval of $\bar{\mathbf{R}}_+$, and $Q = \Omega X]0, T[$. Let $\{B(t)\}_{t \in [0,T]}$ be a set of uniformly elliptic operators in Ω of order m,

$$B(t)u = - \sum_{|p|,|q| \leqslant m} (-1)^{|p|} D^p(b_{pq}(x, t) D^q u), \tag{5.52}$$

where $b_{p,q} \in C(\bar{Q})$, and

$$\begin{aligned} D^p &= \frac{\partial^{p_1+\cdots+p_n}}{\partial x_1^{p_1} \cdots \partial x_n^{p_n}}, \qquad p = (p_1, ..., p_n), \\ |p| &= p_1 + \cdots + p_n, \qquad p_1, ..., p_n \text{ nonnegative integers.} \end{aligned} \tag{5.53}$$

We put

$$b(u, v, t) = - \int_\Omega b_{pq}(x, t) D^q u \overline{D^p v}\, dx. \tag{5.54}$$

Let V be a closed subspace of $H^m(\Omega)$ such that

$$H_0^m(\Omega) \subset V \subset H^m(\Omega). \tag{5.55}$$

We assume that

$$\left.\begin{array}{ll} \text{(i)} & \text{The map } [0, T] \to \mathbf{C},\ t \to b(t, u, v) \text{ is Lipschitzian for every} \\ & u, v \in V; \\ \text{(ii)} & b(t, u, v) = \overline{b(t, v, u)} \qquad \forall u, v \in V; \\ \text{(iii)} & \text{There exists } \alpha \in \mathbf{R}_+ \text{ and } \beta \in \mathbf{R} \text{ such that} \\ & \quad -b(t, u, u) + \beta \| u \|_2^2 \geqslant \alpha \| u \|_m^2 \qquad \forall u \in V. \end{array}\right\} \tag{5.56}$$

Note that condition (5.56) is equivalent to $b_{pq}(x, t)$ being Lipschitzian in t.

It is well-known [22] that $B(t)$, with the domain

$$D_{B(t)} = \{v \in V \text{ such that the map } V \to \mathbf{C},\ u \to b(t, u, v) \text{ is continuous in the topology of } L^2(\Omega)\}, \tag{5.57}$$

is self-adjoint in $L^2(\Omega)$. Therefore $B(t) \in K(L^2(\Omega))\ \forall t \in [0, T]$.

Finally, let E be a linear bounded operator on $L^q(\Omega)$ satisfying (2.12).

We now consider the problem

$$\frac{du}{dt} + iB(t)u = f, \qquad u(0) = 0 \tag{5.58}$$

The following theorem is proved as Theorem 5.4.

THEOREM 5.7. *Let $\{B(t)\}_{t\in[0,T]}$ be a set of uniformly elliptic operators in Ω satisfying* (5.56) (*resp.* (5.56) *with $\beta < 0$*). *Then if $f \in L^p(0, T; L^2(\Omega))$, $p > 1$, the problem* (5.58) *has one and only one weak solution $u \in L^p(0, T; L^2(\Omega))$.*

Remark 5.8. Actually the solution u belongs to $\Lambda_0{}^p(0, T; L^2(\Omega); iB)$ (resp. $\Lambda^p_{\Pi,E}(0, T; L^2(\Omega); iB)$, but for the Schrödinger equation there are no regularization theorems. Therefore also in the case of $p = 2$, $\Lambda_0{}^2(0, T; L^2(\Omega); iB)$ depends on the particular operator $B(t)$.

Now let K be a strongly continuous map:

$$[0, T] \times [0, T] \to \mathscr{L}(L^2(\Omega), L^2(\Omega)).$$

We consider the problem

$$\frac{du}{dt} + iB(t)u + \int_0^t K(t, s)\, u(s)\, ds = f, \qquad u(0) = 0. \tag{5.59}$$

The following result is proved as in Theorem 5.5.

THEOREM 5.8. *Under the same hypotheses as Theorem* 5.7, if $f \in L^p(0, T; L^2(\Omega))$, $p > 1$, *the problem* (5.59) *has one and only one weak solution $u \in L^p(0, T; L^2(\Omega))$.*

Finally, we consider Cauchy's problem for the wave equation,

$$\frac{d^2u}{dt^2} - B(t)u = f, \qquad u(0) = u'(0) = 0, \tag{5.60}$$

and prove

THEOREM 5.9. *Let $\{B(t)\}_{t\in[0,T]}$ be a set of uniformly elliptic operators in Ω satisfying* (5.56) *and the following condition*:

There exists $\omega \in \mathbf{R}$ such that

$$b(t+s,u,u) \leqslant e^{\omega t} b(s,u,u) \qquad \forall u \in V, \quad t, s, t+s \in [0,T]. \tag{5.61}$$

Then if $f \in L^p(0,T;L^2(\Omega))$, the problem (5.60) *has one and only one weak solution $u \in L^p(0,T;V) \cap H^{1,p}(0,T;L^2(\Omega))$, and such that $u(0) = 0$.*

Proof. It is sufficient to observe that (4.1, i, ii) (where $C(t) = B(t)$) are satisfied in view of (5.60, i) and (5.61) and then use Corollary 4.1.

5. EQUATIONS OF HIGHER ORDER IN t

We use the same notations of Section 4. Let $\{B(t)\}_{t\in[0,T]}$ be a set of uniformly elliptic operators in Ω of order m. We assume that the coefficients $b_{ij}(x,t)$ are α-Hölder-continuous in Q, $\alpha \in]0,1]$.

Let V be a closed subspace of $H^m(\Omega)$ satisfying (5.55). We assume that

There exists $\alpha \in \mathbf{R}_+$ and $\beta \in \mathbf{R}_+$ such that

$$-b(t,u,u) + \beta \| u \|_2^2 \geqslant \alpha \| u \|_m^2 \qquad \forall u \in V. \tag{5.62}$$

We now consider the problem

$$\begin{aligned} &\frac{d^l u}{dt^l} + (-1)^l B(t) = f \qquad \text{for} \qquad l \in \mathbf{N}, \\ &u^{(k)}(0) = u^{(k)}(T), \qquad k = 0, 1, \ldots, n-1, \end{aligned} \tag{5.63}$$

and prove

THEOREM 5.10. *Let $\{B(t)\}_{t\in[0,T]}$ be a set of uniformly elliptic operators in Ω satisfying* (5.62). *Then if $f \in L^2(Q)$, the problem* (5.63) *has one and only one weak solution $u \in L^2(Q)$.*

Proof. Due to (5.62) there exists $\omega \in \mathbf{R}_+$ such that

$$B(t) + \omega \in K(L^2(\Omega)) \qquad \forall t[0,T].$$

Then in view of the Hölder continuity of $b_{ij}(x,t)$ and Theorem 5.1 we see that $\{B(t) + \omega\}_{t\in[0,T]}$ satisfies the hypotheses of Theorem 4.5, and the thesis follows.

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