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On 2-arc-transitivity of Cayley graphs

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Dedicated to Crispin, my teacher.

Abstract

The classification of 2-arc-transitive Cayley graphs of cyclic groups, given in (J. Algebra. Combin. 5 (1996) 83–86) by Alspach, Conder, Xu and the author, motivates the main theme of this article: the study of 2-arc-transitive Cayley graphs of dihedral groups. First, a previously unknown infinite family of such graphs, arising as covers of certain complete graphs, is presented, leading to an interesting property of Singer cycles in the group $PGL(2,q)$, q an odd prime power, among others. Second, a structural reduction theorem for 2-arc-transitive Cayley graphs of dihedral groups is proved, putting us—modulo a possible existence of such graphs among regular cyclic covers over a small family of certain bipartite graphs—a step away from a complete classification of such graphs. As a byproduct, a partial description of 2-arc-transitive Cayley graphs of abelian groups with at most three involutions is also obtained.

Keywords: Permutation group; Imprimitive group; Abelian group; Dihedral group; Cayley graph; 2-Arc-transitive graph

1. Introductory remarks

Throughout this paper groups are finite, and graphs are finite, simple, undirected and unless specified otherwise, connected. For the group-theoretic terminology not defined here we refer the reader to [3,9,12,53].

For a graph X we let $V(X)$, $E(X)$ and $A(X)$ denote the vertex set, the edge set, and the arc set of X, respectively. If u and v are adjacent (or neighbours) in X, we denote the corresponding edge by $[u,v]$ (or by its shorter version uv).

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A permutation group G is said to act *semiregularly* on a set V if it has trivial point stabilizers. A transitive and semiregular group is said to be *regular*. Regular group actions are tied to the concept of Cayley graphs. Given a group G and a generating set $Q = Q^{-1}$ of G such that $1 \notin M$, the *Cayley graph* $Cay(G, Q)$ *of* G *relative to* Q has vertex set G and edges of the form $[g, gq], g \in G, q \in Q$. Clearly, the automorphism group of $Cay(G, Q)$ admits a left regular action of G. The converse is also true [45].

Adopting the terminology of Tutte [50], for $k \geqslant 0$, a *k-arc* in a graph X is a sequence of $k+1$ vertices $v_1, v_2, \ldots, v_{k+1}$ of X, not necessarily all distinct, such that any two consecutive terms are adjacent and any three consecutive terms are distinct. Let X be a graph and G be a subgroup of its automorphism group $Aut\,X$. We say that X is (G, k)*-arc-transitive* provided G acts transitively on the set of k-arcs of X. In particular, when $G = Aut\,X$, we say that X is *k-arc-transitive* and we say that it is *exactly k-arc-transitive* if it is k-arc-transitive but not $(k+1)$-arc-transitive. Also, 1-arc-transitive graphs and 0-arc-transitive graphs are usually referred to as *arc-transitive* and *vertex-transitive*, respectively.

When investigating the structure of graphs admitting transitive actions of subgroups of automorphisms one often relies on results which are purely group-theoretic in nature. On the other hand, a graph-theoretic language may often provide useful insights in the study of various problems in permutation groups. A particular case of such a fruitful interplay of group- and graph-theoretic concepts is dealt with in this paper and is motivated by the classical results on B-groups due to Schur and Wielandt (see Propositions 8.2 and 8.3) and by some of the more recent research in 2-arc-transitive Cayley graphs [1,20]. (A group G is a *Burnside group*, in short, a *B-group* if every permutation group containing a regular subgroup isomorphic to G is either imprimitive or doubly transitive—the first example of such a group is due to Burnside [7, p. 343], thus explaining the name.)

Impelled by a complete classification of 2-arc-transitive Cayley graphs of cyclic groups in [1], the main purpose of this article is a thorough study of 2-arc-transitive Cayley graphs of dihedral groups. The fact that cyclic groups of composite order and dihedral groups are B-groups is of vital importance in this respect. Also, note that a vertex-transitive graph is 2-arc-transitive if and only if the restriction of a vertex stabilizer to the set of neighbours of the corresponding (fixed) vertex is a doubly transitive group (see [44, Lemma 9.4]). From a group-theoretic point of view, the objects dealt with in this article are therefore transitive permutation groups containing a regular subgroup isomorphic to a dihedral group, and having point stabilizers acting doubly transitively on some non-trivial connected suborbit, that is, on some suborbit which gives rise to a connected orbital graph. The main result of this article is a structural reduction theorem for 2-arc-transitive Cayley graphs of dihedral groups which comes a step short from a complete classification of these graphs (as well as the corresponding group-theoretic counterparts)—see Theorem 2.1.

This article is organized as follows. In Section 2 we start by giving a brief history of the research on arc-transitive graphs and by laying out the basic strategy behind the analysis of 2-arc-transitive Cayley graphs of dihedral groups. We then present a previously unknown infinite family of 2-arc-transitive Cayley graphs of dihedral groups $D_{4(q+1)}$, q an odd prime power, which are denoted by K_{q+1}^4 and arise as certain 4-fold covers of the complete graph K_{q+1} (and 2-fold covers of $K_{q+1,q+1} - (q+1)K_2$, the complete bipartite graph with a 1-factor removed). We close the section by phrasing Theorem 2.1. In Section 3 we introduce the concept of a symbol of a Cayley graph of a dihedral group, a useful encoding of the set of neighbours of a given fixed vertex in the graph. Section 4 is devoted to special terminology and notation pertaining to covers of graphs. In Section 5 we show that the graphs K_{q+1}^4, q an odd prime power, are indeed Cayley graphs of dihedral groups $D_{4(q+1)}$, and give (as a consequence) an interesting property of Singer cycles in the group $PGL(2,q)$. In Section 6 we obtain a partial classification of 2-arc-transitive Cayley graphs of abelian groups with at most three involutions. These results prove helpful in the investigation of two particular aspects of 2-arc-transitive Cayley graphs of dihedral groups: certain small length blocks of imprimitivity and the above mentioned symbols, a task carried out in Section 7. These results are then used in Section 8, where the proof of Theorem 2.1 is finally given.

Let us remark that, in contrast with some of the other work on 2-arc-transitive graphs [2,11,21,22,30,33–35], where the classification of finite simple groups is a vital and essential part, the underlying philosophy of this article stems from a pursuit of a different goal. With a blending of group-theoretic and combinatorial techniques and results from coverings of graphs we have, for the better part, managed to avoid the use of the classification of finite simple groups. We believe that having consciously opted for a lighter weaponry we have not only succeeded in making this article reasonably self-contained, but have also obtained a clearer insight into the intrinsic peculiarities of the mathematical objects dealt with.

2. Laying out the strategy

The study of arc-transitive graphs has its roots in the seminal paper of Tutte [49], where it was proved that cubic graphs are at most 5-arc-transitive. A number of articles on the subject followed over the years (disguised at times in group-theoretic language), most of them dealing with cubic and quartic arc-transitive graphs, others touching more general grounds (see for example [6,10,13–15,17–19,23–25, 27,32,42,46,47,50,52,54] for a by no means exhaustive list), and perhaps reaching its peak with [51], where using the classification of simple groups, Weiss extended Tutte's bound to graphs of arbitrary valency by showing that no 8-arc-transitive graph exists. However, it was only much later that an infinite family of quartic 7-arc-transitive (finite) graphs was actually found [11].

An interest in this topic of research, in 2-arc-transitive graphs in particular, resurrected quite recently, together with an organized effort that has been set in motion to understand the structure of such graphs based on an analysis of their quasiprimitive and biquasiprimitive quotients. A word of explanation singling out the main feature that makes 2-arc-transitive graphs more tractable than their "less symmetric" arc-transitive counterparts seems appropriate.

Given a transitive group G acting on a set V, we say that a partition $\mathscr{P}$ of V is G-*invariant* (or an *imprimitivity block system* of G) if the elements of G permute the parts, that is, *blocks* of $\mathscr{P}$ blockwise. If the trivial partitions $\{V\}$ and $\{\{v\}\colon v \in V\}$ are the only G-invariant partitions of V, then G is said to be *primitive*, and is said to be *imprimitive* otherwise. Clearly, the set of orbits of any normal subgroup of G gives rise to a G-invariant partition of V; we call it a G-*normal partition*. In particular, if this subgroup is non-trivial and intransitive, then the corresponding partition is non-trivial and the group G is imprimitive. Suppose that every G-normal partition of V consists of at most two blocks. Then the group G is said to be *quasiprimitive* if every G-normal partition is trivial, and is said to be *biquasiprimitive* if it has at least one G-normal partition with two blocks.

Now let X be a $(G,2)$-arc-transitive graph for some subgroup G of $Aut\,X$ and let $\mathscr{P}$ be a G-invariant partition of the vertex set $V(X)$. Let $X_{\mathscr{P}}$ be the associated *quotient graph* of X relative to $\mathscr{P}$, that is, the graph with vertex set $\mathscr{P}$ and edge set induced naturally by the edge set $E(X)$. If $|\mathscr{P}| \geqslant 3$, then the bipartite graphs induced by two adjacent blocks in $\mathscr{P}$ are all isomorphic and moreover, in view of 2-arc-transitivity of G, the blocks in $\mathscr{P}$ are independent sets and, moreover, any vertex $v \in V(X)$ has at most one neighbour in an arbitrary block in $\mathscr{P}$.

Let us apply this fact to the special case where $\mathscr{P}$ coincides with the set of orbits of a non-trivial normal subgroup K of G. If K has at least three orbits, then X is a *normal K-cover* (since K acts regularly, also referred to as a *regular K-cover*, a term we will be using in this article) of a $(G/K,2)$-arc-transitive quotient graph $X_K = X_{\mathscr{P}}$. (In particular, if K is a cyclic group we shall say that X is a *regular cyclic* cover of X_K.) Moreover, assume also that K is a maximal intransitive normal subgroup of G. Then every non-trivial normal subgroup of G/K has at most two orbits and consequently G/K is either quasiprimitive or biquasiprimitive. These observations were made first in [43] in a slightly more general context of locally primitive graphs, that is, vertex-transitive graphs with vertex stabilizers acting primitively on the corresponding neighbours' sets (see [46, Theorem 10.2]). They suggest an obvious (at least in principle) strategy leading to a possible classification of 2-arc-transitive graphs, involving the following two steps. Step 1 would be concerned with obtaining a description of *quasiprimitive* and *biquasiprimitive* 2-arc-transitive graphs, that is, graphs whose every 2-arc-transitive subgroup of automorphisms is either quasiprimitive or biquasiprimitive, in short, *basic* graphs. Step 2 would then consist in finding all 2-arc-transitive regular covers of basic 2-arc-transitive graphs.

It is Step 1 that has received much of the attention thus far, as part of a programme for classifying quasiprimitive 2-arc-transitive graphs. In [30], Ivanov and Praeger have completed the classification of quasiprimitive 2-arc-transitive graphs of affine type, and Baddeley has given many constructions and a detailed description of quasiprimitive graphs of twisted wreath type [2]. More recently, Fang and Praeger have carried out a similar description of 2-arc-transitive graphs associated with the Suzuki groups and Ree groups [21,22]. Finally, let us mention the three most recent papers by Li [33–35], where, among others, a classification of quasiprimitive 2-arc-transitive graphs of odd order and prime power order has been completed.

The importance of the concept of covers to various symmetry problems in graphs is well known and may be measured, among others, by the amount of research that has been done since the first published papers on the subject [16,26,28]. This makes the fact that Step 2 has remained almost unchallenged thus far, so much harder to understand. Part of the reason lies in certain combinatorial difficulties one encounters when faced with classification problems involving covers of graphs. Nevertheless, a few attempts have indeed been made, the most recent one perhaps in [20], where regular covers of complete graphs whose group of covering transformations is either cyclic or isomorphic to $\mathbb{Z}_p^2$, p a prime, and whose fibre-preserving subgroup of automorphisms acts 2-arc-transitively, have been classified. Apart from the obvious canonical double covers, the list contains also certain regular $\mathbb{Z}_4$-covers and $\mathbb{Z}_2^2$-covers of complete graphs which, as will become apparent later on, are one of the essential ingredients of this paper. (Given a positive integer n, we shall use the symbol $\mathbb{Z}_n$ to denote the ring of residues modulo n as well as the cyclic group of order n. This should cause no confusion.)

A second paper dealing, at least implicitly, with Step 2, is [1] where a complete classification of 2-arc-transitive Cayley graphs of cyclic groups is given. Following [1, Theorem 1.1] such a graph is one of the following: the cycle C_n, $n \geqslant 3$, which is k-arc-transitive for any $k \geqslant 2$; or the complete graph K_n, $n \geqslant 3$, which is exactly 2-arc-transitive; or the complete bipartite graph $K_{n/2,n/2}$, $n \geqslant 6$, which is exactly 3-arc-transtive; or $K_{n/2,n/2} - n/2K_2$, the complete bipartite graph minus a 1-factor, that is, the canonical double cover of $K_{n/2}$, where $n/2 \geqslant 5$ is odd, which is exactly 2-arc-transitive. The proof of this theorem is purely combinatorial and completely void of the concepts of basic graphs and regular covers. Nevertheless, it is worth mentioning that, translated into the language of this article, the above theorem says the following: first, complete graphs and complete bipartite graphs are, respectively, the only quasiprimitive and biquasiprimitive 2-arc-transitive circulants of valency greater than 2, and second, the only non-basic 2-arc-transitive circulants of valency greater than 2 are the canonical double covers $K_{m,m} - mK_2$, m odd, of complete graphs K_m.

In this article we continue our investigation of the structure of 2-arc-transitive Cayley graphs by giving a partial extension of the above result to dihedral groups. Also, as a byproduct, we obtain a partial description of 2-arc-transitive Cayley graphs of abelian groups with at most three involutions. (This is not surprising for the intersection of the classes of 2-arc-transitive Cayley graphs of dihedral groups and Cayley graphs of abelian groups with at most three involutions is non-empty. For example, $K_{2m,2m}$, $m \geqslant 2$, is not only a circulant and a Cayley graph of the dihedral group D_{4m}, but also a Cayley graph of the abelian group $\mathbb{Z}_{2m} \times \mathbb{Z}_2$.) When applying the "2-steps" strategy for determining 2-arc-transitive graphs (as layed out above) to the special case of Cayley graphs of dihedral groups, we rely heavily on the fact that cyclic groups of composite order and all dihedral groups are B-groups (see Propositions 8.2 and 8.3). As a consequence, with the exception of complete graphs, all Cayley graphs of dihedral groups must have an imprimitive automorphism group. Moreover, as we shall see, it transpires that in the imprimitive case there always exists an imprimitivity block system associated with a non-trivial intransitive normal subgroup. This enables us to obtain a clean description of basic 2-arc-transitive Cayley graphs of dihedral groups. Apart from the complete graphs, the complete bipartite graphs, and cycles of order twice a prime number—which are also basic 2-arc-transitive circulants—the only new graphs are the incidence/non-incidence graphs of symmetric designs with a group of automorphisms acting doubly transitively on points and containing a regular cyclic subgroup. These are (as it may be deduced from [31, Theorem]: the incidence and non-incidence graphs $B(PG(d,q))$ and $B'(PG(d,q))$, respectively, associated with the projective spaces $PG(d,q)$, $d\geqslant 2$, and the incidence and non-incidence graphs $B(H_{11})$ and $B'(H_{11})$, respectively, of the unique Hadamard design H_{11} on 11 points.

In order to phrase the main result of this article, Theorem 2.1 below, we need to introduce the following additional notation. For an odd prime power q, let V and V' denote two copies of the projective line $PG(1,q)=GF(q)\cup\{\infty\}$. Identify the vertex set of $K_{q+1,q+1}-(q+1)K_2$ with $V\cup V'$ in such a way that the edge set consists of all the edges of the form $[u,v']$, $u\neq v$. Then we let K_{q+1}^4 denote the regular $\mathbb{Z}_2$-cover of $K_{q+1,q+1}-(q+1)K_2$ where the voltage of the arc (u,v') is 1 if and only if $u,v\in GF(q)$ and $u-v$ is a square in $GF(q)$, and is 0 in all other instances. A more detailed discussion of this family of graphs (first discovered in [20], but defined in a less concise manner as certain 4-fold covers of K_{q+1}) is given in Section 5.

Recall that quasiprimitive and biquasiprimitive graphs are referred to as basic graphs.

Theorem 2.1. *Let $n\geqslant 3$ and let X be a connected, 2-arc-transitive Cayley graph of a dihedral group of order $2n$. Then one of the following occurs:*

(i) *X is a basic graph and is isomorphic to one of the following graphs: C_{2n}, n a prime; K_{2n}; $K_{n,n}$; $B(H_{11})$ or $B'(H_{11})$ with $n=11$; $B(PG(d,q))$ or $B'(PG(d,q))$, where $n=(q^d-1)/(q-1)$, $d\geqslant 2$ and q is a prime power.*

(ii) *X is not a basic graph and is isomorphic to $K_{n,n}-nK_2$ or to K_{q+1}^4, with $n=2(q+1)$ and q an odd prime power.*

(iii) *X is not a basic graph and is either a regular cyclic cover of a basic graph in* (i) *other than a cycle or a complete graph, or a regular cyclic cover of a non-basic graph in* (ii).

In the next sections, we carefully prepare the grounds for the proof of Theorem 2.1 (phrased in a slightly modified form as Theorem 8.4) which will be given in Section 8.

Finally, we remark that part (iii) of Theorem 2.1 can be further improved. This task is pursued in a sequel to this article [38]. The preliminary findings suggest that a better understanding of the structure of 2-arc-transitive regular cyclic covers of the incidence and nonincidence graphs $B(PG(d,q))$ and $B'(PG(d,q))$ associated with projective spaces $PG(d,q)$, $d \geqslant 2$, is of vital importance to obtaining a complete classification of 2-arc-transitive Cayley graphs of dihedral groups.

3. Notation, terminology, examples

Let X be a graph and let $u, v \in V(X)$. We let $N^i(u)$ denote the set of all vertices of X at distance i from u. More generally, we let $N^{i,j}(u,v) = N^i(u) \cap N^j(v)$. In particular $N(u,v) = N^{1,1}(u,v)$. For a subset W of $V(X)$ we let $N(W)$ denote the set of all neighbours of vertices in W. For a pair of disjoint subsets U and W of X we let $X[U,W]$, or just $[U,W]$, when the graph X is clear from the context, denote the bipartite subgraph of X with vertex set $U \cup W$ and edge set consisting of all edges of X with one end-vertex in U and the other in W.

Given a ring $\mathcal{R}$ we let $\mathcal{R}^{\#}$ and $\mathcal{R}^{*}$ denote the non-zero elements of R and the group of invertible elements of $\mathcal{R}$, respectively. Moreover, for a subset M of $\mathcal{R}$ let $M^{\#} = M \cap \mathcal{R}^{\#}$ denote the non-zero elements of M. Extending this notation to groups, we let $S^{\#} = S \setminus \{id\}$ denote the set of non-trivial elements in a subset S of a group G. Also, we shall be using additive notation in the case of abelian groups.

For integers $m \geqslant 1$ and $n \geqslant 2$, let $\mathcal{M}(m,n)$ denote the set of all $m \times m$-matrices $\underset{\sim}{S}$ whose (i,j)-entry $S_{i,j} = -S_{j,i}$ is a subset of $\mathbb{Z}_n$ if $i \neq j$ and a subset of $\mathbb{Z}_n^{\#}$ if $i = j$. To each graph X with an (m,n)-*semiregular* automorphism ρ, that is, an automorphism with m orbits W_i, $i \in \mathbb{Z}_m$, of length n, we may associate a member of $\mathcal{M}(m,n)$ in the following way. For each $i \in \mathbb{Z}_n$ choose $w_i \in W_i$. We call the matrix $\underset{\sim}{S} \in \mathcal{M}(m,n)$, with the (i,j)-entry $S_{i,j} = \{s \in \mathbb{Z}_n : [w_i, \rho^s w_j] \in E(X)\}$, the *symbol* of X relative to the $(m+1)$-tuple $(\rho, w_1, \ldots, w_m)$. Conversely, each matrix $\underset{\sim}{S} \in \mathcal{M}(m,n)$ is a symbol of some graph with an (m,n)-semiregular automorphism, namely the graph $X(\underset{\sim}{S})$ with vertex set $\{w_{i,x} : i \in \mathbb{Z}_m, x \in \mathbb{Z}_n\}$ and edge set $\{[w_{i,x}, w_{j,y}] : y - x \in S_{i,j}\}$.

Two instances of the above situation are of particular interest to us. First, the case $m = 1$ gives rise to the so-called *circulants*, that is, Cayley graphs of cyclic groups. For simplicity, reasons we let the set $S = S_{1,1}$ be called a *symbol* of the circulant $X(\underset{\sim}{S})$, that is, the Cayley graph $Cay(\mathbb{Z}_n, S)$, denoted by $Cir(n,S)$. Observe that in this case the symbol depends solely on the automorphism ρ and not on the choice of a particular vertex. Second, the case $m = 2$ with $S_{1,1} = S_{2,2}$ gives rise to Cayley graphs of a dihedral group D_{2n}, for the purpose of this article referred to as *dihedrants*. We let the pair $[S,T] = [S_{1,1}, S_{1,2}]$ be called a *symbol* of the dihedrant $X(\underset{\sim}{S})$; this graph will be denoted by $Dih(2n,S,T)$.

Throughout the rest of this article we let $U = W_1 = \{u_i = \rho^i w_1\colon i \in \mathbb{Z}_n\}$ and $V = W_2 = \{v_i = \rho^i w_2\colon i \in \mathbb{Z}_n\}$ denote the two orbits of the $(2, n)$-semiregular automorphism ρ of $Dih(2n, S, T)$. Thus, a regular dihedral group contained in $Aut\, Dih(2n, S, T)$ is generated by the automorphisms ρ and τ mapping according to the rules:

$$\rho u_i = u_{i+1}, \ \ \rho v_i = v_{i+1}, \quad i \in \mathbb{Z}_n \tag{1}$$

and

$$\tau u_i = v_{-i}, \ \ \tau v_i = u_{-i}, \quad i \in \mathbb{Z}_n. \tag{2}$$

The proof of the following result is straightforward.

Proposition 3.1. *Let $n \geqslant 2$ be an integer, let $S \subseteq \mathbb{Z}_n^*$, $T \subseteq \mathbb{Z}_n$, $a \in \mathbb{Z}_n^*$ and $b \in \mathbb{Z}_n$. Then $Dih(2n, S, T) \cong Dih(2n, aS, aT + b)$.*

Let us give a brief description of some of the known 2-arc-transitive dihedrants in terms of their symbols. For example, the pair $[\emptyset, \{t, t'\}]$, $t, t' \in \mathbb{Z}_n$, is a symbol of the cycle C_{2n} if and only if $t - t' \in \mathbb{Z}_n^*$. On the other hand, $[\mathbb{Z}_n^{\#}, \mathbb{Z}_n]$ is the (only) symbol of the complete graph K_{2n}.

The situation with the complete bipartite graph and the complete bipartite graph with a 1-factor removed is somewhat trickier. Clearly, $[\emptyset, \mathbb{Z}_n]$ and $[\emptyset, \mathbb{Z}_n \setminus \{k\}]$, $k \in \mathbb{Z}_n$, are the obvious symbols of $K_{n,n}$ and $K_{n,n} - nK_2$, respectively. Alternative symbols of the form $[S, T]$, $S \neq \emptyset$ occur when n is even, as will be seen in Section 7, where we give a more detailed discussion of symbols of (2-arc-transitive) dihedrants. Therefore Proposition 3.1 gives only a sufficient condition for the isomorphism of two dihedrants.

As for the graphs associated with projective spaces and the Hadamard design H_{11}, the relative symbols are clearly of the form $[\emptyset, T]$, where T is a corresponding cyclic difference set of $\mathbb{Z}_n$. For example, we may let $T = \{0, 1, 3\}$ in the case of the Heawood graph, the incidence graph $B(PG(3, 2))$ of the smallest projective space, and $T = \{1, 3, 4, 5, 9\}$ in the case of the incidence graph $B(H_{11})$.

4. Graph coverings

In this section we give a more formal definition of various concepts pertaining to graph covers. Let X be a connected graph, and let $\mathscr{P}$ be a partition of $V(X)$ into independent sets of equal size $k \geqslant 2$. The *quotient graph* $X_{\mathscr{P}}$ is the graph with vertex set $\mathscr{P}$, where two vertices P_1 and P_2 of $X_{\mathscr{P}}$ are adjacent if and only if there is at least one edge between a vertex of P_1 and a vertex of P_2 in X. We say that X is a *k-fold cover* of $X_{\mathscr{P}}$ if it has the property that P_1P_2 is an edge if and only if the subgraph $X[P_1, P_2]$ of X is isomorphic to kK_2. (Note that in view of our assumptions all covers are non-trivial and connected.) In this case the quotient graph $X_{\mathcal{P}}$ is called the *base graph* of X and the sets P_i are called the *fibres* of X. The subgroup K of all those automorphisms of X which fix each of the fibres setwise is called the *group of covering transformations*. Note that since X is connected, the action of this group on the fibres of X is necessarily semiregular. In particular, if this action is regular we say that X is a *regular K-cover* or a *normal K-cover* (in short a *regular cover* or a *normal cover*) of $X_{\mathcal{P}}$. (Note that 2-fold covers are necessarily regular covers.) In this case we can reconstruct the graph X from the base graph $X_{\mathcal{P}}$ via a *voltage assignment* $\zeta : A(X_{\mathcal{P}}) \to K$, that is, a function from the set of arcs of $X_{\mathcal{P}}$ into the group K where reverse arcs carry inverse voltages. By ζ_{P_1,P_2} we denote the voltage assigned to the arc $(P_1, P_2) \in A(X_{\mathcal{P}})$. We identify the vertex set of X with the set $\{(P, g)\colon P \in \mathcal{P}, g \in G\}$ and we have that (P_1, g_1) and (P_2, g_2) are adjacent if and only if P_1 and P_2 are adjacent in $X_{\mathcal{P}}$ and $g_1^{-1} g_2 = \zeta_{P_1,P_2}$. Hence a base graph with a voltage assignment gives rise to a graph cover in a natural way (see also [29]). We shall use the notation $X = \mathrm{Cov}(X_{\mathcal{P}}, \zeta)$. Let us stress the double role of the group K of covering transformations, reflected in its left regular action on the vertices of the fibres, and in the voltage multiplication done on the right.

The voltage assignment ζ naturally extends (by group multiplication) to walks in $X_{\mathcal{P}}$. In particular, for any walk W of $X_{\mathcal{P}}$ we let ζ_W denote the voltage of W. By connectedness of X, the voltages of all fundamental closed walks at any vertex $P \in \mathcal{P}$ generate the whole voltage group G. It is also well known [29] that a given voltage assignment can be modified so that the arcs of an arbitrarily prescribed spanning tree receive the trivial voltage, and that the modified assignment is associated with the same graph cover. Namely, the modified voltage of each cotree arc is precisely the voltage of the corresponding fundamental closed walk relative to a fixed chosen base vertex $P \in \mathcal{P}$. Moreover, the following proposition holds.

Proposition 4.1 (Škoviera [48]). *Leaving the voltages of a spanning tree trivial and replacing the voltage assignments on the cotree arcs by their images under an automorphism of the voltage group results in a voltage assignment associated with the same graph cover.*

Let $\alpha \in Aut\, X_{\mathcal{P}}$. If there exists a fibre preserving automorphism $\tilde{\alpha} \in Aut\, X$ which induces α (in a natural way), that is, which *projects* to α, we say that $\tilde{\alpha}$ is a *lift* of α. More generally, if all automorphisms from a group $G \leqslant Aut\, X_{\mathcal{P}}$ have a lift, then the collection of all such lifts constitutes a group $\tilde{G} \leqslant Aut\, X$, the so called *lift* of G. In particular, the group of covering transformations is the lift of the trivial subgroup of $Aut\, X_{\mathcal{P}}$.

The *canonical double cover* of a graph Y is the graph obtained by assigning the voltage $1 \in \mathbb{Z}_2$ to every arc of Y. Note that the canonical double cover of K_n is isomorphic to $K_{n,n} - nK_2$.

The next two propositions provide information about the relationship between automorphisms of graph covers and their base graphs. The first one is taken from

[36, Corollary 4.3]), whereas the second one may be deduced from [37, Corollaries 9.4, 9.7, 9.8].

Proposition 4.2. *Let Y be a graph and let $X = Cov(Y, \zeta)$ be a regular cover of Y with respect to the voltage assignment ζ. Then an automorphism α of Y lifts to an automorphism of X if and only if for each closed walk W in Y the voltage $\zeta_{\alpha W}$ is trivial precisely when ζ_W is trivial.*

Proposition 4.3. *Let X be a regular K-cover of Y. A group $G \leqslant Aut\, Y$, acting semiregularly on $V(Y)$, lifts as a direct product $G \times K$ if and only if there exists a voltage assignment $\zeta : A(Y) \to K$ such that for each $\alpha \in G$ and each walk W of Y we have that $\zeta_W = \zeta_{\alpha W}$.*

5. The graphs K_{q+1}^4 and Singer cycles

In [20, Theorem 1.1] a classification of all regular cyclic covers and regular $\mathbb{Z}_p^2$-covers, p a prime, of complete graphs, with a 2-arc-transitive group of covering transformations, was given. It was shown that, apart from the canonical double covers, the graphs $X_1(4,q)$ and $X_2(4,q)$, q an odd prime power, defined below are the only other such graphs.

For a prime power q, let $\mathbb{F}(q) = GF(q)$ be the Galois field of order q and let $\mathrm{S}\mathbb{F}(q)$ and $\mathrm{N}\mathbb{F}(q)$ be, respectively, the set of all squares and the set of all non-squares in $\mathbb{F}(q)^*$. Further, let Z be a graph isomorphic to K_{q+1} where the vertex set is identified with the projective line $PG(1,q) = \mathbb{F}(q) \cup \{\infty\}$.

Let $q \equiv 3 \pmod 4$. Note that $\mathrm{S}\mathbb{F}(q) \cap -\mathrm{S}\mathbb{F}(q) = \emptyset$, that is, $\mathrm{N}\mathbb{F}(q) = -\mathrm{S}\mathbb{F}(q)$. We define $X_1(4,q)$ to be the 4-fold cover $Cov(Z,\zeta)$, where the voltage assignment $\zeta : A(Z) \to \mathbb{Z}_4$ is given by the following rule:

$$\zeta(x,y) = \begin{cases} 0, & \infty \in \{x,y\}, \\ 1, & y - x \in \mathrm{S}\mathbb{F}(q), \\ 3, & y - x \in \mathrm{N}\mathbb{F}(q). \end{cases} \tag{3}$$

Suppose now that $q \equiv 1 \pmod 4$. Note that $\mathrm{S}\mathbb{F}(q) = -\mathrm{S}\mathbb{F}(q)$. We define $X_2(4,q)$ to be the 4-fold cover $Cov(Z,\zeta)$, where the voltage assignment $\zeta : A(Z) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ is given by the following rule:

$$\zeta(x,y) = \begin{cases} (0,0), & \infty \in \{x,y\}, \\ (1,0), & y - x \in \mathrm{S}\mathbb{F}(q), \\ (0,1), & y - x \in \mathrm{N}\mathbb{F}(q). \end{cases} \tag{4}$$

It was proved in [20, Proposition 3.2] that $X_i(4,q)$, $i = 1,2$, are Cayley graphs. However, the fact that the corresponding groups may be taken to be dihedral was overlooked. There is a more concise way of describing these graphs as covers of

$Y \cong K_{q+1,q+1} - (q+1)K_2$. For an odd prime power q, let V and V' denote two copies of the projective line $PG(1,q)$. Identify $V(Y)$ with $V \cup V'$ and let the edge set consist of all the edges of the form $[u, v']$, $u \neq v$. Then we let K^4_{q+1} be the regular $\mathbb{Z}_2$-cover $Cov(Y, \zeta)$ where the voltage assignment $\zeta : A(Y) \to \mathbb{Z}_2$ is given by the following rule:

$$\zeta(x, y') = \begin{cases} 0, & \infty \in \{x, y\}, \\ 0, & y - x \in \mathrm{S}\mathbb{F}(q), \\ 1, & y - x \in \mathrm{N}\mathbb{F}(q). \end{cases} \tag{5}$$

It may be easily seen that K^4_{q+1} is isomorphic to $X_1(4,q)$ when $q \equiv 3 \pmod 4$ and to $X_2(4,q)$ when $q \equiv 1 \pmod 4$. For example, the corresponding isomorphism $\varphi : X_2(4,q) \to K^4_{q+1}$ in the case $q \equiv 1 \pmod 4$ maps according to the following rules: (The case $q \equiv 3 \pmod 4$ is done in a similar manner.)

$$\varphi(\infty, (i,j)) = \begin{cases} (\infty', 0), & (i,j) = (0,0), \\ (\infty, 0), & (i,j) = (1,0), \\ (\infty, 1), & (i,j) = (0,1), \\ (\infty', 1), & (i,j) = (1,1), \end{cases}$$

and, for each $x \in \mathbb{F}(q)$,

$$\varphi(x, (i,j)) = \begin{cases} (x, 0), & (i,j) = (0,0), \\ (x', 0), & (i,j) = (1,0), \\ (x', 1), & (i,j) = (0,1), \\ (x, 1), & (i,j) = (1,1). \end{cases}$$

We now turn to the main aim of this section and give the proof that the graphs K^4_{q+1} are indeed Cayley graphs of dihedral groups. Proposition 4.3 together with the two lemmas below will play a vital role in that respect.

Lemma 5.1. *Let $Y \cong K_{q+1,q+1} - (q+1)K_2$ with vertex set $V(Y) = V \cup V'$ consisting of two disjoint copies of $PG(1,q)$. Let $u, v, w \in V$ be distinct vertices with u', v', w' as their counterparts in V' and let $C = uv'wu'vw'u$ be the corresponding induced 6-cycle in Y. Let ζ be the voltage assignment on Y giving rise to $K^4_{q+1} = Cov(Y, \zeta)$. Then*

$$\zeta_C = \begin{cases} 0, & q \equiv 1 \pmod 4, \\ 1, & q \equiv 3 \pmod 4. \end{cases}$$

Proof. Observe first that, viewing Y as a canonical double cover of $Z \cong K_{q+1}$, and identifying $V(Z)$ with V, the 6-cycle C arises from the triangle $T = uvwu$ in Z. Letting ζ' be the voltage assignment on Z giving rise to $K^4_{q+1} = Cov(Z, \zeta')$, we may see that the statement of Lemma 5.1 is equivalent to ζ'_T being an element of order 2 or 4 depending on whether $q \equiv 1 \pmod 4$ or $q \equiv 3 \pmod 4$, which is clearly the case in view of (3) and (4). $\square$

The following generalisations of [1, Lemma 2.2], proved in [39, Lemma 2.1], will be quite consistently used throughout the rest of this paper.

Lemma 5.2. *Let A be an abelian group and J be the set of involutions in A. Let $R \subseteq A^{\#} = A \setminus \{0\}$, let $a \in A^{\#}$ and set $\lambda_a = |R \cap (-R + a)|$. Then*

(i) *$a \notin 2R$ implies that λ_a is even;*
(ii) *$a = 2r$ for some $r \in R$ and $|R \cap (J + r)|$ is odd imply that λ_a is even;*
(iii) *$a = 2r$ for some $r \in R$ and $|R \cap (J + r)|$ is even imply that λ_a is odd.*

For an even integer n, we let $\mathbb{E}_n$ and $\mathbb{O}_n$ denote the set $2\mathbb{Z}_n$ of all even and the set $1 + 2\mathbb{Z}_n$ of all odd elements in $\mathbb{Z}_n$, respectively. We are now ready to show that the graphs K_{q+1}^4 are dihedrants.

Theorem 5.3. *Let $q \geqslant 3$ be an odd prime power. Then K_{q+1}^4 is a 2-arc-transitive Cayley graph of the dihedral group $D_{4(q+1)}$ having a symbol of the form $[\emptyset, T]$ for some $T \subseteq \mathbb{Z}_{q+1}$. Moreover, if K is the group of covering transformations (of K_{q+1}^4 viewed as a 4-fold regular cover of $Z \cong K_{q+1}$), then K is normal in $Aut\, K_{q+1}^4$ and $Aut\, K_{q+1}^4 / K \cong P\Gamma L(2, q)$.*

Proof. In view of the isomorphisms $K_{q+1}^4 \cong X_i(4, q)$, $i = 1, 2$, the 2-arc-transitivity of the graphs K_{q+1}^4 follows from [20, Theorem 1.1]. It remains to be shown that these graphs are indeed dihedrants, a fact that went unnoticed in [20], and that the automorphism group $A = Aut\, K_{q+1}^4$ is of the desired form.

Viewing the graphs K_{q+1}^4 as 4-fold covers of Z, we note that the whole group $P\Gamma L(2, q)$ (as a subgroup of $Aut\, Z \cong S_{q+1}$) has a lift [20, p. 288]. But then $P\Gamma L(2, q)$ is the largest group that lifts, for otherwise a 4-transitive group (in $Aut\, Z$) would have a lift, forcing all quadrangles in the quotient Z to have equal voltages, which is clearly not the case. Therefore, to see that A is indeed of the desired form, we just need to show that the fibres are blocks of imprimitivity for the full automorphism group A. To this end, observe that the distance sequence of K_{q+1}^4 is $(1, q, 2q, q + 2, 1)$. Moreover, a fibre containing a given vertex v contains also the vertex w at distance 4 from v and the two vertices in the set $N^3(v) \setminus N(w)$. Since an automorphism of K_{q+1}^4 fixing v necessarily fixes w, and hence it must also fix the set $N^3(v) \setminus N(w)$, the result follows.

We now show that the graphs K_{q+1}^4 are dihedrants. Recalling the definition of K_{q+1}^4 as a regular $\mathbb{Z}_2$-cover of $Y \cong K_{q+1,q+1} - (q + 1)K_2$ in (5), let $V(Y) = V \cup V'$, where V and V' are two copies of $PG(1, q)$. We may assume that V and V' are the two orbits of an isomorphic copy of $PGL(2,q)$ in the automorphism group of the "intermediate" cover Y.

By way of contradiction, assume that $X = K^4_{q+1}$ is not a Cayley graph of $D = D_{4(q+1)}$. Consider a Singer cycle σ of order $q+1$ in $PGL(2,q)$, with V and V' as its two orbits. Since the index 2 subgroup of the lifted group, fixing the two bipartition sets of X, contains a lift of $PGL(2,q)$ and thus a lift of a simple group $PSL(2,q)$, it must act faithfully on each part of the bipartition. Hence a lift $\tilde{\sigma}$ of σ has either two orbits of length $2(q+1)$ or four orbits of length $q+1$. But a graph having an automorphism with two orbits of equal lengths which are independent sets of vertices is clearly a dihedrant. Therefore the former forces X to be a Cayley graph of $D_{4(q+1)}$, and so we may assume that the latter occurs. Hence $\langle \sigma \rangle$ lifts as a direct product, that is, $\langle \tilde{\sigma} \rangle \cong \mathbb{Z}_{q+1} \times \mathbb{Z}_2$. In view of Proposition 4.3 there exists a relabeling of the vertex set in such a way that $V(Y) = \{v_i\colon i \in \mathbb{Z}_{q+1}\} \cup \{v'_i\colon i \in \mathbb{Z}_{q+1}\}$, and there are voltages $\zeta_i \in \mathbb{Z}_2$, for $i \in \mathbb{Z}^{\#}_{q+1}$, such that $K^4_{q+1} \cong Cov(Y, \zeta)$, where $\zeta_{v_i,v'_j} = \zeta_{j-i}$ for all distinct $i,j \in \mathbb{Z}^{\#}_{q+1}$.

Let $i \in \mathbb{Z}^{\#}_{q+1} \setminus \{(q+1)/2\}$, let $j \in \mathbb{Z}^{\#}_{q+1} \setminus \{i\}$ and consider the 6-cycle $C_{i,j} = v_0 v'_i v_j v'_0 v_i v'_j v_0$ in Y. Its voltage $\zeta_{C_{i,j}}$ is $\zeta_i - \zeta_{i-j} + \zeta_{-j} - \zeta_{-i} + \zeta_{j-i} - \zeta_j$ which is, of course, equal to $\zeta_i + \zeta_{i-j} + \zeta_{-j} + \zeta_{-i} + \zeta_{j-i} + \zeta_j$. Setting $j = 2i$ we have that $\zeta_{C_{i,2i}} = \zeta_{2i} + \zeta_{-2i}$. Combining this with Lemma 5.1 we get that, for each $i \in \mathbb{Z}^{\#}_{q+1} \setminus \{(q+1)/2\}$,

$$\zeta_{2i} + \zeta_{-2i} = \begin{cases} 0, & q \equiv 1 \pmod 4, \\ 1, & q \equiv 3 \pmod 4. \end{cases} \tag{6}$$

The case $q \equiv 3 \pmod 4$ is now immediate. Note that $(q+1)/2$ is even, and so letting $i = (q+1)/4$, we get $0 = \zeta_{(q+1)/2} + \zeta_{(q+1)/2} = \zeta_{(q+1)/2} + \zeta_{-(q+1)/2} = 1$, a contradiction.

We may therefore assume that $q \equiv 1 \pmod 4$. Let both i,j be odd. Then, in view of (6), we have $0 = \zeta_{C_{i,j}} = \zeta_i + \zeta_{-i} + \zeta_j + \zeta_{-j}$. Clearly, $(q+1)/2$ is odd, and also $\zeta_{(q+1)/2} = \zeta_{-(q+1)/2}$. Therefore letting $j = (q+1)/2$, we get that, for each $i \in \mathbb{Z}^{\#}_{q+1}$,

$$\zeta_i + \zeta_{-i} = 0. \tag{7}$$

Define now the set $M = \{i \in \mathbb{Z}^{\#}_{q+1}\colon \zeta_i = 0\}$ and let $M^{\mathrm{c}} = \mathbb{Z}^{\#}_{q+1} \setminus M$ be its complement in $\mathbb{Z}^{\#}_{q+1}$. Since X is 2-arc-transitive, any two vertices at distance 2 have the same number of common neighbours, say λ. Let us choose these two vertices from the fibres $\{(v_0,0),(v_0,1)\}$ and $\{(v_i,0),(v_i,1)\}$, respectively. Computing $|N((v_0,0),(v_i,0))|$ we get that, for each $i \in \mathbb{Z}^{\#}_{q+1}$,

$$|M \cap (M+i)| + |M^{\mathrm{c}} \cap (M^{\mathrm{c}}+i)| = \lambda. \tag{8}$$

Similarly, computing $|N((v_0,0),(v_i,1))|$ we have, for each $i \in \mathbb{Z}^{\#}_{q+1}$,

$$|M \cap (M^{\mathrm{c}}+i)| + |M^{\mathrm{c}} \cap (M+i)| = \lambda,$$

which, since M and M^{c} are symmetric by (7), gives us, for each $i\in\mathbb{Z}_{q+1}^{\#}$,

$$2|M\cap(M^{\mathrm{c}}+i)|=\lambda. \tag{9}$$

Taking the sum over all $i\in\mathbb{Z}_{q+1}^{\#}$ in (8) and (9), we obtain $|M|^2+|M^{\mathrm{c}}|^2=(\lambda+1)q$ and $2|M||M^{\mathrm{c}}|=\lambda q$. Consequently, $q^2=(|M|+|M^{\mathrm{c}}|)^2=(2\lambda+1)q$, and so

$$\lambda=(q-1)/2. \tag{10}$$

Moreover, $(|M|-|M^{\mathrm{c}}|)^2=q$. Let $x=|M|$. Then by computation, $x^2-qx+q(q-1)/4=0$, giving us

$$\{|M|,|M^{\mathrm{c}}|\}=\{(q+\sqrt{q})/2,(q-\sqrt{q})/2\}. \tag{11}$$

Now observe that

$$|M\cap(M+i)|+|M\cap(M^{\mathrm{c}}+i)|=|M\cap\mathbb{Z}_{q+1}^{\#}+i|=\begin{cases}|M|-1, & i\in M,\\ |M|, & i\notin M.\end{cases}$$

Therefore

$$|M\cap(M+i)|=\begin{cases}|M|-\lambda/2-1, & i\in M,\\ |M|-\lambda/2, & i\in M^{\mathrm{c}}.\end{cases} \tag{12}$$

Let $M_1=\{m\in M\colon m+(q+1)/2\in M\}$. Then $M_1=M_1+(q+1)/2$ and $M_2\cap(M_2+(q+1)/2)=\emptyset$ where $M_2=M\setminus M_1$. Similarly, let $(M^{\mathrm{c}})_1=\{m\in M^{\mathrm{c}}: m+(q+1)/2\in M^{\mathrm{c}}\}$ and $(M^{\mathrm{c}})_2=M^{\mathrm{c}}\setminus(M^{\mathrm{c}})_1$. Since $(q+1)/2$ is the only involution in $\mathbb{Z}_{q+1}$, Lemma 5.2 implies that $|M\cap(M+i)|$, $i\in\mathbb{Z}_{q+1}^{\#}$, is even if $i\in 2M_1\cup(2M)^{\mathrm{c}}$ and is odd if $i\in 2M_2$. Combining this fact with (12), we see that either $M\subseteq\mathbb{E}_{q+1}^{\#}$ or $M^{\mathrm{c}}\subseteq\mathbb{E}_{q+1}^{\#}$.

Assume first that $M\subseteq\mathbb{E}_{q+1}^{\#}$. In particular, as $(q+1)/2$ is odd, $M_2=M$ and $M_1=\emptyset$, and so $M\cap(M+(q+1)/2)=\emptyset$. Hence $(M^{\mathrm{c}})_1=(\mathbb{E}_{q+1}^{\#}\setminus M)\cup((\mathbb{E}_{q+1}^{\#}\setminus M)+(q+1)/2)$. Letting $j=(q+1)/2$ in (8), we get by (10),

$$\begin{aligned}(q-1)/2&=\lambda=|M\cap(M+(q+1)/2)|+|M^{\mathrm{c}}\cap(M^{\mathrm{c}}+(q+1)/2)|\\&=|M^{\mathrm{c}}\cap(M^{\mathrm{c}}+(q+1)/2)|=|(M^{\mathrm{c}})_1|=2|\mathbb{E}_{q+1}^{\#}\setminus M|\\&=2((q-1)/2-x).\end{aligned}$$

It follows that $|M|=x=(q-1)/4$ and so $|M^{\mathrm{c}}|=(3q+1)/4$. Comparing this with (11) we have $(q-1)/4=(q-\sqrt{q})/2$ and so $(\sqrt{q}-1)^2=0$, forcing $q=1$, a contradiction.

If we assume that $M^{\mathrm{c}}\subseteq\mathbb{E}_{q+1}^{\#}$, then the same argument (with the roles of M and M^{c} interchanged) gives us $|M^{\mathrm{c}}|=(q-1)/4$ and $|M|=(3q+1)/4$, which again leads to a contradiction. This shows that $X=K_{q+1}^4$ is a dihedrant.

Finally, the fact that the symbol is really of the form claimed in the statement of this theorem is now immediate. This completes the proof of Theorem 5.3. $\square$

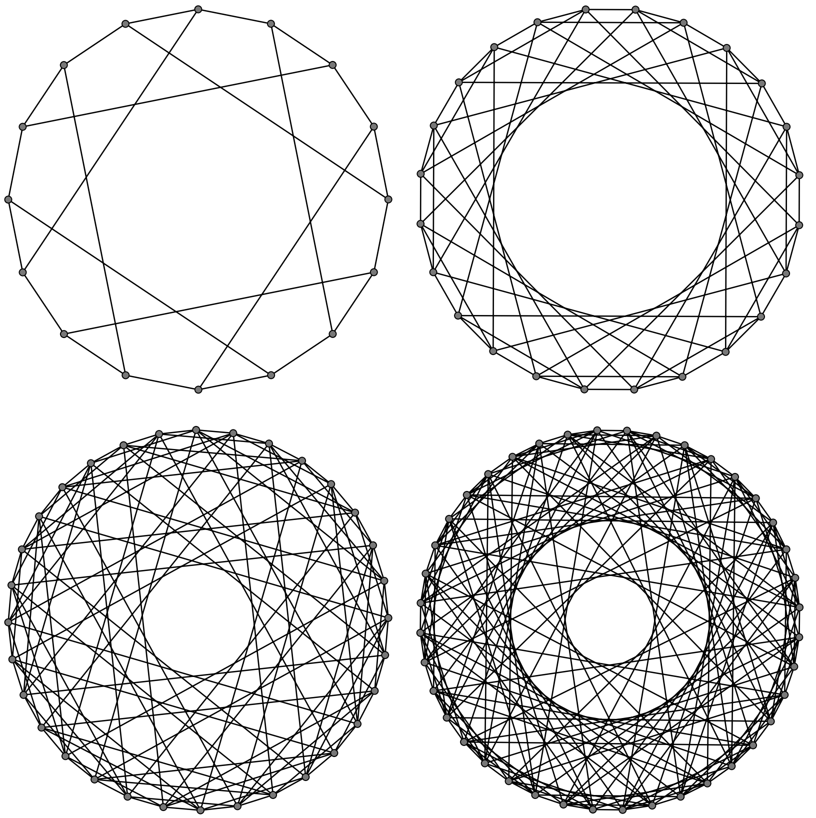

Fig. 1. The 2-arc-transitive dihedrants K_{q+1}^4, $q = 3, 5, 7, 9$.

Although there seems to be no obvious rule for constructing the sets T in the symbols $[\emptyset, T]$ of the graphs K_{q+1}^4, an additional property of these sets will be proved in Section 7. In Corollary 7.6 we shall see that T decomposes into two subsets T_1 and T_2, such that $T_1 = -T_1$ and $T_2 = -T_2 + (q+1)/2$. The smallest four graphs K_{q+1}^4, $q = 3, 5, 7, 9$, of respective orders 16, 24, 32 and 40, are shown in Fig. 1. Note that K_4^4 is the incidence graph of the Moebius-Kantor configuration (the generalized Petersen graph $GP(8,3)$). The fact that this graph is a Cayley graph of the dihedral group D_{16} has been overlooked in the Foster census [5].

Theorem 5.3 has an interesting consequence regarding the structure of Singer cycles in the group $PGL(2,q)$. Recall that $PGL(2,q)$ is sharply 3-transitive and therefore, for any $x \in \mathbb{F}(q)^*$, there is precisely one Singer cycle of the form $\sigma_x = (0\,\infty\,x\ldots)$. We now prove the following fact about this cycle.

Corollary 5.4. *Let $x \in \mathbb{F}(q)^*$ and let y_x be the image of x under the Singer cycle $\sigma_x = (0\,\infty\,x\ldots)$ in $PGL(2,q)$. Then*

$$x(y_x - x) \in \begin{cases} \mathrm{N}\mathbb{F}(q), & q \equiv 1 \pmod 4, \\ \mathrm{S}\mathbb{F}(q), & q \equiv 3 \pmod 4. \end{cases} \tag{13}$$

Proof. The proof is done indirectly by tying (13) to the statement of Theorem 5.3. Note that the vertex set of K_{q+1}^4 is $(V \cup V') \times \mathbb{Z}_2$. Without loss of generality, we may assume that there is a lift $\tilde{\sigma}_x$ of σ_x which maps $(\infty, 0)$ to $(x, 0)$. Let $y = y_x$. Let us first show that

$$\tilde{\sigma}_x(0,0) = \begin{cases} (\infty, 1), & x(y-x) \in \mathrm{N}\mathbb{F}(q), \\ (\infty, 0), & x(y-x) \in \mathrm{S}\mathbb{F}(q). \end{cases} \tag{14}$$

Clearly, $\tilde{\sigma}$ takes the neighbours of $(\infty, 0)$ to the neighbours of $(x, 0)$. But $N((\infty,0)) = \{(0',0)\} \cup \mathrm{S}(\mathbb{F}(q)' \times \{0\}) \cup \mathrm{N}(\mathbb{F}(q)' \times \{1\})$, and $N((x,0)) = \{(\infty',0)\} \cup ((\mathrm{S}\mathbb{F}(q)+x)' \times \{0\}) \cup ((\mathrm{N}\mathbb{F}(q)+x)' \times \{1\})$. In particular, we have that

$$\tilde{\sigma}_x(0',0) = (\infty',0), \tag{15}$$

and moreover $\tilde{\sigma}_x$ takes $(x',0)$ to $(y',0)$ when $y - x \in \mathrm{S}\mathbb{F}(q)$ and to $(y',1)$ when $y - x \in \mathrm{N}\mathbb{F}(q)$. Assume that $y - x \in \mathrm{S}\mathbb{F}(q)$. As above, considering the neighbours of $(x',0)$ and $(y',0)$ we may deduce that $\tilde{\sigma}$ maps $(0,0)$ to $(\infty,0)$ for $x \in \mathrm{S}\mathbb{F}(q)$, and to $(\infty,1)$ for $x \in \mathrm{N}\mathbb{F}(q)$. This proves (14) in the case $y - x \in \mathrm{S}\mathbb{F}(q)$. The case $y - x \in \mathrm{N}\mathbb{F}(q)$ is done in an analogous way.

Now, by (15), it follows that $\tilde{\sigma}_x$ maps $N((0',0))$ to $N((\infty',0))$ and so, in particular, $(z,0)$ is mapped to $(\sigma z, 0)$ if $z \in -\mathrm{S}\mathbb{F}(q)$ and is mapped to $(\sigma z, 1)$ if $z \in -\mathrm{N}\mathbb{F}(q)$. It follows that, for $q \equiv 1 \pmod 4$, the vertex $(z,0)$ is mapped to $(\sigma z, 0)$ if $z \in \mathrm{S}\mathbb{F}(q)$ and to $(\sigma z, 1)$ if $z \in \mathrm{N}\mathbb{F}(q)$. Similarly, for $q \equiv 3 \pmod 4$, the vertex $(z,0)$ is mapped to $(\sigma z, 0)$ if $z \in \mathrm{N}\mathbb{F}(q)$ and to $(\sigma z, 1)$ if $z \in \mathrm{S}\mathbb{F}(q)$. In other words, the number of "switches" between layers 0 and 1, one makes with a continual application of $\tilde{\sigma}_x$ until the fibre $\{(0,0),(0,1)\}$ is reached, equals the cardinality of $\mathrm{N}\mathbb{F}(q)$ if $q \equiv 1 \pmod 4$, and the cardinality of $\mathrm{S}\mathbb{F}(q)$ if $q \equiv 3 \pmod 4$. In short, the number of switches is $(q-1)/2$, which is even for $q \equiv 1 \pmod 4$ and odd for $q \equiv 3 \pmod 4$. Therefore,

$$(\tilde{\sigma}_x)^q(\infty,0) = \begin{cases} (0,0), & q \equiv 1 \pmod 4, \\ (0,1), & q \equiv 3 \pmod 4. \end{cases} \tag{16}$$

By Theorem 5.3, it follows that σ_x lifts as an automorphism with two orbits of length $2(q+1)$ and so, in particular, $(\tilde{\sigma}_x)^{q+1} \neq 1$. Thus $(\tilde{\sigma}_x)^{q+1}(\infty,0) = (\infty,1)$. Combining this with (14) and (16) we get (13), as required. $\square$

6. Abelian groups

In this section we explore the properties of 2-arc-transitive Cayley graphs of abelian groups having at most three involutions. We obtain a partial classification which will be needed in the subsequent sections in our analysis of 2-arc-transitive properties of dihedrants.

The first result is taken from [1].

Proposition 6.1. *Let X be a connected 2-arc-transitive graph and let $v \in V(X)$. Then the bipartite graph $X[N(v), N^2(v)]$ is biregular.*

Proof. Let $v \in V(X)$. Since X is 2-arc-transitive, it follows that both $N(v)$ and $N^2(v)$ are orbits of the vertex stabilizer $Aut\,(X)_v$. The result follows. □

The next proposition is a special case of a result due to Cameron regarding 2-arc-transitive graphs with a "large" number of quadrangles. Namely, it follows from [8, Theorems 4.2 and 4.3] that a 2-arc-transitive graph of valency d satisfying $|N(u,v)| > d/2$ (for any two vertices u, v distance 2 apart) is necessarily the incidence graph of a (possibly degenerate) symmetric design. The degenerate case gives us the following result.

Proposition 6.2. *Let X be a 2-arc-transitive graph of valency $d \geqslant 3$ which is not a complete graph, and let $v \in V(X)$. If there exists a vertex $u \in N^2(v)$ such that $|N(u,v)| = d$ then $X \cong K_{d,d}$, and if there exists a vertex $u \in N^2(v)$ such that $|N(u,v)| = d-1$ then $X \cong K_{d+1,d+1} - (d+1)K_2$.*

We now turn to Cayley graphs of abelian groups. Recall that for an abelian group A, the set $A \setminus \{0\}$ of its non-trivial elements is denoted by $A^{\#}$.

Proposition 6.3. *Let A be an abelian group and let $R = -R$ be a subset of $A^{\#}$ such that the Cayley graph $X = Cay(A, R)$ is 2-arc-transitive. If X is neither a cycle nor a complete graph, then its girth is 4.*

Proof. As X is not a cycle, we can choose $r_1, r_2 \in R$ such that $r_2 \neq r_1, -r_1$ and therefore $(0, r_1, r_1 + r_2, r_2, 0)$ is a 4-cycle in X. If X contains a triangle then, by 2-arc-transitivity, all 2-arcs of X are contained in a triangle, implying that X is a complete graph, a contradiction. □

Note that in a 2-arc-transitive graph X the number of common neighbours of two vertices at distance 2 is constant. It will be denoted by $\lambda(X)$ or just λ when the graph X is clear from the context. In particular, let $X = Cay(A, R)$ be a 2-arc-transitive Cayley graph of an abelian group A with $\lambda \geqslant 2$, that is, of girth 4. Then letting $x \in (R+R)^{\#}$, it follows that $x \in N^2(0)$. Of course, by the above comment, $|N(0,x)| =$

$\lambda(X) = \lambda$. But $N(0,x) = R \cap (R+x)$ and so

$$|R \cap (R+x)| = \lambda \quad \text{for each } x \in (R+R)^{\#}. \tag{17}$$

If λ is odd we have the following general result.

Proposition 6.4. *Let A be an abelian group and let $R = -R$ be a subset of $A^{\#}$ such that $X = Cay(A,R)$ is a 2-arc-transitive Cayley graph of valency $d \geqslant 3$ which is not a complete graph. If $\lambda(X)$ is odd, then X is isomorphic either to $K_{d,d}$ or to $K_{d+1,d+1} - (d+1)K_2$.*

Proof. Since X is not a complete graph, Proposition 6.3 implies that the girth of X is 4 and so $(R+R)^{\#} \subseteq N^2(0)$. Therefore, by (17), it follows that $\lambda = |R \cap (R+t)|$ is odd for any $t \in (R+R)^{\#}$. So, by Lemma 5.2, we have that $N^2(0) = (R+R)^{\#}$ coincides with $(2R)^{\#}$. Therefore, if $0 \in 2R$ then $N^2(0)$ has cardinality $d-1$. Thus, by Proposition 6.1, it follows that each vertex in $N^2(0)$ is adjacent to d vertices in $N(0)$ and so, by Proposition 6.2, we have $X \cong K_{d,d}$. If $0 \notin 2R$ then $|N^2(0)| = d$, and Proposition 6.1 implies that each vertex in $N^2(0)$ is adjacent to $d-1$ vertices in $N(0)$. Hence, by Proposition 6.2, we have $X \cong K_{d+1,d+1} - (d+1)K_2$. $\square$

Proposition 6.5. *Let A be an abelian group, J be the set of involutions in A, and let $R = -R$ be a subset of $A^{\#}$ of cardinality $d \geqslant 3$ such that the Cayley graph $X = Cay(A,R)$ is 2-arc-transitive and not isomorphic to a complete graph. Then either $|J \cap N^2(0)| \geqslant 2$ or X is isomorphic either to $K_{d,d}$ or to $K_{d+1,d+1} - (d+1)K_2$.*

Proof. By Proposition 6.3, the girth of X is 4 and consequently $R \cap (R+R) = \emptyset$ and so $N^2(0) = (R+R)^{\#}$. The proposition clearly holds if $R \setminus J = \emptyset$. Also, by Proposition 6.4, we may assume that $\lambda(X)$ is even. Therefore, in view of Lemma 5.2, we have that $|R \cap (J+r)|$ is odd for each $r \in R \setminus J$, and consequently $J \cap N^2(0) = J \cap (R+R) \neq \emptyset$. Assume that τ is the only element in $J \cap N^2(0)$. By Lemma 5.2, each element in $R \setminus J$ is adjacent to τ, that is, $N(0,\tau)$ contains $R \setminus J$. Also, for any two distinct $\tau_1, \tau_2 \in J \cap R$, we have $\tau_1 + \tau_2 = \tau \in J \cap N^2(0)$. This implies that $|J \cap R| \leqslant 2$. Now if $J \cap R = \emptyset$, then $\lambda(X) = |N(0,\tau)| = d$, and $X \cong K_{d,d}$ by Proposition 6.2. If $|J \cap R| = 1$, then $\lambda = |N(0,\tau)| \geqslant d-1$ and, by Proposition 6.2, X is isomorphic either to $K_{d,d}$ or to $K_{d+1,d+1} - (d+1)K_2$. Finally, if $J \cap R = \{\tau_1, \tau_2\}$, then $\tau = \tau_1 + \tau_2$ and so $N(\tau) = R$ and, by Proposition 6.2, $X \cong K_{d,d}$. Hence proof. $\square$

Proposition 6.6. *Let A be an abelian group with at most one involution, let $R = -R$ be a subset of $A^{\#}$ and let the Cayley graph $X = Cay(A,R)$ be 2-arc-transitive. Then X is either a cycle, a complete graph, a complete bipartite graph or a complete bipartite graph with a 1-factor removed.*

Proof. Assume that X is neither a cycle nor a complete graph. Then its girth is 4 by Proposition 6.3. By Proposition 6.4 we can assume that $\lambda = \lambda(X)$ is even. Hence Lemma 5.2 and (17) together imply that $|R \cap (J + r)|$ is odd for each $r \in R \setminus J \neq \emptyset$, where J is the set of involutions in A. In particular, this forces $J \neq \emptyset$. Let $J = \{\tau\}$. Since X has girth 4, we cannot have $\tau \in R$. It follows that $N(0, \tau) = R$, for given any $r \in R$ there exists $r' \in R$ such that $\tau + r = r'$. By Proposition 6.2, it follows that $X \cong K_{\lambda,\lambda}$. □

The next two propositions deal with the distribution of involutions in Cayley graphs of abelian groups, isomorphic either to the complete bipartite graphs or to the complete bipartite graphs with a 1-factor removed.

Proposition 6.7. *Let $X \cong K_{n,n}$ be a Cayley graph of an abelian group A. Let $H \cong \mathbb{Z}_2^k$ be the largest elementary abelian 2-group contained in A, and $J = H^{\#}$ the set of involutions in A. Then one of the following occurs*:

(i) $J \subseteq N^2(0)$; *or*
(ii) *there exists a basis B of H such that $J \cap N(0) = \bigcup_{i \in \mathbb{Z}} (2i+1)B$ and $J \cap N^2(0) = \bigcup_{i \in \mathbb{Z}} 2iB$.*

Proof. Suppose that $J_1 = J \cap N(0) \neq \emptyset$ and let B be a maximal linearly independent subset of J_1. We claim that B is a basis of H. Assuming this is not the case, let B' be a basis of H containing B, let $\beta \in B$ and let $\beta' \in B' \setminus B$. Since B is a maximal linearly independent subset of J_1, it follows that $\beta + \beta' \notin J_1$. So $\beta + \beta' \in N^2(0)$. But $X[N(0), N^2(0)] \cong K_{n,n-1}$ and so β is adjacent to $\beta + \beta'$. Consequently, $\beta' \in N(0)$, a contradiction, showing that B is indeed a basis of H. The rest of (ii) is now immediate. □

Proposition 6.8. *Let $X \cong K_{n,n} - nK_2$ be a Cayley graph of an abelian group A. Let $H \cong \mathbb{Z}_2^k$ be the largest elementary abelian 2-group contained in A, and $J = H^{\#}$ the set of involutions in A. Then the following statements hold*:

(i) $|J \subseteq N(0)| = 2^{d-1} - 1 = |J \subseteq N^2(0)|$; *and*
(ii) $|J \cap N^3(0)| = 1$.

Proof. Let v be the vertex of X at distance 3 from 0 and let α be the unique element of A which takes 0 to v. Since α preserves distance in X we must have $\alpha v = 0$ and hence $\alpha \in J$, proving (ii). Let $J_1 = J \cap N(0)$ and $J_2 = J \cap N^2(0)$. Clearly α interchanges J_1 and J_2 and hence J_1 and J_2 have equal cardinality, forcing (ii). □

Letting Q_k, $k \geqslant 2$, denote the k-dimensional hypercube, we let FQ_k denote the so called *folded cube* of dimension k, obtained from Q_{k+1} by identifying pairs of antipodal vertices.

Theorem 6.9. *Let A be an abelian group, J be the set of involutions in A such that $|J| \leqslant 3$, and let $R = -R$ be a subset of $A^{\#}$ such that the Cayley graph $X = Cay(A, R)$ is 2-arc-transitive. For each positive integer i let $J_i = J \cap N^i(0)$ be the subset of involutions at distance i from 0. Then one of the following holds*:

(i) *X is either a cycle or a complete graph*; *or*
(ii) *X is a complete bipartite graph and either $J = J_2$ or $|J_1| = 2$ and $|J_2| = 1$*; *or*
(iii) *X is a complete bipartite graph with a 1-factor removed and $|J_i| = 1$ for each $i \in \{1, 2, 3\}$*; *or*
(iv) *$|J_1| = 1$, $A \cong \mathbb{Z}_4^2$, $R = \{(1,0), (3,0), (0,1), (0,3), (2,2)\}$ and $X \cong FQ_5$*; *or*
(v) *$J_1 = \emptyset$, $|J_2| = 2$ and X is a 2-fold cover of a complete bipartite graph*; *or*
(vi) *$|J_2| = 3$.*

Proof. Let us assume that X is neither a cycle nor a complete graph. If X is a complete bipartite graph, then (ii) follows by Proposition 6.7, and if X is a complete bipartite graph with a 1-factor removed, then (iii) follows by Proposition 6.8. We may therefore assume the X is neither of the above graphs.

By Proposition 6.4, we may assume that $\lambda = \lambda(X)$ is even and thus $\lambda \geqslant 2$, and by Proposition 6.6 we may assume that $|J| = 3$. Let $J = \{\tau_1, \tau_2, \tau_3\}$. Also, by Proposition 6.5, we have $|J_2| \geqslant 2$. We distinguish two different cases, depending on whether J_1 is the empty set or not.

Case 1: $J_1 \neq \emptyset$.

Without loss of generality $J_1 = \{\tau_3\}$. Since λ is even, Lemma 5.2 implies that every non-involutory element of R must have precisely one involution as a neighbour. If one of τ_1 or τ_2 is adjacent to all of $R \setminus \{\tau_3\}$, then by Proposition 6.2 we can deduce that the graph X is isomorphic to $K_{\lambda+2,\lambda+2} - (\lambda+2)K_2$. Therefore R decomposes into three (mutually disjoint) subsets: $\{\tau_3\}$, $R_1 = N(0, \tau_1)$ and $R_2 = N(0, \tau_2)$. Clearly $|R_1| = \lambda = |R_2|$.

We now show that $\lambda = 2$. Observe that $N(\tau_1) = R_1 \cup \{\tau_2\} \cup (R_2 + \tau_1)$. Therefore τ_1 and each $r_2 \in R_2$ have τ_2 as a common neighbour and so must have $\lambda - 1$ common neighbours in the set $R_2 + \tau_1$, which implies that $X[R_2, R_2 + \tau_1] \cong K_{\lambda,\lambda} - \lambda K_2$. By symmetry, $X[R_1, R_1 + \tau_2] \cong K_{\lambda,\lambda} - \lambda K_2$. Hence any two vertices in R_1 have precisely $\lambda - 2$ common neighbours in $R_1 + \tau_2$ and their only other common neighbours are 0 and τ_1. Similarly, any two vertices in R_2 have precisely $\lambda - 2$ common neighbours in $S_2 + \tau_1$ and their only other common neighbours are 0 and τ_2. Choose any $r_1 \in R_1$ and any two distinct elements $r_2, r_2' \in R_2$. All three vertices have 0 as a common neighbour. Besides, the remaining $\lambda - 1$ common neighbours of r_1 and r_2 or r_1 and r_2' are not in the sets $R_2 + \tau_1$ and $R_1 + \tau_2$ and hence they must be contained in the set of λ neighbours of r_1 other than 0, τ_1 and its neighbours in $R_2 + \tau_1$. But r_2 and r_2' cannot have a common neighbour in this set of λ vertices, because all of their common neighbours are in $(R_2 + \tau_1) \cup \{\tau_2, 0\}$. It follows that $\lambda = 2$ is the only possibility. Clearly, $R_1 = \{r_1, -r_1\}$ and $R_2 = \{r_2, -r_2\}$ and we obtain a unique graph isomorphic to FQ_5, which is a Cayley graph of $\mathbb{Z}_4^2$ (as well as of $\mathbb{Z}_2^4$) and is 2-arc-transitive but not 3-arc-transitive (with vertex stabilizer isomorphic to S_5). Also, the set of generators is as claimed.

Case 2: $J_1 = \emptyset$. Recall that $|J_2| \geqslant 2$. We assume that $J_2 = \{\tau_1, \tau_2\}$ and show that X is a 2-fold cover of a complete bipartite graph.

Since λ is even and since $J_1 = \emptyset$ and $|J_2| = 2$, it follows by Lemma 5.2 that every element of R must have precisely one involution as a neighbour. Hence R is a disjoint union of sets $R_1 = N(0, \tau_1)$ and $R_2 = N(0, \tau_2)$. Also $|R_1| = \lambda = |R_2|$. (Note that $R_i = -R_i$.) Observe also that $R_i + \tau_i = R_i$ for $i = 1, 2$. Define $R_1 + R_2 = \{r_1 + r_2\colon\, r_1 \in R_1, r_2 \in R_2\}$ and let $R_1 + R_1 = \{r_1 + r_1'\colon\, r_1, r_1' \in R_1\} \setminus ((R_1 + R_2) \cup \{0, \tau_1\})$ and let $R_2 + R_2 = \{r_2 + r_2'\colon\, r_2, r_2' \in R_2\} \setminus ((R_1 + R_2) \cup \{0, \tau_2\})$.

We will now show that X has diameter 4 and that $N^4(0) = \{\tau_3\}$. This will then imply that X is a 2-fold cover of a complete bipartite graph.

First, observe that $N(\tau_3)$ is a disjoint union of the sets $R_1 + \tau_3 = R_1 + \tau_2$ and $R_2 + \tau_3 = R_2 + \tau_1$. Therefore $N(\tau_1, \tau_3) = R_2 + \tau_1$ and $N(\tau_2, \tau_3) = R_1 + \tau_2$. Let us prove that $(R_1 + \tau_2) \cup (R_2 + \tau_1) \subseteq N^3(0)$. By symmetry we need only consider the set $R_1 + \tau_2$. If there was a vertex $v = r_1 + \tau_2$ in $(R_1 + \tau_2) \cap N^2(0)$, then we would have $|N(0, v)| = \lambda$. But since the girth of X is 4, we must have $N(0, v) = R_1$ which implies that, in particular, r_1 is adjacent to $r_1 + \tau_2$ and so $\tau_2 \in R$, a contradiction. Therefore τ_i, $i = 1, 2$, has no neighbours in $N^2(0)$ and hence by 2-arc-transitivity the subgraph induced by $N^2(0)$ is the null-graph.

Next we will see that $\tau_3 \in N^4(0)$. Suppose that $\tau_3 \in N^3(0)$. Then without loss of generality $\tau_1 + \tau_2 = \tau_3 = r_1 + r + r'$ where $r_1 \in R_1$ and $r, r' \in R$. But then we have $r_1' = \tau_1 - r_1 = r + r' + \tau_2$, implying that τ_2 has a neighbour in $N^2(0)$, a contradiction. We conclude that $\tau_3 \in N^4(0)$.

Our next step will be to show that $N(R_1 + R_2)$ is contained in the union $\mathscr{U}$ of the four sets R_1, R_2, $R_1 + \tau_2$ and $R_2 + \tau_1$. To that end note that a translation by τ_3 fixes the set $R_1 + R_2$ while interchanging the sets R_1 and $R_1 + \tau_2$ and the sets R_2 and $R_2 + \tau_1$. But each vertex in $R_1 + R_2$ has λ neighbours in R and thus, since its image under the translation by τ_3 is in $R_1 + R_2$, it also has λ neighbours in $R + \tau_3$. Hence all of its neighbours are in $\mathscr{U}$. Now we show that also $N(R_1 + R_1)$ is contained in $\mathscr{U}$. If a vertex $v = r_1 + r_1' \in R_1 + R_1$ has a neighbour in $N(\tau_3) = (R_1 + \tau_2) \cup (R_2 + \tau_1)$, then it must have λ neighbours in this set in view of the fact that $v \in N^2(\tau_3)$. But then $N(v)$ is contained in $\mathscr{U}$ as v already has λ neighbours in R. So assume that v has no neighbours in $N(\tau_3)$ and consider the set $v + R_2$. Now, for each $r_2 \in R_2$, $v + r_2 = r_1 + (r_1' + r_2)$ is in $\mathscr{U}$ but cannot be in R_1 as this would imply that $v = r_1^* - r_2$ for some $r_1^* \in R_1$, which is not possible for $v \in R_1 + R_1$. Hence $v + R_2 = R_2$ and as v has at least one neighbour in R_1 we have that $|N(0, v)| \geqslant \lambda + 1$, a contradiction. Similarly we can show that $N(R_2 + R_2)$ is contained in $\mathscr{U}$. It follows that $N^3(0) = (R_1 + \tau_2) \cup (R_2 + \tau_1)$.

Finally, we will show that $N(R_1 + \tau_2)$ (and equivalently $N(R_2 + \tau_1)$) is contained in $N^2(0) \cup \{\tau_3\}$. If not then there exists a vertex $v \in N^{j,2}(0, \tau_3)$ for $j \in \{3, 4\}$. But then $v + \tau_3 \in N^{2,j}(0, \tau_3)$, contradicting the fact that every vertex in $N^2(0)$ is also in $N^2(\tau_3)$. We conclude that $N^4(0) = \{\tau_3\}$ as well as that the diameter of X is 4.

As X is vertex-transitive, *Aut* X has blocks of imprimitivity $\mathscr{B} = \{\{v, v + \tau_3\colon\, v \in V(X)\}$. Clearly, if u is adjacent to v then $u + \tau_3$ is adjacent to $v + \tau_3$ and so by 2-arc-transitivity of X we must have that X is a 2-fold cover of a 2-arc-transitive graph, say Y. But $|V(X)| = 8\lambda$ with $|N(0)| = 2\lambda = |N^3(0)|$, $|N^2(0)| = 4\lambda - 2$ and $|N^4(0)| = 1$. Also, X is bipartite with one half of the bipartition consisting of all blocks of the form $\{r, r+\tau_3\}$, $r \in R$. Therefore Y is a bipartite graph with 4λ vertices and valency 2λ and so must be isomorphic to $K_{2\lambda,2\lambda}$. This completes the proof of Theorem 6.9. □

An investigation of 2-arc-transitivity properties of Cayley graphs of abelian groups, leading to a generalisation of the results of this section is continued in [40].

7. Blocks and symbols of dihedrants

Throughout the rest of this article we shall be assuming the notation of Section 3. Letting $S = -S$ be a symmetric subset of $\mathbb{Z}_n^{\#}$, $n \geqslant 3$, and T be a subset of $\mathbb{Z}_n$, we let $X = Dih(2n, S, T)$ be the dihedrant with symbol $[S, T]$, that is, the graph with vertex set $U \cup V$, where $U = \{u_i\colon\ i \in \mathbb{Z}_n\}$ and $V = \{v_i\colon\ i \in \mathbb{Z}_n\}$, and edges of the form $u_i u_{i+s}$, $v_i v_{i+s}$, for all $i \in \mathbb{Z}_n$ and $s \in S$, and $u_i v_{i+t}$, for all $i \in \mathbb{Z}_n$ and $t \in T$. Also, we let ρ and τ denote the generators of a regular dihedral group in $Aut\, Dih(2n, S, T)$—see (1) and (2). In particular, U and V are the two orbits of ρ. We emphasize that these automorphisms are assumed to play the same role for an arbitrary dihedrant of order $2n$, even if (for simplicity reasons) the sets S and T are not explicitly mentioned in the definition.

As already mentioned in the introductory section, by a classic result of Wielandt, every dihedral group is a B-group (see Proposition 8.3). The full automorphism group of a dihedrant is therefore necessarily imprimitive, unless the graph or its complement is a complete graph. In this section we explore the nature of the corresponding imprimitivity block systems.

Let X be a dihedrant of order $2n$, which is not a complete graph or its complement. We shall say that a block B of $Aut\, X$ is *cyclic* if there exists a vertex $w \in V(X)$ and $m \in \mathbb{Z}_n^{\#}$ such that B coincides with the orbit $\langle \rho^m \rangle w$. Further, we shall say that a block B of $Aut\, X$ is *dihedral* provided there exist two vertices u and v belonging to distinct orbits of ρ and $m \in \mathbb{Z}_n \setminus \mathbb{Z}_n^*$ such that B coincides with the union $\langle \rho^m \rangle u \cup \langle \rho^m \rangle v$. Our first lemma states that all blocks are of one of these two kinds.

Lemma 7.1. *Let $n \geqslant 3$, $X \neq K_{2n}$ be a dihedrant of order $2n$, and let $\mathcal{B}$ be an imprimitivity block system of $A = Aut\, X$, let ρ generate the cyclic subgroup of index 2 in a regular dihedral subgroup of A, and let K be the kernel of the action of A on $\mathcal{B}$. Then one of the following occurs*:

(i) *the blocks in $\mathcal{B}$ are all cyclic and there exists $m \in \mathbb{Z}_n^{\#}$ such that $K = \langle \rho^m \rangle$, or*
(ii) *the blocks in $\mathcal{B}$ are all dihedral and there exists $m \in \mathbb{Z}_n \setminus \mathbb{Z}_n^*$ such that $\langle \rho^m \rangle$ is of index 2 in K.*

Proof. The proof is straightforward. The two possibilities depend on whether a given block in $\mathcal{B}$ contains solely vertices from one orbit of $\langle \rho \rangle$, or it contains two vertices from different orbits of $\langle \rho \rangle$. Now, just take an arbitrary vertex $u \in B$ and let m be the smallest positive integer with the property that $\rho^m u \in B$. If $B = \langle \rho^m \rangle u$, then the blocks are all cyclic and they coincide with the orbits of $\langle \rho^m \rangle$. Hence $K = \langle \rho^m \rangle$. If $B \neq \langle \rho^m \rangle u$ then there exists $v \in B \setminus \langle \rho^m \rangle u$. In this case $B = \langle \rho^m \rangle u \cup \langle \rho^m \rangle v$ and the blocks are all dihedral. Moreover, $m \in \mathbb{Z}_n \setminus \mathbb{Z}_n^*$. Clearly, $\langle \rho^m \rangle$ is of index 2 in K. $\square$

Lemma 7.2. *Let $n \geqslant 3$, $X \neq K_{2n}$ be a 2-arc-transitive dihedrant of order $2n$, let $\mathcal{B}$ be an imprimitivity block system of $A = Aut\,X$, let $D = \langle \rho, \tau \rangle$ be a regular dihedral subgroup of A, and let K be the kernel of the action of A on $\mathcal{B}$. Then the quotient group A/K acts 2-arc-transitively on $X_{\mathcal{B}}$, and moreover, the following statements hold*:

(i) *if the blocks in $\mathcal{B}$ are cyclic then $X_{\mathcal{B}}$ is a dihedrant admitting a regular dihedral action of the group D/K;*
(ii) *if the blocks in $\mathcal{B}$ are dihedral then $X_{\mathcal{B}}$ is a circulant admitting a regular cyclic action of the group $\langle \rho \rangle/(K \cap \langle \rho \rangle)$.*

Proof. The fact that $\bar{A} = A/K$ acts 2-arc-transitively is immediate. Next, to prove (i) observe that if the blocks in $\mathcal{B}$ are cyclic, then by Lemma 7.1 there exists $m \in \mathbb{Z}_n^{\#}$ such that the blocks coincide with the orbits of the subgroup $\langle \rho^m \rangle$ and hence $K = \langle \rho^m \rangle$. It follows that $D/K = DK/K = D/D \cap K$ is a regular dihedral subgroup of $\bar{A}$, and so $X_{\mathcal{B}}$ is a 2-arc-transitive dihedrant.

As for (ii), observe that if the blocks are dihedral, then there exists $m \in \mathbb{Z}_n \setminus \mathbb{Z}_n^*$ such that the subgroup $\langle \rho^m \rangle$ is of index 2 in K. Moreover, we have that $\langle \rho \rangle / K \cap \langle \rho \rangle = \langle \rho \rangle / \langle \rho^m \rangle$ is a regular cyclic subgroup of $\bar{A}$ and so $X_{\mathcal{B}}$ is a 2-arc-transitive circulant. $\square$

Lemma 7.3. *Let $n \geqslant 3$, $X \ncong K_{2n}, C_{2n}$ be a 2-arc-transitive dihedrant of order $2n$ and let $\mathcal{B}$ be an imprimitivity block system of $Aut\,X$ with blocks of length $k \geqslant 2$. If $|\mathcal{B}| > 2$, then X is a k-fold cover of the quotient $X_{\mathcal{B}}$. In particular, if the blocks in $\mathcal{B}$ are cyclic then X is a regular $\mathbb{Z}_k$-cover of $X_{\mathcal{B}}$.*

Proof. Clearly, because of 2-arc-transitivity, we have that for any two blocks $B, B' \in \mathcal{B}$, a vertex in B has at most one neighbour in B'. The result will be proved if we show that either there are no edges with one end-vertex in B and the other end-vertex in B' or the bigraph $X[B, B']$ is isomorphic to kK_2.

Let K denote the kernel of the action of $A = Aut\,X$ on $\mathcal{B}$. If the blocks in $\mathcal{B}$ are cyclic, the result is immediate by Lemma 7.1. In this case there exists $m \in \mathbb{Z}_n^{\#}$ such that the kernel K coincides with a non-trivial subgroup $\langle \rho^m \rangle$ of $\langle \rho \rangle$ of order k and so X is a regular $\mathbb{Z}_k$-cover of $X_{\mathcal{B}}$. We may therefore assume that the blocks in $\mathcal{B}$ are dihedral and that, with no loss of generality, there is a block $B \in \mathcal{B}$ containing vertices u_0 and v_0. Then B coincides with the set $\langle \rho^m \rangle u \cup \langle \rho^m \rangle v = \{u_0, u_m, \dots, u_{(k-1)m}, v_0, v_m, \dots, v_{(k-1)m}\}$. Let $[S, T]$, where $S \subseteq \mathbb{Z}_n^*$ and $T \subseteq \mathbb{Z}_n$, be the corresponding symbol of X relative to the triple (ρ, u_0, v_0), and let $B_i = \rho^i B$ for each $i \in \mathbb{Z}_n$. If $S \neq \emptyset$, then the bigraphs $X[B_i, B_j]$ are clearly of the desired form. So assume that $S = \emptyset$. By Lemma 7.2, the quotient $X_{\mathscr{B}}$ is a 2-arc-transitive circulant (of valency at least 3) and is therefore, by [1, Theorem 1.1], isomorphic to one of K_m, $K_{m/2,m/2}$, or $K_{m/2,m/2} - m/2K_2$, where $m = n/k$.

Suppose first that $X_{\mathscr{B}} \cong K_{m/2,m/2}$ or $X_{\mathscr{B}} \cong K_{m/2,m/2} - m/2K_2$. The symbol of the circulant $X_{\mathscr{B}}$ is therefore either $\mathbb{O}_m$ in the first case or $\mathbb{O}_m \setminus \{m/2\}$ with $m/2$ odd in the second case. But then it easily follows that $T \subseteq \mathbb{O}_n$ and hence $\langle T - T \rangle \subseteq \mathbb{E}_n$. This implies that X is disconnected, a contradiction.

We may therefore assume that $X_{\mathscr{B}} \cong K_m$. Consider two neighbours v_i and v_j of u_0 and the corresponding three blocks B_0, B_i, B_j. By 2-arc-transitivity there exists $\alpha \in A$ which fixes u_0 and interchanges v_i and v_j. By the nature of the blocks, precisely one of v_i and v_j has a neighbour in B_j and B_i, respectively. Say v_i has a neighbour $u_{j+lm} \in B_j$ for some $l \in \mathbb{Z}_k$. But then α takes the edge $v_i u_{j+lm}$ into a non-edge, a contradiction. Hence X is a k-fold cover of $X_{\mathscr{B}}$. This completes the proof of Lemma 7.3. □

We now turn our attention to special kinds of imprimitivity block system, namely those with blocks of length 2 or 4.

Lemma 7.4. *Let $n \geq 3$, let $X = X(2n, S, T) \ncong K_{2n}, C_{2n}$ be a connected 2-arc-transitive dihedrant of order $2n$, let U and V be the two orbits of the automorphism ρ mapping according to the rule* (1), *and let $\mathscr{B}$ be an imprimitivity block system of* Aut X *with blocks of length 2 such that, for each $B \in \mathscr{B}$, we have $|B \cap U| = 1$ and $|B \cap V| = 1$. Then $X_{\mathscr{B}} \cong K_n$ and $X \cong K_{n,n} - nK_2$.*

Proof. Clearly, modulo a possible relabelling of the vertices of X, we may assume that the blocks in $\mathscr{B}$ have the form $\{u_i, v_i\}$, $i \in \mathbb{Z}_n$.

By Lemma 7.2, it follows that $X_{\mathscr{B}}$ is a 2-arc-transitive circulant, and hence, in view of [1, Theorem 1.1], isomorphic to one of K_n, $K_{n/2,n/2}$ or $K_{n/2,n/2} - n/2K_2$. Also, Lemma 7.3 implies that X is a 2-fold cover of $X_{\mathscr{B}}$, and as such a regular $\mathbb{Z}_2$-cover with the permutation ω mapping according to the rule

$$\omega u_i = v_i, \quad \omega v_i = u_i, \quad i \in \mathbb{Z}_n \tag{18}$$

as the generator of the kernel K of the action of $A = Aut\, X$ on $\mathscr{B}$. But ω commutes with ρ and so X is also a Cayley graph of the abelian group $\langle \rho, \omega \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_2$. If we identify the trivial element of this group with the vertex u_0, then the three involutions ω, $\rho^{n/2}$ and $\omega\rho^{n/2}$ are identified, respectively, with vertices v_0, $u_{n/2}$ and $v_{n/2}$. Furthermore $T = -T$ is a symmetric subset of $\mathbb{Z}_n$ and $S \cap T = \emptyset$.

If $X_{\mathscr{B}} \cong K_n$ then, by [20, Theorem 1.1], X is necessarily the canonical double cover of $X_{\mathscr{B}}$ and thus isomorphic to $K_{n,n} - nK_2$. In what follows, we may therefore assume that $X_{\mathscr{B}}$ is isomorphic either to $K_{n/2,n/2}$ or to $K_{n/2,n/2} - n/2K_2$. We derive a contradiction in both cases.

Case 1: $X_{\mathscr{B}} \cong K_{n/2,n/2} - n/2K_2$. In particular, since $X_{\mathscr{B}}$ is a circulant, we have that $n/2$ is odd and $S \cup T = \mathbb{O}_n \setminus \{n/2\}$. In view of the conditions on S and T we have that none of vertices v_0, $u_{n/2}$ and $v_{n/2}$ is in $N(u_0)$. Next, note that $N^2(u_0) = \{u_x\colon x \in (S+S) \cup (T+T)\} \cup \{v_y\colon y \in S+T\}$, and since, by assumption, $n/2$ is odd we have that $n/2 \notin (S+S) \cup (T+T) \cup (S+T)$. It follows that none of these vertices is in the set $N(u_0) \cup N^2(u_0)$. But these three vertices correspond to the three involutions in $\langle \rho, \omega \rangle$, and so the above fact is a clear contradiction with Theorem 6.9.

Case 2: $X_{\mathscr{B}} \cong K_{n/2,n/2}$. Note that $S \cap T = \emptyset$ and that $S \cup T = \mathbb{O}_n$. First, we show that

$$S + T = \mathbb{E}_n^{\#}. \tag{19}$$

Assume the contrary. Then for some $2j \in \mathbb{Z}_n^{\#}$ we have $(S+2j) \cap T = \emptyset$. Consequently, $S + 2j \subseteq S$, and similarly $T + 2j \subseteq T$. In fact, $S + 2j = S$ and $T + 2j = T$. Consider the vertices u_0 and u_{2j}. It follows that $|N(u_0) \cap N(u_{2j})| = |S| + |T| = n/2$, the valency of X. But then the only possibility is that $X \cong 2K_{n/2,n/2}$, contradicting connectedness of X, and thus proving (19).

We now consider the condition on the number λ of common neighbours of two vertices at distance 2, say u_0 in U and the other, say v_x, in V. We have that $\lambda = |N(u_0) \cap N(v_x)| = |S \cap T + x| + |T \cap S + x| = 2|S \cap T + x|$. Then, by (19),

$$|S \cap (T+x)| = \begin{cases} \lambda/2, & x \in \mathbb{E}_n^{\#}, \\ 0, & x \in \mathbb{O}_n \cup \{0\}. \end{cases}$$

We may translate this into the functional equation

$$\chi_S * \chi_T = \lambda \chi_{\mathbb{E}_n^{\#}}, \tag{20}$$

where χ_M denotes the characteristic function of a subset M of $\mathbb{Z}_n$ and $*$ denotes the convolution of functions. Let i be the imaginary unit and $\xi = e^{2\pi i/n}$ be the nth root of unity. Applying the discrete Fourier transform (which converts convolutions to products) to (20) we get, for each $j \in \mathbb{Z}_n$,

$$2 \sum_{s \in S} (\xi^j)^s \sum_{t \in T} (\xi^j)^t = \lambda((\xi^j)^2 + (\xi^j)^4 + \cdots + (\xi^j)^{n-2}). \tag{21}$$

Note that if $n/2$ is odd, then precisely one of the two vertices $u_{n/2}$ and $v_{n/2}$ is a neighbour of u_0. Besides, the third vertex v_0 is at distance at least 3 from u_0. Hence, $N^2(u_0)$ contains at most one of these three vertices (corresponding to involutions in $\langle \rho, \omega \rangle$. It can be seen that FQ_5 is not a dihedrant and so Theorem 6.9 implies that $X \cong K_{n,n} - nK_2$ is the only possibility. A contradiction, as the latter is clearly not a 2-fold cover of $K_{n/2,n/2}$.

We may therefore assume that $n/2$ is even. Plugging into (21) the value $j = n/4$, we get 0 on the left-hand side of the equation. Namely, as n is divisible by 4, we have with each $i = \xi^{4k+1}$ also $-i = \xi^{n-4k-1}$ inside each of the two sums. But the right-hand side of this equation is non-zero; in fact it equals $-\lambda$.

These contradictions show that neither this case can occur, completing the proof of Lemma 7.4. □

The next lemma deals with blocks of length 4.

Lemma 7.5. *Let $n \geqslant 3$, let $X = X(2n, S, T) \ncong K_{2n}, C_{2n}$ be a connected 2-arc-transitive dihedrant of order $2n$, let U and V be the two orbits of the automorphism ρ mapping according to the rule* (1), *let $\mathscr{B}$ be an imprimitivity block system of $A = Aut\,X$, and let K be the kernel of the action of A on $\mathscr{B}$.*

If $|\mathscr{B}| \geqslant 2$ and if the blocks in $\mathscr{B}$ are of length 4 and are such that $|B \cap U| = 2$ and $|B \cap V| = 2$ for each $B \in \mathscr{B}$, and if K acts transitively on each $B \in \mathscr{B}$, then the following statements hold:

(i) $n = 2(q+1)$, *where q is an odd prime power, $X_{\mathscr{B}} \cong K_{q+1}$ and $X \cong K_{q+1}^4$; and*
(ii) *X has a symbol of the form $[\emptyset, T_1 \cup T_2]$, where $T_1 \cap T_2 = \emptyset$, $T_1 = -T_1$, $T_2 = -T_2 + q + 1$ and $\{|T_1|, |T_2|\} = \{(q-1)/2, (q+1)/2\}$.*

Proof. Since $|\mathscr{B}| \geqslant 2$ and $|B| = 4$ for $B \in \mathscr{B}$ and since K is transitive on each $B \in \mathscr{B}$, we have that X is a regular 4-fold cover of $X_{\mathscr{B}}$. Without loss of generality, we can assume that $B = \{u_0, u_{n/2}, v_0, v_{n/2}\}$ is a block in $\mathscr{B}$. Among others this implies that

$$0, n/2 \notin S \cup T \tag{22}$$

and that $S \cap (S + n/2) = \emptyset$ and

$$T \cap (T + n/2) = \emptyset. \tag{23}$$

For each $i \in \mathbb{Z}_n$ let $B_i = \rho^i B$. Since the blocks in $\mathscr{B}$ are dihedral, Lemma 7.2 implies that the quotient $X_{\mathscr{B}}$ is a 2-arc-transitive circulant (of order $n/2$) and thus, in view of [1, Theorem 1.1], isomorphic to one of $K_{n/2}$, $K_{n/4,n/4}$ or $K_{n/4,n/4} - (n/4)K_2$. In particular, the valency $val\,X$ of X attains one of the following values:

$$val\,X \in \{n/4 - 1, n/8, n/8 - 1\}. \tag{24}$$

There exists an automorphism σ of X belonging to the kernel K such that the restriction σ^B is either $(u_0 v_0)(u_{n/2} v_{n/2})$ or $(u_0 v_0 u_{n/2} v_{n/2})$. We split the argument accordingly.

Case 1: $\sigma^{B_0} = (u_0 v_0)(u_{n/2} v_{n/2})$. If σ interchanges u_i with v_i for each $i \in \mathbb{Z}_n$, then X is a Cayley graph of the abelian group $\langle \rho, \sigma \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_2$, which has precisely three involutions. If we identify the trivial element of this group with the vertex u_0, then the three involutions σ, $\rho^{n/2}$ and $\sigma\rho^{n/2}$ are identified, respectively, with vertices v_0, $u_{n/2}$ and $v_{n/2}$. But all of these vertices are in the same block as u_0 and hence at the same distance from u_0. In view of (22) this distance is at least 3, contradicting Theorem 6.9.

We may therefore assume that there exists a decomposition $\{L, M\}$ of $\mathbb{Z}_n$ such that $0 \in L$ and such that $K = \langle \rho^{n/2}, \sigma \rangle$, where σ maps according to the following rule.

$$\sigma^{B_i} = \begin{cases} (u_i v_i)(u_{i+n/2} v_{i+n/2}), & i \in L, \\ (u_i v_{i+n/2})(u_{i+n/2} v_i), & i \in M. \end{cases}$$

Clearly, $L = L + n/2$ and $M = M + n/2$. Choose any two adjacent blocks B_i and B_j in $\mathscr{B}$. Then we may easily see that there exists a decomposition $\{T_1, T_2\}$ of T such that $T_1 = -T_1$ and $T_2 = -T_2 + n/2$ and that, moreover,

$$j - i \in S \cup T_1 \Leftrightarrow i, j \in L \text{ or } i, j \in M$$

and

$$j - i \in T_2 \Leftrightarrow i \in L,\ j \in M \text{ or } i \in M,\ j \in L.$$

Consequently,

$$S \cup T_1 \subseteq L \quad \text{and} \quad T_2 \subseteq M. \tag{25}$$

Set $L_0 = L \cap \mathbb{E}_n$, $L_1 = L \cap \mathbb{O}_n$, $M_0 = M \cap \mathbb{E}_n$ and $M_1 = M \cap \mathbb{O}_n$. Furthermore, let $\mathscr{L}_0 = \{B_i\colon i \in L_0\}$, $\mathscr{L}_1 = \{B_i\colon i \in L_1\}$, $\mathscr{M}_0 = \{B_i\colon i \in M_0\}$ and $\mathscr{M}_1 = \{B_i\colon i \in M_1\}$. We have two possibilities.

Subcase 1.1: $X_{\mathscr{B}} \ncong K_{n/2}$. In this case $X_{\mathscr{B}}$ is isomorphic either to $K_{n/4,n/4}$ or to $K_{n/4,n/4} - n/4K_2$. Consequently, the symbol of the circulant $X_{\mathscr{B}}$ is either $\mathbb{O}_m$ in the first case or $\mathbb{O}_m \setminus \{n/4\}$, and moreover $n/4$ odd, in the second case. This means that the bigraphs $[\mathscr{L}_0, \mathscr{L}_1]$ and $[\mathscr{M}_0, \mathscr{M}_1]$ which are $|S \cup T_1|$-regular, and the bigraphs $[\mathscr{L}_0, \mathscr{M}_1]$ and $[\mathscr{M}_0, \mathscr{L}_1]$, which are $|T_2|$-regular graphs, exhaust all the edges of $X_{\mathscr{B}}$. In particular, neither the subgraph generated by $S \cup T_1$-edges nor the subgraph generated by T_2-edges is connected. But this is impossible as $\mathbb{Z}^*_{n/2} \subseteq \mathbb{O}_{n/2}$ and each unit in the symbol of the circulant $X_{\mathscr{B}}$ gives rise to a connected subgraph of $X_{\mathscr{B}}$. Consequently, some of the sets L_0, L_1, M_0, M_1 must be empty. But by assumption $0 \in L$ and so $L_0 \neq \emptyset$. This leaves us with one possibility only: $L = \mathbb{E}_n$ and $M = \mathbb{O}_n$. It follows that $S \cup T_1 = \emptyset$ and so $T = T_2$. Hence a symbol of X has the form $[\emptyset, T]$ with $T \subseteq \mathbb{O}_n$. But then $\langle T - T \rangle \subseteq \mathbb{E}_n$, and so X is disconnected, a contradiction.

Subcase 1.2: $X_{\mathscr{B}} \cong K_{n/2}$. Since any two B_i, B_j, $i \neq j$, are adjacent, (25) gives us

$$(S \cup T_1) \cup (S \cup T_1 + n/2) = L^{\#} \setminus \{n/2\} \quad \text{and} \quad T_2 \cup (T_2 + n/2) = M. \tag{26}$$

Besides, $L + L \subseteq L$, $M + M \subseteq L$ and $L + M \subseteq M$. Since both L and M are non-empty subsets of $\mathbb{Z}_n$, it follows that L is a subgroup of $\mathbb{Z}_n$ of index 2. In other words, $L = \mathbb{E}_n$ and $M = \mathbb{O}_n$. Note that in view of [20, Theorem 1.1], we have $n = 2(q+1)$, where $q \equiv 1 (\mathrm{mod}\, 4)$ is a prime power, and $X \cong K^4_{q+1}$. But then, by Theorem 5.3, we may assume that $S = \emptyset$. Applying (23) we have $T_1 \subseteq \mathbb{E}^{\#}_n$, $T_2 \subseteq \mathbb{O}_n$ and so, in view of (23), their cardinalities $|T_1|$ and $|T_2|$ are, respectively, $(q-1)/2$ and $(q+1)/2$.

Case 2: $\sigma^{B_0} = (u_0 v_0 u_{n/2} v_{n/2})$. If the restriction σ^{B_i} is $(u_i v_i u_{i+n/2} v_{i+n/2})$ for each $i \in \mathbb{Z}_n$, then it easily follows that X is a Cayley graph of the abelian group $\langle \rho, \sigma \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_2$, which has precisely three involutions. These involutions may be identified with vertices $u_{n/2}$, $v_{n/4}$ and $v_{3n/4}$, provided the trivial element is identified with u_0. Clearly, $u_{n/2}$ is at distance at least 3 from u_0. We may then apply Theorem 6.9 to deduce that X is either isomorphic to the complete bipartite graph with a 1-factor removed (the three involutions at distance 1,2 and 3 from u_0) or is isomorphic to some 2-fold cover of a complete bipartite graph (with two involutions at distance 2 and one at distance 4 from u_0). Consequently, $val\, X$ is either $n-1$ or $n/2$, contradicting (24).

The rest of the argument is almost identical to the one used in Case 1. Namely, we can assume that there exists a decomposition $\{L, M\}$ of $\mathbb{Z}_n$ such that $0 \in L$ and that $K = \langle \rho^{n/2}, \sigma \rangle$, where σ maps according to the following rule:

$$\sigma^{B_i} = \begin{cases} (u_i v_i u_{i+n/2} v_{i+n/2}), & i \in L, \\ (u_i v_{i+n/2} u_{i+n/2} v_i), & i \in M. \end{cases}$$

We have $L = L + n/2$ and $M = M + n/2$. Choosing any two adjacent blocks B_i and B_j in $\mathscr{B}$, we see that there exists a decomposition $\{T_1, T_2\}$ of T such that $T_1 = -T_1$ and $T_2 = -T_2 + n/2$ and that, moreover,

$$j - i \in S \cup T_2 \Leftrightarrow i, j \in L \text{ or } i, j \in M$$

and

$$j - i \in T_1 \Leftrightarrow i \in L, j \in M \text{ or } i \in M, j \in L.$$

Consequently,

$$S \cup T_2 \subseteq L \quad \text{and} \quad T_1 \subseteq M. \tag{27}$$

Comparing (27) with (25) we see that the only distinction between Cases 1 and 2 is that the roles of T_1 and T_2 are interchanged. As in Subcase 1.1 we get that $X_{\mathscr{B}} \not\cong K_{n/2}$ forces X to have a symbol of the form $[\emptyset, T]$ with $T \subseteq \mathbb{O}_n$. Hence $\langle T - T \rangle \subseteq \mathbb{E}_n$, and so X is disconnected, a contradiction. On the other hand, if $X_{\mathscr{B}} \cong K_{n/2}$ then, by [20, Theorem 1.1], we have $n = 2(q+1)$, where $q \equiv 3 \pmod 4$, and $X \cong K^4_{q+1}$. Moreover, $S = \emptyset$, $T_1 \subseteq \mathbb{O}_n$ and $T_2 \subseteq \mathbb{E}^{\#}_n$, and in view of (23), the cardinalities $|T_1|$ and $|T_2|$ are, respectively, $(q+1)/2$ and $(q-1)/2$. □

Let us mention that if (T_1, T_2) is a pair of subsets satisfying part (ii) of Lemma 7.5, then so are the pairs $(T_1 + x, T_2 + x)$, for $x = (q+1)/2, q+1, 3(q+1)/2$. In particular, this means that the cardinalities of the sets T_1 and T_2 may take both values $(q-1)/2$ and $(q+1)/2$. Namely, if say $|T_1| = (q+1)/2$ and $|T_2| = (q-1)/2$, then by letting $T'_1 = T_2 + (q+1)/2$ and $T'_2 = T_1 + (q+1)/2$, we have that $T'_1 = -T'_1$, $T'_2 = -T'_2 + q + 1$, and $|T'_1| = (q-1)/2$, and $|T'_2| = (q+1)/2$. Now, it can be easily seen that precisely one of the above mentioned four pairs of subsets has the property that the first component, that is the symmetric one, contains 0. We call this pair the *canonical form* of the corresponding symbol of the graph K^4_{q+1}.

We summarize the above discussion in the following corollary of Lemma 7.5.

Corollary 7.6. *Let q be an odd prime power. Then the* 2-*arc-transitive dihedrant* K^4_{q+1} *has a symbol of the form* $[\emptyset, T_1 \cup T_2]$, *where* $T_1 \subseteq \mathbb{E}_{2(q+1)} \setminus \{q+1\}$, $T_2 \subseteq \mathbb{O}_{2(q+1)}$, $0 \in T_1$, $T_1 = -T_1$, *and* $T_2 = -T_2 + q + 1$. *Moreover,*

(i) *if* $q \equiv 1 \pmod 4$ *then* $|T_1| = (q+1)/2$ *and* $|T_2| = (q-1)/2$; *and*
(ii) *if* $q \equiv 3 \pmod 4$ *then* $|T_1| = (q-1)/2$ *and* $|T_2| = (q+1)/2$.

Table 1 below gives the canonical forms (T_1, T_2) of symbols $[\emptyset, T_1 \cup T_2]$ of the graphs K^4_{q+1} for all prime powers $q \leqslant 25$. The computations have been carried out using MAGMA [4]. See also Fig. 1 where the graphs K^4_{q+1}, $q = 3, 5, 7, 9$ are shown.

We wrap up this section with two propositions. The first one gives a description of all possible symbols of the complete bipartite graphs and the complete bipartite graphs with a 1-factor removed. The second one suggests a uniqueness of these two classes of graphs among 2-arc-transitive dihedrants with symbols of the form $[\emptyset, T]$.

Proposition 7.7. *Let* $n \geqslant 3$ *and let* X *be a dihedrant of order* $2n$ *with symbol* $[S, T]$ *such that* $S \neq \emptyset$ *and* $0 \in T$. *Then the following statements hold.*

(i) *If* $X \cong K_{n,n}$, *then* n *is even, and* $[S, T] = [\mathbb{O}_n, \mathbb{E}_n]$.
(ii) *If* $X \cong K_{n,n} - nK_2$, *then* n *is even and either* $[S, T] = [\mathbb{O}_n, \mathbb{E}_n \setminus \{2k\}]$ *for some* $2k \neq 0$, *or* $n/2$ *is odd and* $[S, T] = [\mathbb{O}_n \setminus \{n/2\}, \mathbb{E}_n]$.

Proof. That n is even in both cases is clear. Since both $K_{n,n}$ and $K_{n,n} - nK_2$ are bipartite graphs, and hence triangle-free graphs, we have

$$S \cap (S + S) = \emptyset \text{ and } S \cap (T + T) = \emptyset. \tag{28}$$

In particular, since $0 \in T$, we have

$$S \cap T = \emptyset. \tag{29}$$

The proof of (i) is now at hand. Since the valency of $K_{n,n}$ is $n = |S| + |T| = |S \cup T|$, it follows by (29), that $S \cup T = \mathbb{Z}_n$. In particular, $T = -T$ and therefore, by (28),

Table 1
The sets T_1 and T_2

q	n	T_1	T_2
3	8	$\{0\}$	$\{1,3\}$
5	12	$\{0,2,10\}$	$\{7,11\}$
7	16	$\{0,2,14\}$	$\{3,5,9,15\}$
9	20	$\{0,2,6,14,18\}$	$\{1,3,7,9\}$
11	24	$\{0,8,10,14,16\}$	$\{1,3,5,11,13,23\}$
13	28	$\{0,4,6,12,16,22,24\}$	$\{1,3,5,9,11,13\}$
17	36	$\{0,10,12,14,16,20,22,24,26\}$	$\{1,3,7,11,15,17,23,31\}$
19	40	$\{0,2,4,8,14,26,32,36,38\}$	$\{3,5,7,11,15,17,21,29,31,39\}$
23	48	$\{0,2,4,8,10,18,30,38,40,44,46\}$	$\{1,3,5,9,15,19,21,23,31,35,37,41\}$
25	52	$\{0,6,14,16,18,22,24,28,30,34,36,38,46,\}$	$\{9,17,27,29,31,33,37,41,45,47,49,51\}$

$S \cap 2(S \cup T) = \emptyset$ and so $S \cap \mathbb{E}_n = \emptyset$. Hence $S \subseteq \mathbb{O}_n$ and so $\mathbb{E}_n \subseteq T$. Now assume that there exists $t = 2i + 1 \in T$ and take $s = 2j + 1 \in S$. Then $s - t = 2(j - i) \in \mathbb{E}_n \subseteq T$. So $s \in T + t = -T + t$, contradicting (28). We conclude that $S = \mathbb{O}_n$ and $T = \mathbb{E}_n$.

As for part (ii), suppose first that $\mathbb{E}_n \nsubseteq (S + S) \cup (T + T)$. Let, in view of (29), x be the sole element of $\mathbb{Z}_n \setminus S \cup T$. Then either $x = -x = n/2$ or $-x \in T$. The first case cannot occur for it implies $T = -T$ which clearly forces $\mathbb{E}_n \subseteq (S + S) \cup (T - T)$. As for the second case it implies that $T \setminus \{x\}$ is a symmetric subset of $\mathbb{Z}_n$, forcing $\mathbb{E}_n \setminus \{2x\} \subseteq (S + S) \cup (T - T)$. We may therefore assume that $2x \notin (S + S) \cup (T - T)$. Then $S \cap (S + 2x) = \emptyset$ as well as $T \cap (T + 2x) = \emptyset$. It follows that $u_{2x} = u_{-2x}$ is the vertex at distance 3 from u_0. Hence $4x = 0$ and so $x \in \{n/2, 3n/2\}$. In particular, $n/2$ is even and moreover $\mathbb{E}_n \setminus \{n/2\} \subseteq (S + S) \cup (T - T)$. But then, by (28), $S \subseteq \mathbb{O}_n \cup \{n/2\}$ and so $\mathbb{E}_n \setminus \{x, n/2\} \subseteq T$. But $T \cap (T + n/2) = \emptyset$ and so $\mathbb{E}_n \setminus \{x, n/2\} \cap (\mathbb{E}_n \setminus \{x, n/2\} + n/2) = \emptyset$. It follows that $n/2 - 2 \leqslant n/4$, forcing $n = 8$. Without loss of generality $x = n/4 = 2$ and so this gives us the following four possibilities for the sets S and T: either $S = \{4\}$ and $T = \{0, 1, 3, 5, 6, 7\}$, or $S = \{1, 4, 7\}$ and $T = \{0, 3, 5, 6\}$, or $S = \{3, 4, 5\}$ and $T = \{0, 1, 6, 7\}$, or $S = \{1, 3, 4, 5, 7\}$ and $T = \{0, 6\}$. But in all of these cases there is a triangle in X, contradicting (28).

We may therefore assume that $\mathbb{E}_n \subseteq (S + S) \cup (T + T)$. Then, by (28), we have $S \subseteq \mathbb{O}_n$. Clearly, if $S = \mathbb{O}_n$ then $T = \mathbb{E}_n \setminus \{2k\}$ for some $2k \neq 0$. So assume that $S \neq \mathbb{O}_n$ and take $z \in \mathbb{O}_n \setminus S$. Assume first that $z \in T$ and consider the sums $s + z$ and $-s + z$ for some $s \in S$. Both $s + z$ and $-s + z$ are even, and so if $s \neq -s$ then one of the above two sums is in T, contradicting (28). So $S = \{n/2\}$, as the only remaining possibility. But then, since at most one of 1, -1, $1 + n/2$ and $-1 + n/2$ does not belong to T, we have again a contradiction with (28). All of these show that $z \notin T$ and so $T \subseteq \mathbb{E}_n$. Therefore $S = \mathbb{O}_n \setminus \{z\}$ and $T = \mathbb{E}_n$. Since $S = -S$, it follows that $z = -z$ and so $z = n/2$. In particular, $n/2$ is odd. Hence proof. □

Lemma 7.8. *Let $n \geqslant 3$ and let $X \ncong C_{2n}$ be a connected 2-arc-transitive Cayley graph of a dihedral group of order $2n$ having a symbol $[\emptyset, T]$, where $T = -T$ is a symmetric subset of $\mathbb{Z}_n$. Then X is a isomorphic either to $K_{n,n}$ or to $K_{n,n} - nK_2$.*

Proof. Since $T = -T$, it follows that X is also a Cayley graph of the abelian group $\mathbb{Z}_n \times \mathbb{Z}_2$. More precisely, the permutation ω, mapping according to the rule (18) together with ρ generates a subgroup of $Aut\,X$ isomorphic to $\mathbb{Z}_n \times \mathbb{Z}_2$. As in the proofs of Lemmas 7.4 and 7.5, we identify the trivial element of the group and the three involutions ω, $\rho^{n/2}$ and $\omega\rho^{n/2}$, respectively, with vertices u_0, v_0, $u_{n/2}$ and $v_{n/2}$. We then apply Theorem 6.9 to deduce that X is isomorphic either to $K_{n,n}$ or to $K_{n,n} - nK_2$. □

8. Proving Theorem 2.1

The following group-theoretic results will be needed in the proof of Theorem 2.1, or rather in the proof of its modified version Theorem 8.4.

Proposition 8.1 (Burnside; [41, Theorem 7.3]). *A transitive permutation group of prime degree p is either doubly transitive or solvable, in which case it may be identified with the group of all affine transformations* $f_{a,b}: x \to ax + b (a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p)$ *of the field* $\mathbb{Z}_p$.

Proposition 8.2 (Schur; [53, Theorem 25.3]). *A primitive group containing a regular cyclic subgroup of composite order is doubly transitive.*

Proposition 8.3 (Wielandt; [53, Theorem 25.6]). *A primitive group containing a regular dihedral subgroup is doubly transitive.*

We now restate Theorem 2.1 in a slightly modified form. Let $\mathcal{G}_1$ denote the class of graphs containing: C_{2n}, $n \geqslant 3$ and K_{2n}, $n \geqslant 3$, $n \geqslant 3$ and let $\mathcal{G}_2$ be the class of graphs containing: $K_{n,n}$, $n \geqslant 3$, $K_{n,n} - nK_2$, $n \geqslant 3$, K_{q+1}^4, q an odd prime power, $B(H_{11})$, $B'(H_{11})$, and $B(PG(d,q))$ and $B'(PG(d,q))$, with $d \geqslant 2$ and q a prime power, where we follow the notation introduced in Section 2 in the paragraphs preceding the statement of Theorem 2.1.

Theorem 8.4. *Let* $n \geqslant 3$, *let* X *be a connected, 2-arc-transitive Cayley graph of a dihedral group* $D = \langle \rho, \tau \rangle$ *of order* $2n$, *where* ρ *and* τ *map according to the respective rules* (1) *and* (2). *Then one of the following occurs*:

(i) *either* $X \in \mathcal{G}_1 \cup \mathcal{G}_2$; *or*
(ii) X *is a regular cyclic cover of a graph in* $\mathcal{G}_2$; *more precisely*: *there exists a proper divisor* m *of* n *such that the set* $\mathcal{B}$ *of orbits of* $\langle \rho^m \rangle$ *is an imprimitivity block system of* $Aut\, X$ *relative to which* X *is a regular* $\mathbb{Z}_{n/m}$*-cover of* $X_{\mathcal{B}}$, *the latter being a graph in* $\mathcal{G}_2$ *admitting a regular dihedral group* $D/\langle \rho^m \rangle$.

Proof. Assume that, by contradiction, X is a minimal counterexample to the statement of Theorem 8.4. Since $X \ncong C_{2n}$, we have that

$$val\, X \geqslant 3. \tag{30}$$

Next, since $X \ncong K_{2n}$ the girth of X is at least 4, and moreover, Proposition 8.3 implies that $A = Aut\, X$ is imprimitive. Choose a minimal block of A, say of cardinality $k \geqslant 2$, and let $\mathcal{B}$ be the corresponding imprimitivity block system of A. Let K denote the kernel of the action of A on $\mathcal{B}$ and let $\bar{A} = A/K$ denote the corresponding quotient group. Clearly, in view of the minimality of B, the restriction A_B^B of the setwise stabilizer A_B to B is primitive. Now $\mathcal{B}$ is also an imprimitivity block system for D. Besides, D_B^B is a regular (cyclic or dihedral) subgroup of a primitive group A_B^B. Hence Propositions 8.2 and 8.3 together imply that either

$$A_B^B \text{ is doubly transitive or } k = p \text{ is prime.} \tag{31}$$

We distinguish two different cases.

Case 1: $|\mathscr{B}| = 2$. Then $\mathscr{B} = \{B, B'\}$ and we have that the kernel K coincides with the setwise stabilizers $A_B = A_{B'}$. Apply (31) and Proposition 8.1 to have that

$$K^B \text{ and } K^{B'} \text{ are either solvable of prime degree or doubly transitive.} \quad (32)$$

Suppose first that K acts unfaithfully on the two blocks B and B'. Then we may assume $K^B_{B'} \neq id$. But $K^B_{B'}$ is a normal subgroup of K^B, a primitive group, and so $K^B_{B'}$ is transitive by [53, Theorem 8.8]. This forces $X = X[B, B'] \cong K_{k,k}$ and so $X \in \mathscr{G}_2$.

Assume that K acts faithfully on the two blocks B and B'. We claim that in this case K^B and $K^{B'}$ are doubly transitive. Note that, for $v \in B$, the set of neighbours $N(v)$ of v must be an orbit of $K_v = A_v$. Moreover, 2-arc-transitivity of X forces a doubly transitive action of K_v on $N(v)$. So assume now that k is a prime and that K^B and $K^{B'}$ are solvable. Combining together Proposition 8.1 with the faithfulness of the action of K on B and B' we see that K_v is cyclic of order d, where d divides $p-1$. Therefore K_v cannot be doubly transitive on $N(v)$ as $|N(v)| = val\,X \geqslant 3$ in view of (30). So (32) implies that K^B and $K^{B'}$ are indeed doubly transitive.

If K^B and $K^{B'}$ are permutationally equivalent then, for each $v \in B$, the vertex stabilizer K_v has a fixed vertex $v' \in B'$ and two orbits $B \setminus \{v\}$ and $B' \setminus \{v'\}$. It follows that $N(v) = B' \setminus \{v'\}$ and so $X = [B, B'] \cong K_{k,k} - kK_2$ and $X \in \mathscr{G}_2$.

We may now assume that K^B and $K^{B'}$ are permutationally inequivalent. Recall that K^B and $K^{B'}$ are doubly transitive groups which contain a regular cyclic or dihedral group. Hence X is isomorphic to the incidence/non-incidence graph of a symmetric design with a group of automorphisms acting doubly transitively on points and containing a regular cyclic or dihedral group. It may be deduced from [31, Theorem] that X is isomorphic to one of the following graphs: the incidence and nonincidence graphs $B(PG(d,q))$ and $B'(PG(d,q))$, respectively, associated with the projective spaces $PG(d,q)$, $d \geqslant 2$, and the incidence and nonincidence graph $B(H_{11})$ and $B'(H_{11})$, respectively, of the unique Hadamard design H_{11} on 11 points. Hence $X \in \mathscr{G}_2$.

All of the above contradict the choice of X and show that this case cannot occur.

Case 2: $|\mathscr{B}| > 2$. Of course, in this case X covers $X_{\mathscr{B}}$. Moreover, by (30), $X_{\mathscr{B}}$ is not a cycle. Let us consider the setwise stabilizer D_B. It follows that either there exists $m \in \mathbb{Z}_n$ such that $D_B = \langle \rho^m \tau \rangle$, or there exists $m \in \mathbb{Z}_n^{\#} \setminus \mathbb{Z}_n^*$ such that either $D_B = \langle \rho^m \rangle$ or $D_B = \langle \rho^m, \tau \rangle$.

Suppose first that $D_B = \langle \rho^m \tau \rangle$. Then X is a 2-fold cover of $X_{\mathscr{B}}$. Also, a block in $\mathscr{B}$ contains one vertex from the orbit U and one vertex from the orbit V. By Lemma 7.4 it follows that $X \cong K_{n,n} - nK_2$ and so X belongs to $\mathscr{G}_2$.

Suppose next that $D_B = \langle \rho^m, \tau \rangle$. Note that $id \neq \langle \rho^m \rangle \neq \langle \rho \rangle$ and that $\langle \rho^m \rangle \leqslant K$. In particular, K^B is a 2-extension of the cyclic group $\langle \rho^m \rangle^B$ and thus the minimality of the block B forces $|B| = 4$, $m = n/2$ and so $K \cong \mathbb{Z}_2^2$, for in all other cases the orbits of $\langle \rho^m \rangle$ would form an imprimitivity block system of a doubly transitive group A_B^B. Moreover, the blocks in $\mathscr{B}$ are dihedral consisting of two vertices from each of the two orbits of ρ. By Lemma 7.5 we have that $n = 2(q+1)$, and $X_{\mathscr{B}} \cong K_{q+1}$ and $X \cong K_{q+1}^4$, where $q \equiv 1 \pmod 4$ is an odd prime power. Thus $X \in \mathscr{G}_2$.

Suppose finally that $D_B = \langle \rho^m \rangle$. As above $id \neq \langle \rho^m \rangle \neq \langle \rho \rangle$. Clearly, D_B is a normal subgroup of D and so $\langle \rho^m \rangle \leqslant K$. Of course, D_B is regular on B and so $id \neq K = \langle \rho^m \rangle$. Since K is a cyclic normal subgroup of A, it follows that every subgroup of K is normal in A, and thus the orbits of this subgroup form an imprimitivity block system of A. The minimality of blocks in $\mathscr{B}$ then forces $k = p$, a prime and $K \cong \mathbb{Z}_p$. Thus X is a regular $\mathbb{Z}_p$-cover of $X_{\mathscr{B}}$. Moreover, since the blocks in $\mathscr{B}$ are cyclic, Lemma 7.2 implies that $X_{\mathscr{B}}$ is itself a 2-arc-transitive dihedrant, of order $2m$, where $m = n/p < n$, admitting a dihedral regular subgroup $D/\langle \rho^m \rangle$, The choice of X as a minimal counterexample now leaves us with the following three possibilities.

First, if $X_{\mathscr{B}}$ belongs to $\mathscr{G}_1$ then it is isomorphic to K_{2m}. But then X is a 2-arc-transitive regular $\mathbb{Z}_p$-cover of K_{2m}. Such graphs were classified in [20, Theorem 1.1], and it follows that $X \cong K_{2m,2m} - 2mK_2$. Hence $X \in \mathscr{G}_2$.

Second, if $X_{\mathscr{B}}$ belongs to $\mathscr{G}_2$ then X is a regular cyclic cover of a graph in $\mathscr{G}_2$.

Finally, suppose that $X_{\mathscr{B}}$ is itself a regular cyclic cover of a graph Y in $\mathscr{G}_2$. Then there exists a non-trivial proper subgroup of $\langle \rho \rangle / \langle \rho^m \rangle$ whose orbits form an imprimitivity block system of $\bar{A}$ relative to which $X_{\mathscr{B}}$ is a regular cyclic cover of Y. It can be easily seen that, in view of Lemma 7.3, this implies the existence of a proper divisor l of m such that the set $\mathscr{C}$ of orbits of $\langle \rho^l \rangle$ is an imprimitivity block system of A relative to which X is a regular $\mathbb{Z}_{n/l}$-cover of $X_{\mathscr{C}} \cong Y$.

As in Case 1, all of the above contradict the choice of X and show that neither this case can occur. This completes the proof of Theorem 8.4. $\square$

Proof of Theorem 2.1. Theorem 2.1 is an immediate consequence of Theorem 8.4.

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