JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **199**, 243–255 (1996)
ARTICLE NO. 0139

On the Convergence of Normalizing Transformations in the Presence of Symmetries

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Received December 9, 1994; revised June 6, 1995

It is shown that, under suitable conditions, involving in particular the existence of analytic constants of motion, the presence of Lie point symmetries can ensure the convergence of the transformation of a vector field (or dynamical system) into normal form.

1. INTRODUCTION

The technique of transforming a system of first-order ordinary differential equations (also called dynamical system) into ‘‘normal form’’ (NF) is an old and well-known method of investigation, going back to the classical work of Poincaré (and then of Dulac and Birkhoff), and developed in more recent times by several authors (see, e.g., [2, 3, 5–7, 22, 23 and references therein]). Its connection with symmetry properties (precisely, Lie-point symmetries (see [20, 21])) of the dynamical system have been also pointed out [1, 6, 11, 12, 16]. One of the main troubles with this procedure is given by the problem of the convergence of the normalizing transformation: it is known in fact that these transformations are performed by means of recursive techniques, and only special conditions can ensure their convergence [7].

A possible way out which is usually adopted is that of considering these transformations ‘‘up to some finite order,’’ i.e., of considering ‘‘approximate normal forms’’ (and, correspondingly, approximate solutions, approximate symmetries, etc. of the given system; see, e.g., [4, 14]). In this paper, instead, we deal with the case of converging normalizing transformations

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0022-247X/96 $18.00

and with some conditions ensuring convergence. Our discussion is related to a theorem by Markhashov [19]: even if the original proof of this theorem has appeared to be not complete, a similar result has been proved at least in some particular case (see [8]). A very remarkable result in this context is given in a recent paper by Bruno and Walcher [10], in the case of 2-dimensional systems. The main idea is that the presence of some symmetry of the dynamical system can ensure—under suitable conditions—the convergence of the normalizing transformations.

Our paper presents some considerations in the same direction. In particular, we give a direct proof of a ‘‘Markhashov-type'' theorem in a well-defined and simple case and analyze some applications of this idea, which includes a generalization of the Bruno–Walcher result [10].

2. A ‘‘MARKHASHOV-TYPE'' THEOREM

We will consider dynamical systems (DS) of the form

$$\dot{u} = f(u), \qquad u = u(t) \in R^n, \tag{1}$$

where $\dot{u} = du/dt$, f is assumed to be analytic in a neighbourhood of $u = 0$, with $f(0) = 0$, and

$$f(u) = Au + F(u), \tag{2}$$

where the matrix $A \equiv (\nabla f)(0)$ is assumed to be nonzero and diagonalizable. Let us remark that most of the results below could be extended to the non-diagonalizable case, apart from some complications in the notations and statements (see [1, 13]).

As is well known [2, 7], a normalizing transformation is a nonlinear formal transformation:

$$u \to \hat{u} = u + \cdots \tag{3}$$

transforming (1)–(2) into a new DS which we write in the form (to avoid cumbersome notations, we will denote by u both the ‘‘original'' and the transformed coordinates)

$$\dot{u} = \hat{f}(u) = Au + \hat{F}(u) \tag{4}$$

where the nonlinear part $\hat{F}(u)$ is in NF. To define this notion, we introduce in the space of analytic functions, defined in a neighbourhood of $u = 0$, the Lie–Poisson bracket

$$\{f, g\}_k = (f \cdot \nabla) g_k - (g \cdot \nabla) f_k \qquad (k = 1, \ldots, n) \tag{5}$$

and, given any $n \times n$ matrix A, the "homological operator" $\mathscr{A}$

$$\mathscr{A}(f) = \{Au, f\} = (Au) \cdot \nabla f - Af. \tag{6}$$

Then, a nonlinear vector function $\hat{F}$ is said to be in NF with respect to A (or resonant with A) if

$$\mathscr{A}(\hat{F}) = 0. \tag{7}$$

In the basis where A is diagonal, with eigenvalues $a_1, \ldots, a_n$, a monomial $\hat{F}_k(u) = u_1^{m_1} \cdot \cdots \cdot u_n^{m_n}$ of degree j (with m_i integer numbers such that $\sum_i m_i = j$, $m_i \geq 0$) is resonant if $(m, a) \equiv \sum_i m_i a_i = a_k$. As is well known, the relevance of the above definitions is essentially due to the fact that, given a vector function f, all nonresonant terms can be removed by means of a formal coordinate transformation.

We also say that a vector function

$$g(u) = Bu + G(u) \tag{8}$$

is a (Lie-point time-independent) symmetry for the DS (1)–(2) if

$$\{f, g\} = 0. \tag{9}$$

In terms of Lie algebras, one says that the vector field operator $g \cdot \nabla$ generates a symmetry of the DS.

A scalar function $\rho = \rho(u)$ is a (time-independent) constant of motion (or first integral) for the DS (1)–(2) if the Lie derivative along f vanishes:

$$\left.\frac{d\rho}{dt}\right|_f \equiv f \cdot \nabla\rho = 0. \tag{10}$$

The above definitions of symmetry and of constant of motion can be clearly applied both to analytic functions and to formal power series.

Our discussion needs a few preliminary results, some of which are rather simple or well known; however, for clarity and completeness, we give all of them: some of these introductory results may also have an independent interest.

LEMMA 1. *If g is a symmetry for the DS, and ρ a constant of motion for it, then also*

$$h = \rho g$$

is a symmetry for the DS. More precisely, the algebra of the symmetries g of a DS is a finite-dimensional module over the constants of motion of the DS.

The first part of this lemma is immediate; the other statement describes the general property [13, 15, 16, 21] of the solutions g to the system of

PDEs (9) which gives the symmetries of (1)–(2). See below for some remarks concerning the number of the "admissible" constants of motion. An important step in the discussion is provided by the following lemma.

LEMMA 2 [1, 12, 23]. *If the DS* (1)–(2) *admits a symmetry g* (8), *and f is put in NF*

$$f \to \hat{f} = Au + \hat{F}, \qquad \hat{F} \in \operatorname{Ker} \mathscr{A}, \tag{11}$$

by a formal normalizing transformation, then g is transformed into a new form

$$g \to \tilde{g} = Bu + \tilde{G} \tag{12}$$

(*not necessarily normal and possibly formal*; *we reserve the notation* $\hat{\cdot}$ *only to NF*), *obviously satisfying* $\{\hat{f}, \tilde{g}\} = 0$ *as a direct consequence of* (9), *and*

$$\tilde{G} \in \operatorname{Ker} \mathscr{A} \qquad \text{or equivalently} \qquad \{Au, \tilde{G}\} = 0. \tag{13}$$

Proof. (see [12] for other details). Expanding $\hat{F}$ in formal power series

$$\hat{F} = \sum_{j \geq 2} \hat{F}_j, \tag{14}$$

where $\hat{F}_j$ are homogeneous polynomials of degree j, and expanding in a similar way $\tilde{G}$, one has from $\{\hat{f}, \tilde{g}\} = 0$ at order 1

$$[A, B] = 0 \tag{15}$$

and at order 2

$$\{Au, \tilde{G}_2\} = \{Bu, \hat{F}_2\}. \tag{16}$$

Applying to this equality the operator $\mathscr{A}$, one has $\mathscr{A}^2(\tilde{G}_2) = 0$ which gives, thanks to the assumption on the matrix A, $\mathscr{A}(\tilde{G}_2) = 0$. At the order $k > 2$, one has

$$\{Au, \tilde{G}_k\} - \{Bu, \hat{F}_k\} = \sum_{j=2}^{k-1} \{\hat{F}_j, \tilde{G}_{k-j+1}\}.$$

On the other hand, an immediate application of the Jacobi identity shows that if

$$\hat{F}_j, \tilde{G}_i \in \operatorname{Ker} \mathscr{A},$$

the same is true for $\{\hat{F}_j, \tilde{G}_i\}$. This allows us to proceed inductively to obtain (13).

Remarks. (a) [12]. By means of a further formal transformation, one may also normalize $\tilde{g} \to \hat{g}$, thus obtaining, if also B is assumed to be a diagonalizable matrix, the "joint" NF, where $\mathscr{B}$ is the homological operator $\mathscr{B}(\cdot) = \{Bu, \cdot\}$:

$$\hat{F}, \hat{G} \in \operatorname{Ker} \mathscr{A} \cap \operatorname{Ker} \mathscr{B}. \tag{17}$$

(b) [13] If B is not diagonalizable, we can decompose it into a semisimple (diagonalizable) and a nilpotent part:

$$B = B^{(s)} + B^{(n)}.$$

In this case, one can show that $\hat{F} \in \operatorname{Ker} \mathscr{B}$ in (17) can be substituted by

$$\hat{F} \in \operatorname{Ker}(\mathscr{B}^{(s)}) \qquad \text{or equivalently} \qquad \{B^{(s)}u, \hat{F}\} = 0.$$

An immediate consequence of the notion of joint NF and of Lemma 2 (in particular Remark (a)) is the following proposition.

PROPOSITION 1. *A DS* (1)–(2) *is in NF* (11) *if and only if it admits the linear symmetry Au. If the original DS* (1)–(2) *admits a symmetry* $g = Bu + G$ *with diagonalizable B, then the normalized DS in joint NF* (17) *also admits the linear symmetry Bu* (*or* $B^{(s)}u$ *in the case of non-diagonalizable B, see* [13]).

Let us now recall that there are two basic conditions, called Condition A and Condition ω [7], which ensure that a vector function $f = Au + F$ can be put in NF by a converging transformation [7]. For convenience, we state Condition A in a quite restrictive form; however, we shall use it only in the case of linear NF $\hat{f} = Au$, then the present formulation coincides with Condition A in [7].

Condition A. There is a coordinate transformation changing f to $\hat{f}$, where $\hat{f}$ has the form

$$\hat{f} = Au + \alpha(u) Au$$

and $\alpha(u)$ is some scalar-valued power series (with $\alpha(0) = 0$).

Condition ω. Let $\omega_k = \min|(q, a)|$ for all positive integers q_i such that $\sum_{i=1}^{n} q_i < 2^k$ and $(q, a) \neq 0$: then one has

$$\sum_{k=1}^{\infty} 2^{-k} \ln \omega_k^{-1} < \infty.$$

The first condition will play a key role in our discussion; the other one is a weaker condition, controlling the appearance of small divisors [7], and we explicitly assume that it is always satisfied here. In particular, it is satisfied in the cases considered in Section 4.

We are now in position to give a simple and direct proof of the following result. Let us preliminarily note that any DS admits an obvious symmetry, namely $g \equiv f$, which is in fact the generator of the dynamical flow. Accordingly, when considering the symmetries of a DS, it is always understood that none of them (nor their linear combinations) is proportional to f. Also, we may clearly exclude the case (which may be considered here as "trivial") that the DS, once in NF, takes the form $\dot{u} = (1 + \alpha(u))Au$: in this case indeed the Condition A is satisfied, and the convergence of the normalizing transformation is automatically guaranteed.

THEOREM 1. *Assume that the DS* (1)–(2) *possesses a finite number* l (≥ 1) *of analytic symmetries* $g_j = B_j u + G_j$, *where all the matrices* B_j *are linearly independent* (*and not zero*). *Assume also that, once in NF, the DS admits exactly* l *linearly independent* (*possibly formal*) *symmetries. Then, there exists a converging normalizing transformation for* f.

Proof. Let us start by writing the DS in NF by means of some formal normalizing transformation

$$f \to \hat{f} = Au + \hat{F}.$$

Under this transformation the l symmetries become

$$g_j \to \tilde{g}_j = B_j u + \tilde{G}_j, \tag{18}$$

which, together with the other symmetry Au, provide $l + 1$ symmetries for the DS in NF. By assumption, our problem in NF admits exactly l symmetries (plus the trivial one $\hat{f}$), which implies that Au must be a linear combination (with constant coefficients) of the $\tilde{g}_j$, i.e.,

$$Au = \sum_{j=1}^{l} \beta_j \left(B_j u + \tilde{G}_j \right),$$

which implies in turn $A = \sum \beta_j B_j$ and $\sum \beta_j \tilde{G}_j = 0$. Considering now the symmetry g_A of the initial DS defined precisely by

$$g_A = \sum_{j=1}^{l} \beta_j g_j = Au + \sum_{j=1}^{l} \beta_j G_j$$

one has that, under the above transformation,

$$g_A \to \tilde{g}_A = Au.$$

This shows that Condition A is satisfied by the symmetry $g_A \to \tilde{g}_A$; then there is a normalizing transformation which is convergent. Under this transformation, f is transformed into $\hat{f} = Au + \hat{F}$ which is in NF according to Lemma 2 (indeed $\hat{F} \in \operatorname{Ker} \mathscr{A}$, being $g_A = Au + \sum \beta_j G_j$). Notice in particular that, if all the matrices B_j are assumed to be diagonalizable, then, according to Proposition 1, their linear parts $B_j u$ directly provide l symmetries for the DS in NF.

The above theorem looks quite "formal" and not easily applicable in concrete cases: in fact, it may be difficult to check in practice that the required properties of the NF (which is usually *a priori* not explicitly known) are verified. In the two next sections, we will give more concrete versions of this result and study some cases in which the above hypotheses can be fulfilled.

3. THE NUMBER OF CONSTANTS OF MOTION AND SYMMETRIES OF A DS

First, let us remark that one of the crucial hypotheses of Theorem 1 is that the DS in NF admits a finite number of independent symmetries. According to Lemma 1, the finiteness of this number depends in an essential way on the number of independent constants of motion of the DS. In fact, it is clear that the presence of constants of motion of the problem in NF precludes the application of our above argument: indeed, even assuming the existence of a symmetry of the form $g_A = Au + G_A$, the corresponding nonlinear part $\tilde{G}_A$ could in this case be obtained as a combination of $\hat{f}$ and of the other $\tilde{g}_j$ multiplied by suitable constants of motion, and Condition A for the symmetry g_A fails to be verified.

To carefully discuss this point, let us recall the two following relevant results.

LEMMA 3 [10, 23]. *If $\hat{f} = Au + \hat{F}$ is in NF, i.e., $\hat{F} \in \operatorname{Ker} \mathscr{A}$, then any (formal) constant of motion ρ of the DS $\dot{u} = \hat{f}$ is also a (formal) constant of motion of the linear part $\dot{u} = Au$.*

LEMMA 4 [13, 16]. *Given the matrix A, the set $\operatorname{Ker} \mathscr{A}$ of terms $\hat{F}(u)$ resonant with A is given by $M(\mu(u))u$, where M is the most general matrix such that $[M, A] = 0$ and its entries M_{ij} are functions of the constants of motion $\mu = \mu(u)$ of the linear system $\dot{u} = Au$; or also, choosing a basis M_j in the space of the matrices M commuting with A, the most general $\hat{F}$ in NF is*

$$\hat{F} = \sum_j \mu_j(u) M_j u \qquad \text{with } Au \cdot \nabla \mu_j = 0 \text{ and } \mu_j(0) = 0. \tag{19}$$

According to Lemma 3, there are two essentially different ways in which a DS may admit a finite number of constants of motion, namely:

(1) the linear part $\dot{u} = Au$ admits a finite number of linearly independent constants of motion (then the same is true, *a fortiori*, for the full DS);

(2) the linear part does admit infinite constants of motion (functionally dependent, of course), but only a finite number of them (possibly none) are admitted by the full DS.

Both cases deserve some consideration.

Consider, as an example of case (1), a 2-dimensional system with $A = \operatorname{diag}(1, 2)$: then a constant of motion is $\rho = u_1^2/u_2$. This is *not* an analytic or a formal constant of motion, but it can be admitted in this context, because we are interested in analytic symmetries, which are described by vector functions of the form $g = Bu + G$, i.e., with $g(0) = 0$: then ρg may be analytic even if ρ is not. In the hypotheses of Theorem 1 we assumed $B_j \neq 0$; then the only admitted constants of motion are of the type $u_1^{m_1} \cdot \cdots \cdot u_n^{m_n}$ where *at most one* of the m_i may be -1. Therefore, only a finite number of independent constants of motion are admitted in this case. A situation of this type happens, e.g., when the resonant eigenvalues are real and have the same sign. From the point of view of the convergence of the normalizing transformations, this case is actually more conveniently treated with the notion of the Poincaré domain (which provides direct criteria of convergence [2]); the present discussion offers simply a different approach to the problem.

A more interesting situation occurs in case (2), which we are going to consider.

Assume, e.g., that the DS takes in NF the following special form (where—in the sum appearing in (19)—only the matrix A and some other matrix M, commuting with A and not proportional to A, are present):

$$\dot{u} = \hat{f} = Au + \alpha(u) Au + \mu(u) Mu. \tag{20}$$

According to Lemma 3, any (formal) constant of motion of (20) is also a constant of motion of the linear problem $\dot{u} = Au$. Therefore, to find a constant of motion of (20), we can start with a constant of motion ρ of the linear problem, i.e., $Au \cdot \nabla\rho = 0$; then we get

$$\left.\frac{d\rho}{dt}\right|_{\hat{f}} = \hat{f} \cdot \nabla\rho = \mu(u) Mu \cdot \nabla\rho \tag{21}$$

Now $Mu \cdot \nabla\rho = 0$ is verified if and only if ρ is a constant of motion of the other linear problem $\dot{u} = Mu$; therefore, if there are no common constants of motion of the two linear problems $\dot{u} = Au$ and $\dot{u} = Mu$, one has $d\rho/dt|_{\hat{f}} \neq 0$. In particular, this happens, e.g., if $M = I$ (identity) and the eigenvalues of A are nondegenerate: indeed the constants of motion of $\dot{u} = Iu$ are of the form u_i/u_j (or functions thereof), but the assumption on the eigenvalues of A excludes that these can be constants of motion of $\dot{u} = Au$. Then we can state the following:

THEOREM 1′. *Assume that*:

(i) *the DS* (1)–(2) *possesses a finite number* l (≥ 1) *of analytic symmetries* $g_j = B_j u + G_j$, *where all the matrices* B_j *are linearly independent* (*and not zero*), *and where* l *is precisely the number of the linearly independent linear symmetries admitted by the DS once in NF*;

(ii) *once in NF, the DS has the form* (20), *and the two linear problems* $\dot{u} = Au$ *and* $\dot{u} = Mu$ *do not admit common constants of motion.*

Then the DS can be put in NF by means of a convergent normalizing transformation.

THEOREM 1″. *Condition* (i) *in Theorem* 1′ *can be substituted by one of the two following*:

(i_1) *the DS* (1)–(2) *admits a nontrivial analytic symmetry* $g = Bu + G$ *such that* $B \neq 0$ *is proportional to* A;

(i_2) *the DS* (1)–(2) *admits an analytic symmetry* $g = G(u)$ *with vanishing linear part and* G *not proportional to* F.

Remarks. (a) Assumption (ii) in Theorem 1′ ensures that the NF (20) has no other symmetry apart from the linear ones $B_j u$. Instead, in the "trivial" case that Condition A is satisfied by the DS, so that the NF has the form $\dot{u} = (1 + \alpha(u))Au$, any constant of motion of the linear problem $\dot{u} = Au$ is clearly also a constant of motion of the full problem in NF. Then, in this case—in contrast with the case covered by Theorem 1′—the DS in NF does admit, in general, an infinite number of (functionally dependent) polynomial constants of motion and of symmetries as well.

(b) In the assumption (ii) of Theorem 1′ the request that no common constants of motion are present can be substituted by the request that the only common constants of motion are rational functions of degree 0. Our final example is an application of this assumption.

(c) In the case the DS satisfies condition (i_2) of Theorem 1″, it is clearly sufficient to consider the new nontrivial symmetry $g = G + f = Au + F + G$ to recover (i_1). The argument then proceeds as in Theorem 1.

4. APPLICATIONS AND EXAMPLES

We want first to show that the theorem by Bruno and Walcher [10] for 2-dimensional DS can be reobtained as a corollary of the above approach. We have in fact:

COROLLARY. *Consider a 2-dimensional DS* $\dot{u} = Au + F$ *such that the eigenvalues* a_1, a_2 *of the matrix* A *satisfy a relation* $k_1 a_1 + k_2 a_2 = 0$, *where* k_1, k_2 *are non-negative relatively prime integers, not simultaneously zero. Then, if the DS possesses an analytic symmetry, it can be put in NF by means of a converging transformation.*

Proof. Let $\dot{u} = Au + F$ the DS, and $g = Bu + G$ its symmetry. If $B = 0$ then consider the new symmetry $g + f = Au + (F + G)$, so that the linear part is now $\neq 0$. From $[A, M] = 0$ and observing that the eigenvalues are in this case necessarily distinct, one has that M must be a combination of A and I. Then, thanks to Lemma 4, the DS in NF takes just the form of (20):

$$\dot{u} = Au + \alpha(u) Au + \mu(u) u.$$

If $\mu = 0$, the NF satisfies Condition A and there is a convergent normalizing transformation. If $\mu \neq 0$, the same result follows from Theorem 1′: indeed, there are no constants of motion for the NF, and the only admitted symmetry is Au; then the argument proceeds just as before.

It can be remarked that in the case of dimension $n = 2$, one gets directly $l = 1$, and the occurrence of the special form (20) of the NF is guaranteed by Lemma 4. Then, all assumptions are automatically fulfilled in dimension $n = 2$. Instead, if $n > 2$ and the NF has the general form (19), it is clear that a constant of motion ρ of the linear part $\dot{u} = Au$ may be a constant of motion of the full problem $\dot{u} = Au + \hat{F}$, even if it is not a constant of motion of each one of the single problems $\dot{u} = M_j u$.

One of the possibilities which can guarantee the special form (20) of the NF and allows us to repeat the above argument on the number of constants of motion is the presence of some additional symmetries $g_j = B_j u + G_j$: thanks to Lemma 2, one has that $\hat{F}$ must satisfy $\{B_j u, \hat{F}\} = 0$, and this condition may exclude some of the matrices M_j in the expression (19). The next example will show this possibility and will also be a good illustration of the above discussion.

EXAMPLE. Consider the space R^{2m}, and put $u \equiv (x_1, \ldots, x_m, y_1, \ldots, y_m) \in R^{2m}$; assume that a Lie group Γ acts "diagonally" on both the m-dimensional spaces of the vectors x and y through the same linear representation $\mathscr{D}$, i.e., $x \to x' = \mathscr{D}x$, $y \to y' = \mathscr{D}y$. Consider then a DS of the form

$$\dot{u} = Au + F, \tag{22}$$

where

$$A = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix} \tag{22'}$$

and assume that $F(u)$ admits the symmetries $B_i u$, where B_i are the (matrix representatives in the direct sum $\mathscr{D} \oplus \mathscr{D}$ of the) Lie generators of this group Γ (the linear part Au fulfils this symmetry requirement, so that the full DS (22) admits this symmetry). Let us assume here for concreteness (the general case could be relevant for the study of Hamiltonian DS, see [17, 18], but we do not consider here this situation) the case $m = 3$, $\Gamma = SO(3)$ and $\mathscr{D}$ its fundamental representation. A DS satisfying the above assumptions is for instance

$$\dot{u} = Au + p(u)u, \tag{23}$$

where $p(u)$ is an analytic function which, thanks to the $SO(3)$ symmetry, depends on the quantities, $x^2 = (x, x)$, $y^2 = (y, y)$, $x \cdot y = (x, y)$ (the parentheses stand for the scalar product in R^3). Once in NF, this DS takes necessarily the form

$$\dot{u} = Au + \alpha Au + \mu u, \tag{24}$$

where α and μ are functions of $r^2 = x^2 + y^2$ only, thanks to Proposition 1, which ensures that the linear symmetries $B_i u$ are preserved, and to Lemma 4 as well. This NF has precisely the special form of (20), and it is easy to check that there are no constants of motion for (24), apart from 0-degree rational functions as $(x_1^2 + y_1^2)/(x_2^2 + y_2^2)$. It is important to notice here that, if our problem would not possess the symmetry $SO(3)$, the NF (24) would contain many other matrices M_i (according to Lemma 4), but that it is precisely the presence of the symmetry $SO(3)$ which forces the NF to contain only A and the identity. This seems to confirm the conjecture [9, 10, 17, 18] that the presence of a "sufficient" number of symmetries may be an essential request in order to guarantee the convergence of a normalizing transformation. Let us now assume (this example can in fact be viewed as a multidimensional extension of an example given in [10]) that in the initial DS (22) the function p is a homogeneous polynomial of degree k of the quantities $x^2, y^2, x \cdot y$: then the vector function

$$g = r^{2k} u \tag{25}$$

is a nontrivial analytic symmetry for our example (22) (in the case that $p = (x^2 + y^2)^k$, we can choose e.g., $g = (x \cdot y)^k u$); indeed

$$\{Au + pu, r^{2k} Iu\} = \{Au, r^{2k} Iu\} + p(u \cdot \nabla r^{2k})u - r^{2k}(u \cdot \nabla p)u = 0$$

and so we can conclude, e.g., from i_2) of Theorem 1″ and Theorem 1′, that the DS can be normalized by a convergent transformation.

ACKNOWLEDGMENTS

I am grateful to Professor A. D. Bruno for very useful and clarifying discussions and remarks and for his kind interest in this argument. Professor Bruno informed me that he succeeded in extending, under suitable conditions, the results in [10] to the case of dimension $n = 3$. I am also indebted to my friend Giuseppe Gaeta for valuable discussions.

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